



Foundations of Knowledge Representation

Lecture 9: Inconsistency Handling

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Inconsistency: Motivational Example

Suppose that you have a database that contains, among other things, the following pieces of information (in some form of code):

a : All European swans are white.

b : The bird caught in the trap is a swan.

c : The bird caught in the trap comes from Sweden.

d : Sweden is part of Europe.

From *a – d*, infer *e*: The bird caught in the trap is white.

Now suppose that: The bird caught in the trap turns out to be black.

Problem: By adding $\neg e$: the database becomes inconsistent.

Inconsistency Handling - Overview

We can distinguish between two general approaches to handle inconsistency:

- (1) **Paraconsistent Reasoning**, i.e. reasoning from inconsistent knowledge bases:
 - One option is to consider only consistent subsets of the database (*i.e. maximal consistent subsets*).
 - Another option is to leave the database inconsistent, but to prohibit the logics from deriving trivial inferences (*i.e. paraconsistent logics*).
- (2) **Belief Revision**, integrate knowledge into a given knowledge base and preserve consistency.

Inconsistency Handling- Outline

- 1 Paraconsistent Reasoning**
 - Maximal Consistent Subsets
 - Four-Valued Logic

- 2 Belief Revision**
 - AGM
 - Contraction
 - Epistemic Entrenchment

Maximal Consistent Subsets

A very natural approach to draw plausible conclusions from an inconsistent belief base amounts to taking advantage of the **maximal consistent subsets** of the base.

Definition (Consistency)

A set Γ of propositions is consistent if there is no φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

Definition (Maximal Consistent)

A set Γ of propositions is maximal consistent iff

- (1) Γ is consistent.
- (2) $\Gamma \subseteq \Gamma'$ and Γ' consistent $\Rightarrow \Gamma = \Gamma'$.

Maximal Consistent Subsets

Goal: define an **inference relation** based on an **inference mechanism** that is based on maximal consistent subsets.

Consider the following ones:

- (1) **Skeptical inference**, where a formula is a consequence of the base if it is implied by each maximal consistent subset,
- (2) **Credulous inference**, where a formula is a consequence of the base if it is implied by at least one maximal consistent subset,
- (3) **Argumentative inference**, where a formula is a consequence of the base if it is implied by at least one maximal consistent subset and its negation is not.

Maximal Consistent Subsets - Formally

Denote the set of all maximal consistent subsets of Γ with $MC(\Gamma)$.

Then:

Skeptical inference : $\Gamma \vdash_{\text{skeptical}} \varphi$ iff $\Sigma \vdash \varphi$ for all $\Sigma \in MC(\Gamma)$

Credulous inference : $\Gamma \vdash_{\text{credulous}} \varphi$ iff $\Sigma \vdash \varphi$ for at least one $\Sigma \in MC(\Gamma)$

Argumentative inference : $\Gamma \vdash_{\text{argumentative}} \varphi$ iff $\Sigma \vdash \varphi$ for at least one $\Sigma \in MC(\Gamma)$
and there is no $\Sigma \in MC(\Gamma)$ such that $\Sigma \vdash \neg\varphi$

Maximal Consistent Subsets - Example

Suppose that we have the following (contradictory) pieces of information that were gathered concerning an incoming model of a car from your favorite brand:

Example (Inconsistent Database K)

We have six pieces of information ($\varphi_1 - \varphi_6$):

$$\varphi_1 = \mathit{esc}$$

$$\varphi_2 = \mathit{mg}$$

$$\varphi_3 = \mathit{t} \wedge (\mathit{esc} \rightarrow \neg \mathit{sc})$$

$$\varphi_4 = \mathit{lc} \wedge \neg \mathit{mg}$$

$$\varphi_5 = \mathit{sc} \wedge \neg \mathit{mg} \wedge (\mathit{sc} \rightarrow \neg \mathit{lc}) \wedge (\mathit{esc} \rightarrow \neg \mathit{wdf})$$

$$\varphi_6 = \mathit{wdf} \wedge \neg \mathit{mg} \wedge (\mathit{wdf} \rightarrow \neg \mathit{lc}) \wedge (\mathit{esc} \rightarrow \neg \mathit{t})$$

Let: **esc**: has a 6-cylinder engine ; **mg**: has a manual gearbox ; **t**: has a turbo; **sc**: is a sport car; **lc**: has low fuel consumption ; **wdf**: has a 4-wheel drive

Maximal Consistent Subsets - Example

From this, we can derive the following maximal consistent subsets:

$$(K1) = \{\varphi_1, \varphi_6\},$$

$$(K2) = \{\varphi_1, \varphi_5\},$$

$$(K3) = \{\varphi_1, \varphi_2, \varphi_3\},$$

$$(K4) = \{\varphi_1, \varphi_3, \varphi_4\},$$

$$(K5) = \{\varphi_3, \varphi_5, \varphi_6\}$$

Inference:

Using **skeptical inference** from all these maximal consistent subsets, none of the six formulae in K can be derived as a conclusion.

Using **credulous inference** from all these maximal consistent subsets, all of the six formulae in K can be derived as a conclusion.

Using **argumentative inference** from all these maximal consistent subsets, none of the six formulae in K can be derived as a conclusion.

Paraconsistent Logics

There is a range of so-called **paraconsistent logics** for reasoning with inconsistency. Selecting an appropriate paraconsistent logic for an application depends on the requirements of the application.

Types of paraconsistent logics that have proved to be of use for knowledge representation and reasoning:

- (1) Weakly-negative logics
- (2) Four-valued logics
- (3) Signed systems
- (4) Quasi-classical logic

For more information on systems (1), (3), and (4) see: Anthony Hunter: Paraconsistent Logics, 1998)

Four-Valued Logic

Idea: Use a subset of the classical language and a subset of the classical proof theory, together with an intuitive four-valued semantics.

Definition (Language)

The language for four-valued logic is a subset of classical logic. Let P be the usual set of formulae of classical logic that is formed using the connectives \neg , \wedge , and \vee . Then the set of formulae of the language, denoted Q , is $P \cup \{\alpha \rightarrow \beta \mid \alpha, \beta \in P\}$. That means, implication is not nestable.

Definition (Values)

A formula in the language can be one of "true", "false", "both" or "neither", which we denote by the symbols T, F, B, and N, respectively.

Four-Valued Logic - Example

Imagine an **artificial reasoner (AR)** to operate as follows:

- (1) The AR receives assertions and denials of atomic sentences. Upon receiving an assertion, it is to mark the item **"told true"**. Upon receiving a denial, it is to mark the item **"told false"**.
- (2) The AR has exactly four possibilities for any particular atomic sentence:
 - Told true but never told false*, i.e. {"told true"}
 - Told false but never told true*, i.e., {"told false"};
 - Never told true and never told false*, i.e., {};
 - Told true and told false*, i.e., {"told true", "told false"}.
- (3) The four possibilities correspond to our four values: T, F, None, Both.

Four-Valued Logic - Example

- (4) The AR may have questions put to it, which it is to answer. In terms of answering a question of the form "p?", it answers as follows:

If the item has the value T then answer "yes";

If the item has the value F then answer "no";

If the item has the value None then answer "don't know"

If the item has the value Both then answer "yes and no".

- (5) If it receives additional information, it incorporates it into the information it already has. It does this by representing what it has been told by a [setup](#).

Four-Valued Logic - Example

Definition (Setup)

A setup maps atomic sentences into the set $4 = \{T, F, \text{None}, \text{Both}\}$, according to the following rules:

If an atomic formula p is **affirmed**, the setup is revised as follows:

*If the current value of the atomic formula p is **None**, it is mapped to **T**.*

*If the current value of the atomic formula p is **Both**, it is mapped to **Both**.*

*If the current value of the atomic formula p is **T**, it is mapped to **T**.*

*If the current value of the atomic formula p is **F**, it is mapped to **Both**.*

If an atomic formula p is **denied**, the setup revised as follows:

What do you think? Left as an exercise.

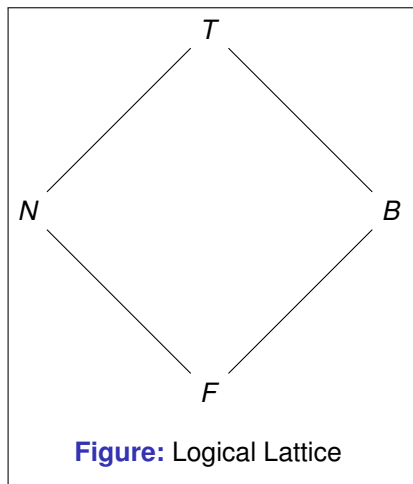
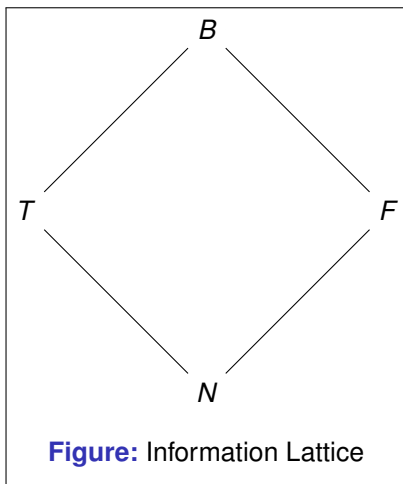
Four-Valued Logic - Example

Thus we envision that the database we are working with can be represented by a **list of statements**.

Consider the **2000 presidential election** in the US and a hypothetical setup for the states (*NC*), (*FL*), (*CA*) as well as the candidates *Bush*, *Gore*, and *Nader*.

Statement	Truth Value Assigned
Bush won NC	{told true} = T
Gore won NC	{told false} = F
Nader won NC	{told false} = F
Bush won FL	{told true, told false} = Both
Gore won FL	{told true, told false} = Both
Nader won FL	{told false} = F
Bush won CA	{ } = None
Gore won CA	{ } = None
Nader won CA	{ } = None

Four-Valued Logic - Natural Orderings



Four-Valued Logic - Semantics

Basic Assumption: For the semantics we assume the logical lattice.

Definition (Semantic Assignment Function, Properties)

The semantic assignment function observes monotonicity and complementation in the logical lattice. So $\neg x$ is the complement of x , $x \wedge y$ is the meet of $\{x, y\}$ and $x \vee y$ is the join of $\{x, y\}$, giving the following truth tables for the \neg, \wedge, \vee connectives:

Four-Valued Logic - Semantics

Table: Truth table for conjunction

\wedge	N	F	T	B
N	N	F	N	F
F	F	F	F	F
T	N	F	T	B
B	F	F	B	B

Table: Truth table for negation

α	N	F	T	B
$\neg\alpha$	N	T	F	B

Table: Truth table for disjunction

\vee	N	F	T	B
N	N	N	T	T
F	N	F	T	B
T	T	T	T	T
B	T	B	T	B

Four-Valued Logic - Semantics

Intuitive Idea: An inference in our four-valued logic never takes one farther away from the truth:

i.e., never takes one to a conclusion that is "less true" than the premises.

Definition (Inference)

The inference β from α is valid iff $\alpha \leq \beta$, where \leq is the ordering relation for the logical lattice. Let $\alpha \rightarrow \beta$ signify that the inference from α to β is valid in our four values, i.e. that α entails β .

Belief Revision

Consider again the following situation:

a : All European swans are white.

b : The bird caught in the trap is a swan.

c : The bird caught in the trap comes from Sweden.

d : Sweden is part of Europe.

From *a – d*, infer *e*: The bird caught in the trap is white.

Now suppose that: The bird caught in the trap turns out to be black.

Problem: By adding $\neg e$: the database becomes inconsistent.

This is a typical belief revision scenario: A rational agent receives new information that makes her change her beliefs.

Belief Revision - Outline

- (1) Preliminaries
- (2) AGM Postulates
- (3) Contraction Postulates
- (4) AGM and Contraction
- (5) Epistemic Entrenchment

Methodological Problems and Kinds of Belief Change

Methodological Problems:

- (1) How are the beliefs in the database represented?
- (2) What is the relation between the elements explicitly represented in the database and the beliefs that may be derived from these elements?
- (3) How are the choices concerning what to retract made?

Kinds of Belief Change:

- (1) **Expansion:** A new sentence is added to a belief system \mathcal{B} together with the logical consequences of the addition (regardless of whether the larger set so formed is consistent).
- (2) **Revision:** A new sentence that is inconsistent with a belief system \mathcal{B} is added, but, in order to maintain consistency in the resulting belief system, some of the old sentences in \mathcal{B} are deleted.
- (3) **Contraction:** Some sentence in \mathcal{B} is retracted without adding any new facts. In order for the resulting system to be closed under logical consequences some other sentences from \mathcal{B} must be given up.

Formal Preliminaries

For our framework, we are working with a language \mathcal{L} based on first-order logic.

\mathcal{L} is closed under application of boolean operators:

\neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (implication)

\mathcal{L} is identified by its consequence relation \vdash :

- (1) $\vdash \varphi$ for all truth-functional tautologies.
- (2) If $\vdash (\varphi \rightarrow \psi)$ and $\vdash \varphi$, then $\vdash \psi$.
- (3) \vdash is consistent, i.e. $\not\vdash \perp$.
- (4) \vdash satisfies the deduction theorem. that is, $\{\varphi_1, \varphi_2, \dots, \varphi_n\} \vdash \psi$ iff $\vdash \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \rightarrow \psi$.
- (5) \vdash is compact.

The AGM Paradigm

In the AGM paradigm, **beliefs** are represented as sentences of \mathcal{L} and **belief sets** as theories of \mathcal{L} are closed under logical consequence.

The process of belief revision is modeled as a **function** $*$ mapping a **belief set** \mathcal{B} and a sentence φ to a new belief set $\mathcal{B} * \varphi$.

We want $*$ to capture the notion of rational belief revision correctly. Hence, we need to impose certain constraints. These constraints (next slide) are based on the principle of minimal change:

Principle of minimal change:

A rational agent ought to change her beliefs as little as possible in order to (consistently) accommodate the new information.

The AGM Postulates for Belief Revision

($\mathcal{B} * 1$) $\varphi \in \mathcal{B} * \varphi$.

($\mathcal{B} * 2$) $\mathcal{B} * \varphi \subseteq \mathcal{B} + \varphi$.

($\mathcal{B} * 3$) If $\neg\varphi \notin \mathcal{B}$ then $\mathcal{B} + \varphi \subseteq \mathcal{B} * \varphi$.

($\mathcal{B} * 4$) If φ is consistent then $\mathcal{B} * \varphi$ is also consistent.

($\mathcal{B} * 5$) If $\vdash \varphi \leftrightarrow \psi$ then $\mathcal{B} * \varphi = \mathcal{B} * \psi$.

($\mathcal{B} * 6$) $\mathcal{B} * (\varphi \wedge \psi) \subseteq (\mathcal{B} * \varphi) + \psi$.

($\mathcal{B} * 7$) If $\neg\psi \notin \mathcal{B} * \varphi$ then $(\mathcal{B} * \varphi) + \psi \subseteq \mathcal{B} * (\varphi \wedge \psi)$.

The AGM Postulates - Explanation

- ($\mathcal{B} * 1$) : Says that the new information on φ should always be included in the new belief set.
- ($\mathcal{B} * 2$) and ($\mathcal{B} * 3$) : Together, they state that whenever the new information φ does not contradict the initial belief set \mathcal{B} , there is no reason to remove any of the original beliefs at all; the new belief state will contain \mathcal{B} , the new information, and what follows from the logical closure.
- ($\mathcal{B} * 4$) : Says that the agent should aim for consistency.
- ($\mathcal{B} * 5$) : Says that the syntax of the new information has no effect on the revision process.
- ($\mathcal{B} * 6$) and ($\mathcal{B} * 7$) : They say that for any two sentences φ and ψ , if in revising the initial belief set \mathcal{B} by φ one can reach a belief set $\mathcal{B} * \varphi$ that is consistent with ψ , then to produce $\mathcal{B} * (\varphi \wedge \psi)$ all that one needs to do is to expand $\mathcal{B} * \varphi$ with ψ .

Belief Contraction

Contraction can be described as the process of rationally removing a certain belief φ from a belief set \mathcal{B} .

It is also formally defined as a function $\dot{-}$ mapping a belief set \mathcal{B} and a sentence φ to a new belief set $\mathcal{B} \dot{-} \varphi$.

Contraction typically occurs when an agent loses faith in φ and decides to give it up.

Problem: It does not suffice to simply take out φ from \mathcal{B} . Other sentences in \mathcal{B} could reproduce φ through logical closure.

Example

Consider the belief set $\mathcal{B} = \text{Cn}(\{p \rightarrow q, p, q\})$ and assume that we want to contract \mathcal{B} by q . Then, not only do we have to remove q from \mathcal{B} , but we also need to give up (at least) one of $p \rightarrow q$ or p , for otherwise q will resurface via logical closure.

Contraction Postulates

(B $\dot{-}$ 1) $\mathcal{B} \dot{-} \varphi \subseteq \mathcal{B}$.

(B $\dot{-}$ 2) If $\varphi \notin \mathcal{B}$ then $\mathcal{B} \dot{-} \varphi = \mathcal{B}$.

(B $\dot{-}$ 3) If $\not\vdash \varphi$ then $\varphi \notin \mathcal{B} \dot{-} \varphi$.

(B $\dot{-}$ 4) If $\varphi \in \mathcal{B}$, then $\mathcal{B} \subseteq (\mathcal{B} \dot{-} \varphi) + \varphi$.

(B $\dot{-}$ 5) If $\vdash \varphi \leftrightarrow \psi$ then $\mathcal{B} \dot{-} \varphi = \mathcal{B} \dot{-} \psi$.

(B $\dot{-}$ 6) $(\mathcal{B} \dot{-} \varphi) \cap (\mathcal{B} \dot{-} \psi) \subseteq \mathcal{B} \dot{-} (\varphi \wedge \psi)$.

(B $\dot{-}$ 7) If $\psi \notin \mathcal{B} \dot{-} (\varphi \wedge \psi)$ then $\mathcal{B} \dot{-} (\varphi \wedge \psi) \subseteq \mathcal{B} \dot{-} \varphi$.

Contraction Postulates - Explanation

- ($\mathcal{B} \dot{-} 1$) Says that contraction produces a belief set smaller than or equal to the original one.
- ($\mathcal{B} \dot{-} 2$) Says that if φ is not in the initial belief set \mathcal{B} to start with, then there is no reason to change anything at all.
- ($\mathcal{B} \dot{-} 3$) Tells us that all sentences but tautologies can be removed from the initial beliefs \mathcal{B} .
- ($\mathcal{B} \dot{-} 4$) Says that contracting and then expanding by φ will give us back (at least) the initial theory \mathcal{B} .

Contraction Postulates - Explanation

- ($\mathcal{B} \dot{-} 5$) Tells us that contraction by logically equivalent sentences produces the same result. That is, contraction is not syntax-sensitive.
- ($\mathcal{B} \dot{-} 6$) Tells us that if there is some belief $\chi \in \mathcal{B}$ that is neither related to φ nor ψ , it should not be affected by the contraction of \mathcal{B} by $\varphi \wedge \psi$.
- ($\mathcal{B} \dot{-} 7$) Since $\mathcal{B} \dot{-} \varphi$ is the minimal change of \mathcal{B} to remove φ , it follows that $\mathcal{B} \dot{-} (\varphi \wedge \psi)$ cannot be larger than $\mathcal{B} \dot{-} \varphi$. This postulates makes it smaller or equal to it.

Relation between revision and contraction functions

Idea: Define revision in terms of contraction as follows:

To revise \mathcal{B} by φ , first contract \mathcal{B} by $\neg\varphi$ (thus removing anything that may contradict the new information) and then expand the resulting theory with φ .

This is known as the *Levi Identity* :

Levi Identity

$$\mathcal{B} * \varphi = ((\mathcal{B} \dot{-} \neg\varphi) + \varphi)$$

Relation between revision and contraction functions

Now: Define contraction in terms of revision. The idea is that:

A sentence ψ is accepted in the contraction $\mathcal{B} \dot{-} \varphi$ iff ψ is accepted both in \mathcal{B} and in $\mathcal{B} \dot{-} \neg\varphi$.

This is known as the *Harper Identity* :

$$\text{Harper Identity } \mathcal{B} \dot{-} \varphi = (\mathcal{B} * \neg\varphi) \cap \mathcal{B}.$$

Epistemic Entrenchment

Motivation: Agents can perceive their individual beliefs to have different epistemic values. Consider, for instance:

ψ All swans are white. (law-like belief)

χ Lucy is a swan. (simple belief)

Idea: Introduce the notion of *epistemic entrenchment*. Intuitively, this notion can be understood as follows:

Epistemic Entrenchment:

The epistemic entrenchment of a belief ψ is the degree of resistance that ψ exhibits to change: the more entrenched ψ is, the less likely it is to be swept away during contraction by some other belief φ .

Epistemic Entrenchment - Formally

Epistemic entrenchment is defined as a preorder \leq on L encoding the "retractibility" of individual beliefs.

Preorder \leq :

$\chi \leq \psi$ iff the agent is at least as (or more) reluctant to give up ψ than she is to give up χ .

Constraints on \leq in relation to a given \mathcal{B} :

- (EE1)** If $\varphi \leq \psi$ and $\psi \leq \chi$ then $\varphi \leq \chi$.
- (EE2)** If $\varphi \vdash \psi$ then $\varphi \leq \psi$.
- (EE3)** $\varphi \leq \varphi \wedge \psi$ or $\psi \leq \varphi \wedge \psi$.
- (EE4)** When \mathcal{B} is consistent, $\varphi \notin \mathcal{B}$ iff $\varphi \leq \psi$ for all $\psi \in L$.
- (EE5)** If $\psi \leq \varphi$ for all $\psi \in L$, then $\vdash \varphi$.

Epistemic Entrenchment Postulates - explanation

- (EE1) States that \leq is transitive.
- (EE2) Says that the stronger a belief is logically, the less entrenched it is.
- (EE3) Says that if one wants to retract $\varphi \wedge \psi$ from \mathcal{B} , this can only be achieved by giving up either φ or ψ .
- (EE4) Says that in the principle case where \mathcal{B} is consistent, all non-beliefs (all sentences that are not in \mathcal{B}) are minimally entrenched.
- (EE5) Says that tautologies are the only maximal elements of \leq and therefore the hardest to remove.

Epistemic Entrenchment and Contraction

In a next step, we define contraction in terms of epistemic entrenchment by defining the condition (C-):

Definition (C-)

$\psi \in \mathcal{B} \dot{-} \varphi$ iff $\psi \in \mathcal{B}$ and either $\varphi < \psi$ or $\vdash \varphi$.

With that definition at hand, we get the following *representation theorem*:

Theorem

Let \mathcal{B} be a theory of L . If \leq is a preorder in L that satisfies the axioms (EE1)-(EE5) then the function defined by (C-) is an AGM contraction function. Conversely, if $\dot{-}$ is an AGM contraction function, then there is a preorder \leq in L that satisfies the axioms (EE1)-(EE5) as well as condition (C-).