Project Thesis

Generalized Ultrametric Spaces in Quantitative Domain Theory

by

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Abstract

Domains and metric spaces are two central tools for the study of denotational semantics in computer science, but are otherwise very different in many fundamental aspects. A construction that tries to establish links between both paradigms is the space of formal balls, a continuous poset which can be defined for every metric space and that reflects many of its properties. On the other hand, in order to obtain a broader framework for applications and possible connections to domain theory, generalized ultrametric spaces (gums) have been introduced. In this paper, we employ the space of formal balls as a tool for studying these more general metrics by using concepts and results from domain theory. It turns out that many properties of the metric can be characterized by conditions on its formal-ball space. Furthermore, we can state new results on the topology of gums as well as two modified fixed point theorems, which may be compared to the Prieß-Crampe and Ribenboim theorem and the Banach fixed point theorem, respectively. Deeper insights into the nature of formal-ball spaces are gained by applying methods from category theory. Our results suggest that, while being a useful tool for the study of gums, the space of formal balls cannot provide the hoped-for general connection to domain theory.

Contents

1 Introduction

Domain theory and the theory of metric spaces are the two central utilities in the study of denotational semantics in computer science. Although both formalisms are capable of capturing the relevant aspects of computation and approximation, they do so in very different ways. Consequently, various methods for relating both paradigms have been sought, establishing a line of research that is now known as *quantitative domain theory*.

In [EH98], a construction for obtaining a partially ordered set from a given (classical) metric space was introduced. This order was called the *space of formal balls* and was shown to be a continuous poset with properties closely related to the metric from which it originated. It could also be shown that the space of formal balls can be employed as a tool for proving known results for metric spaces, and that it yields a *computational model* for the metric topology. In [HS03], the concept of a formal-ball space was then extended to the setting of *generalized ultrametric spaces* (gums), i.e. non-Archimedian metrics that can have sets of distances other than the real numbers. Restricting to gums with (linearly ordered) ordinal distance sets, a constructive proof of the Prieß-Crampe and Ribenboim fixed point theorem (see [PCR99]) could be obtained.

However, beyond this result, little is known about the space of formal balls for generalized ultrametric spaces, especially in the situation where no additional restrictions are imposed on the distance set. Thus, an initial objective of this work will be to establish detailed relations between these concepts, following the lines of [EH98]. In Section 3, we shall see that completeness and continuity of formal ball spaces can indeed be characterized in a convenient way. Furthermore, as in the case of classical metric spaces, the space of formal balls can serve as a computational model for the metric topology of a gum. As a side effect, this will shed some light on the role of the open ball topology for gums.

Another central question that is to be addressed in this paper is whether the construction of spaces of formal balls can connect the theory of gums and domain theory in a general sense. For this purpose, we will call upon the formalism of category theory in Section 4 and establish a categorical equivalence between suitable categories of gums on the one hand and partially ordered sets on the other. It will turn out that the spaces of formal balls actually form a very restricted class of partial orders and the utility of this approach to quantitative domain theory may thus be doubted.

Finally, in Section 5, we present two fixed point theorems for gums, which are compared with the Prieß-Crampe and Ribenboim theorem and the Banach fixed point theorem, respectively. Together with the former application of the space of formal balls for the investigation of the metric topology, this demonstrates the use of this construction as a tool for obtaining proofs.

During our considerations, we will also introduce numerous restrictions on the very general definition of gums. Since these restrictions often give tight characterizations of certain desirable situations, they may turn out to be useful for choosing reasonable settings for future investigations of gums.

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2 Preliminaries and notation

In this section, we provide basic definitions of various concepts that are needed below. Beside some remarks on notation, it is concerned with the fundamentals of generalized ultrametric spaces, domain theory, topology, and category theory.

2.1 Partial orders

For the basic notions of order theory we recommend [DP02] as a standard reference. We assume the reader to be familiar with the corresponding notions and restrict to some remarks on the notation that we will employ below.

For a partially ordered set Γ, we use Γ^∂ to denote the order dual of Γ. Care will be taken to clarify to what version of a poset a given order-theoretic property or limit-construction refers to. For this purpose, we will sometimes use notations such as $\leq^\partial$.

Since we will have to deal with more than one order most of the time, we will be careful to distinguish between the according constructions. For instance, least upper bounds within the orders $\leq$, $\sqsubseteq$, and $\leq^\partial$ will be denoted by $\bigvee$, $\bigsqcup$, and $\bigvee^\partial$, respectively.

For a partial order $\leq$, $<$ will be used to denote the strict order induced by $\leq$.

2.2 Generalized ultrametric spaces

Definition 2.1 Let X be a set and let $(\Gamma, \leq)$ be a partially ordered set with least element $\bot$. (X, d, Γ) is a *generalized ultrametric space* (*gum*) if $d : X \times X \to \Gamma$ is a function such that, for all x,y,$z \in X$ and all $\gamma \in \Gamma$, we have:

(U1) $d(x, y) = \bot$ implies $x = y$.

(U2) $d(x, x) = \bot$.

(U3) $d(x, y) = d(y, x)$.

(U4) If $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$.

These properties will be called *identity of indiscernibles* (U1), *reflexivity* (U2), *symmetry* (U3), and the *strong triangle inequality* (U4), respectively. The poset Γ will be referred to as the *set of distances* of a gum. In the following we will only consider gums where the set of points X is non-empty.

The next definition introduces an important tool in our study of generalized ultrametric spaces, which was first defined for the general case in [Hit01] and [HS03]. It is motivated by a similar construction for classical metric spaces, that was introduced in [EH98].

Definition 2.2 Let (X, d, Γ) be a generalized ultrametric space. We define an equivalence relation $\approx$ on $X \times \Gamma$ by setting $(x, \alpha) \approx (y, \beta)$ iff $\alpha = \beta$ and $d(x, y) \leq \alpha$.

The *space of formal balls* $(\mathbf{B}X, \sqsubseteq)$ is an ordered set, where $\mathbf{B}X = (X \times \Gamma)|_{\approx}$ is the set of all $\approx$-equivalence classes and, for all $[(x, \alpha)]$, $[(y, \beta)] \in \mathbf{B}X$, we have $[(x, \alpha)] \sqsubseteq [(y, \beta)]$ iff $\beta \leq \alpha$ and $d(x, y) \leq \alpha$.

It is easy to see that $(\mathbf{B}X, \sqsubseteq)$ is a well-defined partially ordered set. In the following, (X, d, Γ) will be a generalized ultrametric space and $\mathbf{B}X$ will be used to abbreviate its space of formal balls. Sets of the form $\{y \mid d(x, y) \leq \alpha\}$ will be called *closed ball* with center x and radius α and are denoted by $\overline{B}_\alpha(x)$. Similarly, *open balls* are sets of the form $B_\alpha(x) = \{y \mid d(x, y) < \alpha\}$.

Definition 2.3 A gum (X, d, Γ) is

(i) *spherically complete* if every non-empty chain C of closed balls of X, ordered by subset inclusion, has non-empty intersection $\bigcap C \neq \emptyset$,

(ii) *chain-spherically complete* if every non-empty chain C of closed balls of the form $C = \{B_\beta(x_\beta) \mid \beta \in \Lambda\}$, where Λ is a chain in Γ, has non-empty intersection.

Note that any chain of closed balls whose set of radii is a chain in Γ has a form as in (ii), since any two $\subseteq$-comparable balls with the same radius coincide. This is an immediate consequence of the fact that every point inside a closed ball is also its center, a well-known fact for ultrametrics (see also [Hit01]).

It is clear from the definition that every spherically complete gum is also chain-spherically complete. To see that the converse is not true, consider the following example.

Example 2.4 Let $X = \{x_i \mid i \in \mathbb{N}\}$ be a set of points and let $\Gamma = \{\perp\} \cup \{\rho_i \mid i \in \mathbb{N}\} \cup \{\delta_{ij} \mid i < j;\ i, j \in \mathbb{N}\}$ be a set of distances, where we assume all elements in these sets to be mutually distinct. To define an ordering $\leq$ on Γ, let $\alpha < \beta$ iff either $\alpha = \perp \neq \beta$ or $\alpha = \delta_{ij}$, $\beta = \rho_k$, and $k \leq i$ (see Figure 1). We define a gum (X, d, Γ) by setting

$$d(x_i, x_j) = \begin{cases} \perp & \text{if } i = j \\ \delta_{ij} & \text{if } i < j \\ \delta_{ji} & \text{if } j < i \end{cases}$$

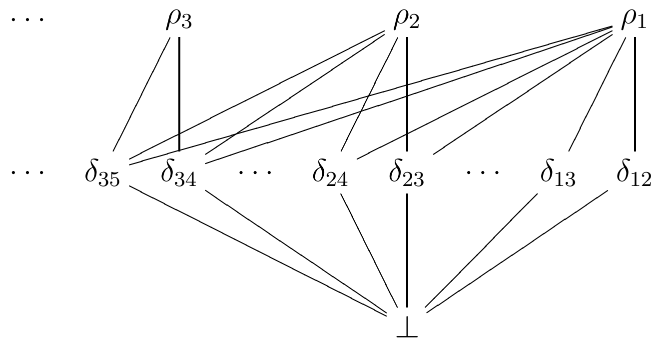

Figure 1: A diagram indicating the order on Γ from Example 2.4.

To see that this definition indeed yields a gum, first note that properties (U1), (U2), (U3) of Definition 2.1 follow directly from the definition of d. For the strong triangle inequality, consider points x_i, x_j, $x_k \in X$ and a distance $\alpha \in \Gamma$. Assume $d(x_i, x_j) \leq \alpha$ and $d(x_j, x_k) \leq \alpha$. We distinguish several cases:

(i) If $i = k$ then $d(x_i, x_k) = \perp \leq \alpha$ follows immediately.

(ii) If $i = j$ then $d(x_i, x_k) = d(x_j, x_k) \leq \alpha$. The case $j = k$ is treated similarly.

(iii) If i, j and k are mutually distinct then α clearly cannot be $\perp$. Furthermore, α cannot be of the form δ_{lm}, since this would require the distances $d(x_i, x_j)$ and $d(x_j, x_k)$ to be δ_{lm} or $\perp$, which both would cause some of the indices to be equal. Thus $\alpha = \rho_l$ for some $l \in \mathbb{N}$. Since $\perp < d(x_i, x_j) \leq \alpha$ and $\perp < d(x_j, x_k) \leq \alpha$, we obtain $l \leq \min(i, j)$ and $l \leq \min(j, k)$. Especially, $l \leq i$ and $l \leq k$, which implies $l \leq \min(i, k)$. By the definition of d and $(\Gamma, \leq)$ this entails $d(i, k) \leq \rho_l$.

Thus we have proven x to be a gum. Next we want to show that X is chain-spherically complete. But this follows immediately, since the longest chains Λ in Γ contain just three elements and finite chains of balls will always have non-empty intersection.

Now consider the family C of closed balls $\left(\overline{B}_{\rho_i}(x_i)\right)_{i \in \mathbb{N}}$. From the definition of (X, d, Γ), we derive $\overline{B}_{\rho_i}(x_i) = \{x_j \mid j \geq i\}$. Now it is easy to see that C is a chain of balls of X with $\bigcap C = \emptyset$. Thus X is not spherically complete.

2.3 Domains

In the following, we briefly introduce the very basics of domain theory and some results we will need in the subsequent sections. For a more extensive treatment of the subject, we refer to [AJ94] and [GHK+03].

Definition 2.5 Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$ be non-empty.

- A is *directed* if, for every $a, b \in A$, there is $c \in A$, such that $a \leq c$ and $b \leq c$.
- P is a *directed complete partial order* (*dcpo*), if every directed subset of P has a supremum.
- P is a *complete partial order* (*cpo*), if P is a dcpo with least element.

We will consider continuity for arbitrary posets without any additional assumption of completeness.

Definition 2.6 Let P be a poset, let $a, b \in P$, and let $B \subseteq P$.

- *a approximates b*, written $a \ll b$, if, for every directed set $A \subseteq P$ that has a supremum, $\bigvee A \geq b$ implies $c \geq a$ for some $c \in A$.
- a is *compact* if $a \ll a$.
- The set $\{c \in P \mid \ c \ll a\}$ is denoted $\twoheaddownarrow a$. The set $\twoheaduparrow a$ is defined analogously.
- B is a *base* of P if, for all $c \in P$, there is a directed subset $A \subseteq B \cap \twoheaddownarrow c$ that has the supremum c.
- P is (ω-) continuous if it has a (countable) base.
- P is (ω-) algebraic if it has a (countable) base of compact elements.

Lemma 2.7 Let P be a continuous dcpo with greatest element $\top$. For any base B of P, $(B \cap \twoheaddownarrow \top)$ is also a base. Especially, $\twoheaddownarrow \top$ is a base of P.

Proof. Consider some base B and an element $p \in P$. There is a directed set $A \subseteq B \cap \twoheaddownarrow p$ with supremum p. For any element $a \in A$, we find that $a \ll p$ and $p \leq \top$ imply $a \ll \top$. Thus, $A \subseteq B \cap \twoheaddownarrow \top \cap \twoheaddownarrow p$. Since p has been arbitrary, this shows that $B \cap \twoheaddownarrow \top$ is a base of P. The rest of the claim follows, since P is a base of P by continuity. $\square$

The appropriate homomorphisms between dcpos are Scott-continuous functions:

Definition 2.8 Let P and Q be dcpos and let $f : P \to Q$ be a monotonic mapping. f is *(Scott-) continuous* if, for every directed set $A \subseteq P$, $\bigvee f(A) = f(\bigvee A)$.

Finally, we give some basic results without proofs.

Proposition 2.9 ([AJ94, Proposition 2.1.15]) A partially ordered set P is a dcpo iff each chain in P has a supremum.

However, this result depends on the Axiom of Choice. The next result is also known as the *dcpo fixed point theorem.*

Proposition 2.10 ([AJ94, Proposition 2.1.19]) Let P be a cpo with least element $\perp$ and let $f : D \to D$ be Scott-continuous. Then f has a least fixed point given by $\bigvee_{n\in\mathbb{N}} f^n(\perp)$.

One can, however, also obtain fixed points if f is not Scott-continuous.

Proposition 2.11 ([DP02, Theorem 8.22]) Let P be a cpo and let $f : D \to D$ be monotonic. Then f has a least fixed point.

2.4 Topological spaces

In this section, we summarize some concepts and results from topology that are needed below. Our main reference for these topics is [Smy92].

A topology $\mathcal{T}$ on a set X is a system of subsets of X that is closed under arbitrary unions and finite intersections, and that contains both X and the empty set. In this situation, $(X, \mathcal{T})$ is called a *topological space* and the elements of $\mathcal{T}$ are called *open* sets. A set is *closed* if it is the complement of an open set and the *closure* of a set S is the smallest closed set that contains S.

Let B be a set of subsets of X. The smallest topology $\mathcal{T}$ that contains B is called the topology *generated* by B, and B is then a *subbase* of $\mathcal{T}$. If the set of all (possibly infinite) unions of sets from B forms a topology $\mathcal{T}$, then B is a *base* of $\mathcal{T}$. Given a topological space $(X, \mathcal{T})$, a subset $D \subseteq X$ is *dense* in $\mathcal{T}$ if it meets every open set. A *separable* topological space is one that has a countable dense subset.

A function f between the sets of points of two topological spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ is *continuous*, if the inverse image of every open set of $\mathcal{T}$ of f yields an open set of $\mathcal{S}$. If f is a bijective mapping and both f and f^{-1} are continuous, then f is a *homeomorphism.*

Next, we will specify some special topological spaces which will appear in our treatment.

Definition 2.12 Consider a gum (X, d, Γ). The topology generated by the subbase $\{B_\alpha(x) \mid x \in X,\ \alpha \in \Gamma\}$ is called the *metric topology* or the *topology of open balls* of X.

This definition is motivated by the definition for the standard topology for classical metric spaces. However, in the general case, open balls have no reason to form a base for a topology and merely yield a subbase. This already suggests that, for the metric topology of a gum to be a useful notion, it is required to impose further restrictions on gums. This will be detailed in the following section.

Unless otherwise stated, topological concepts of some gum X will always refer to the metric topology of X.

Definition 2.13 Let P be a dcpo. A subset $O \subseteq P$ is *Scott-open* if

(i) $x \in O$ implies $\uparrow x \in O$ (O is an upper set),

(ii) for any directed set $S \subseteq P$, $\bigvee S \in O$ implies $S \cap O \neq \emptyset$ (O is inaccessible by directed suprema).

The *Scott-topology* is the topology of Scott-open sets.

Definition 2.14 Let P be a dcpo. The *Lawson-topology* is the topology generated by the base $\{U \backslash \uparrow F \mid U \text{ Scott-open}, \ F \subseteq P \text{ finite}\}$.

We finish by quoting some results about the Scott-topology on continuous domains. Details can be found in [AJ94, Section 2.3.2].

Proposition 2.15 Let P be a continuous dcpo with base B.

(i) For any $p \in P$, $\twoheaduparrow p$ is Scott-open.

(ii) Every Scott-open set $O \subseteq P$ is of the form $O = \bigcup_{p \in O \cap B} \twoheaduparrow p$.

2.5 Categories

Next we will introduce some basic notions of category theory that we will need later on. For a more detailed exposition we refer to [Bor94].

Definition 2.16 A category $\mathbf{C}$ consists of the following:

(i) a class $|\mathbf{C}|$ of *objects* of the category,

(ii) for every A, $B \in |\mathbf{C}|$, a set $\mathbf{C}(A, B)$ of *morphisms* from A to B,

(iii) for every A, B, $C \in |\mathbf{C}|$, a composition operation

$$\circ : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C),$$

(iv) for every $A \in |\mathbf{C}|$, an *identity morphism* $\mathrm{id}_A \in \mathbf{C}(A, A)$,

such that, for all $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$, $h \in \mathbf{C}(C, D)$, $h \circ (g \circ f) = (h \circ g) \circ f$ (*associativity axiom*), $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$ (*identity axiom*).

A morphism $f \in \mathbf{C}(A, B)$ is an *isomorphism* if there is a (necessarily unique) morphism $g \in \mathbf{C}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

The structure preserving mappings between categories are called functors:

Definition 2.17 Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A *functor* $\mathbf{F}$ from $\mathbf{A}$ to $\mathbf{B}$ consists of the following:

(i) a mapping $|\mathbf{A}| \to |\mathbf{B}|$ of objects, where the image of an object $A \in |\mathbf{A}|$ is denoted by $\mathbf{F}A$,

(ii) for every A, $A' \in |\mathbf{A}|$, a mapping $\mathbf{A}(A, A') \to \mathbf{B}(\mathbf{F}A, \mathbf{F}A')$, where the image of a morphism $f \in \mathbf{A}(A, A')$ is denoted by $\mathbf{F}f$,

such that, for every $f \in \mathbf{A}(A, A')$ and $g \in \mathbf{A}(A', A'')$, $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$ and $\mathbf{F}\,\mathrm{id}_A = \mathrm{id}_{\mathbf{F}A}$.

For a category $\mathbf{C}$, the *identity functor*, that maps all objects and morphisms to themselves, will be denoted by $\mathrm{id}_{\mathbf{C}}$. The following definition introduces a way to "pass" from one functor to another:

Definition 2.18 Let $\mathbf{A}$ and $\mathbf{B}$ be categories. Consider functors $\mathbf{F}, \mathbf{G} : \mathbf{A} \to \mathbf{B}$. A *natural transformation* $\eta : \mathbf{F} \Rightarrow \mathbf{G}$ is a class of morphisms $(\eta_A : \mathbf{F}A \to \mathbf{G}A)_{A \in |\mathbf{A}|}$ such that, for every morphism $f \in \mathbf{A}(A, A')$, $\eta_{A'} \circ \mathbf{F}f = \mathbf{G}f \circ \eta_A$.

We will call a natural transformation a *natural isomorphism* if all of its morphisms are isomorphisms. Now we can introduce the most important notion for our subsequent considerations:

Definition 2.19 A functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is an *equivalence of categories* if there is a functor $\mathbf{G} : \mathbf{B} \to \mathbf{A}$ and two natural isomorphisms $\eta : \mathrm{id}_{\mathbf{B}} \Rightarrow \mathbf{FG}$ and $\epsilon : \mathbf{GF} \Rightarrow \mathrm{id}_{\mathbf{A}}$.

Note that, due to the use of isomorphisms, this definition is symmetric and $\mathbf{G}$ is an equivalence of categories as well. We also remark that our definition is only one of many equivalent statements (see [Bor94, Proposition 3.4.3]), most of which employ the notion of an *adjoint functor*. Although we do not want to define this concept here, we will sometimes call the functor $\mathbf{G}$ the *left adjoint* of $\mathbf{F}$. For more information we refer to the indicated literature.

3 The poset $\mathbf{B}X$

In this section, we investigate the relation between a generalized ultrametric space and its set of formal balls. The following two results will be useful tools for this purpose, since they establish close connections between suprema in $\mathbf{B}X$ and infima in Γ.

Proposition 3.1 Let x be any element of X and define $\pi_x : \Gamma^\partial \to \downarrow[(x, \perp)]$ by $\pi_x(\beta) = [(x, \beta)]$. Then π_x is an order-isomorphism. In addition, for any $\Lambda \subseteq \Gamma^\partial$ with least upper bound α, $\pi_x(\alpha)$ is the least upper bound of $\pi_x(\Lambda)$ with respect to $\mathbf{B}X$.

Proof. Since $\perp$ is the greatest element of Γ^∂, it is clear by the definition of $\sqsubseteq$ that π_x is an order-isomorphism.

Now let $[(y, \gamma)]$ be an upper bound of $\pi_x(\Lambda) = \{[(x, \beta)] \mid \beta \in \Lambda\}$ in $\mathbf{B}X$. Then, for all $\beta \in \Lambda$, $\gamma \leq \beta$ and $d(x, y) \leq \beta$. Since α is assumed to be the greatest lower bound of Λ in Γ^∂, these imply that $\gamma \leq \alpha$ and $d(x, y) \leq \alpha$, i.e. $[(x, \alpha)] \sqsubseteq [(y, \gamma)]$. □

The next corollary shows a strong relationship between least upper bounds in $\mathbf{B}X$ and greatest lower bounds in Γ. Thus it may be compared with [EH98, Theorem 5], where a similar result is obtained for the case of metric spaces.

Corollary 3.2 Let A be a subset of $\mathbf{B}X$, define $\Lambda = \{\beta \mid [(y, \beta)] \in A\}$, and let $[(x, \alpha)]$ be an upper bound of A. Then $[(x, \alpha)]$ is the least upper bound of A in $\mathbf{B}X$ iff α is the greatest lower bound of Λ in Γ.

Proof. For all $y, z \in X$ and $\beta, \gamma \in \Gamma$, $[(y, \beta)] \sqsubseteq [(z, \gamma)]$ implies $[(y, \beta)] = [(z, \beta)]$, since $d(y, z) \leq \beta$ by definition of $\sqsubseteq$. Thus, A is a subset of $\downarrow[(x, \perp)]$ and we can apply Proposition 3.1. If $[(x, \alpha)]$ is the least upper bound of A in $\mathbf{B}X$, then α is the greatest lower bound of Λ in Γ, because of the given order-isomorphism. The converse direction has been shown in the second part of Proposition 3.1. □

Hence, to guarantee the existence of least upper bounds for sets $A \subseteq \mathbf{B}X$ from a given class (such as ascending chains or directed sets) one needs to ensure that the respective subsets of distances have a greatest lower bound in Γ and that A has some upper bound in $\mathbf{B}X$.

One immediately obtains the following result. Part of the proof is taken from [Hit01, Proposition 3.3.1].

Proposition 3.3 The space of formal balls $\mathbf{B}X$ is chain complete iff X is chain-spherically complete and Γ^∂ is chain complete.

Proof. Assume that $\mathbf{B}X$ is chain complete and let $\left(\overline{B}_\beta(y_\beta)\right)_{\beta\in\Lambda}$ be a chain of closed balls in X, where Λ is a chain in Γ^∂. Then $[(y_\beta,\beta)]_{\beta\in\Lambda}$ is an ascending chain in $\mathbf{B}X$ and thus has a least upper bound $[(x,\alpha)]$. Hence $\overline{B}_\alpha(x) \subseteq \bigcap_{\beta\in\Lambda}\overline{B}_\beta(y_\beta)$.

For a chain $\Lambda \subseteq \Gamma^\partial$, for any $x \in X$, $[(x,\beta)]_{\beta\in\Lambda}$ is again a chain in $\mathbf{B}X$ and has a least upper bound $[(x,\alpha)]$. By Corollary 3.2, α is the supremum of Λ.

Now assume that X is chain-spherically complete and Γ^∂ is chain complete. Consider a chain $[(y_\beta,\beta)]_{\beta\in\Lambda}$ in $\mathbf{B}X$ and note that all chains have to be of this form. Indeed, for any two elements $[(y_1,\beta_1)]$ and $[(y_2,\beta_2)]$ of some chain, $\beta_1=\beta_2$ implies $[(y_1,\beta_1)]=[(y_2,\beta_2)]$, since $d(y_1,y_2)\le\beta_1=\beta_2$ by linearity of the chain. According to Definition 2.2, this shows that $[(y_1,\beta_1)]=[(y_2,\beta_2)]$.

A chain of closed balls in X with non-empty intersection is now given by $\left(\overline{B}_\beta(y_\beta)\right)_{\beta\in\Lambda}$. Let x be any element of $\bigcap_{\beta\in\Lambda}\overline{B}_\beta(y_\beta)$ and let α be the least upper bound of the chain Λ with respect to Γ^∂. By Corollary 3.2, $[(x,\alpha)]$ is the supremum of $[(y_\beta,\beta)]_{\beta\in\Lambda}$. □

Using Proposition 2.9 one can go from chain completeness to directed completeness.

Corollary 3.4 The space of formal balls $\mathbf{B}X$ is a dcpo iff X is chain-spherically complete and Γ^∂ is a dcpo.

However, the proof of the theorem we use here needs the Axiom of Choice. For a direct proof, one has to extend the notion of chain-spherically complete from chains to directed sets of balls. Using directed sets instead of chains in the proof of Proposition 3.3 will then yield an analogous result.

For the details, consider any set $D=\left(\overline{B}_\beta(y_\beta)\right)_{\beta\in\Lambda}$ of closed balls of X, such that, for any $\beta,\ \beta'\in\Lambda$, there is $\gamma\in\Lambda$ with $\gamma\le\beta$, $\gamma\le\beta'$, $\overline{B}_\gamma(y_\gamma)\subseteq\overline{B}_\beta(y_\beta)$, and $\overline{B}_\gamma(y_\gamma)\subseteq\overline{B}_{\beta'}(y_{\beta'})$. We say that X is *directed-spherically complete* if $\bigcap D$ is non-empty for any such set D. The following is straightforward.

Proposition 3.5 The space of formal balls $\mathbf{B}X$ is a dcpo iff X is directed-spherically complete and Γ^∂ is a dcpo.

Proof. Assume that $\mathbf{B}X$ is directed complete and let $\left(\overline{B}_\beta(y_\beta)\right)_{\beta\in\Lambda}$ be a directed set of closed balls in the above sense. Then $[(y_\beta,\beta)]_{\beta\in\Lambda}$ is a directed set in $\mathbf{B}X$ and thus has a least upper bound $[(x, \alpha)]$. Hence $\overline{B}_\alpha(x) \subseteq \bigcap_{\beta\in\Lambda} \overline{B}_\beta(y_\beta)$.

For a directed set $\Lambda \subseteq \Gamma^\partial$, for any $x \in X$, $[(x, \beta)]_{\beta\in\Lambda}$ is again a directed set in $\mathbf{B}X$ and has a least upper bound $[(x, \alpha)]$. By Corollary 3.2, α is the supremum of Λ.

Now assume that X is directed-spherically complete and Γ^∂ is directed complete. Consider a directed set $[(y_\beta, \beta)]_{\beta\in\Lambda}$ in $\mathbf{B}X$ and note that all directed sets have to be of this form. Indeed, for any two elements $[(y_1, \beta_1)]$ and $[(y_2, \beta_2)]$ of some directed set, $\beta_1 = \beta_2$ implies $[(y_1, \beta_1)] = [(y_2, \beta_2)]$. To see this, note that there is some element $[(y_3, \beta_3)]$ with $[(y_3, \beta_3)] \sqsubseteq [(y_1, \beta_1)]$ and $[(y_3, \beta_3)] \sqsubseteq [(y_2, \beta_2)]$ by directedness. But then $[(y_1, \beta_1)] = [(y_3, \beta_1)]$ and $[(y_2, \beta_2)] = [(y_3, \beta_2)]$, as demonstrated in the proof of Corollary 3.2. This finishes the proof of the claim and thus elements of a directed set can indeed be indexed by their respective radii.

A directed set of closed balls in X with non-empty intersection is now given by $\left(\overline{B}_\beta(y_\beta)\right)_{\beta\in\Lambda}$. Let x be any element of $\bigcap_{\beta\in\Lambda} \overline{B}_\beta(y_\beta)$ and let α be the least upper bound of the directed set Λ with respect to Γ^∂. By Corollary 3.2, $[(x, \alpha)]$ is the supremum of $[(y_\beta, \beta)]_{\beta\in\Lambda}$. $\square$

3.1 Continuity of BX

Next, we want to investigate continuity of $\mathbf{B}X$. We point out that we do not require $\mathbf{B}X$ to be a dcpo, since we can work with the notion of continuity introduced in Definition 2.6. Therefore, we do not need to impose any preconditions on the gum X to state the following results.

Also note that $\ll^\partial$ on Γ generally does not coincide with $\ll$ on Γ^∂. However, when studying domain theoretic properties, we are always interested in the order Γ^∂, not in Γ itself. Hence, when dealing with distances, $\ll$ will denote the approximation order on Γ^∂ exclusively.

Lemma 3.6 Consider points x, $y \in X$ and distances α, $\beta \in \Gamma$. Then

(i) $[(x, \alpha)] \ll [(y, \beta)]$ in $\mathbf{B}X$ iff $\alpha \ll \beta$ in Γ^∂ and $d(x, y) \leq \alpha$,

(ii) $[(x, \alpha)]$ is compact in $\mathbf{B}X$ iff α is compact in Γ^∂.

Proof. To show (i), let $[(x, \alpha)] \ll [(y, \beta)]$ and let $\Lambda \subseteq \Gamma^\partial$ be directed with $\bigvee^\partial \Lambda = \gamma \geq^\partial \beta$. Obviously, $d(x, y) \leq \alpha$ and thus $[(x, \alpha)] = [(y, \alpha)]$. By Proposition 3.1, we find a directed set $A = \pi_y(\Lambda)$ with supremum $[(y, \gamma)] \sqsupseteq [(y, \beta)]$. This implies that $[(x, \alpha)] \sqsubseteq [(y, \delta)]$, for some $[(y, \delta)] \in A$. But then $\delta \in \Lambda$ with $\alpha \leq^\partial \delta$.

The other direction of the statement can be shown in a similar way. Just assume $\alpha \ll \beta$ (in Γ^∂) and $d(x,y) \leq \alpha$. This implies $[(x,\alpha)] \sqsubseteq [(y,\beta)]$. Now consider a directed set $A \subseteq \mathbf{B}X$ with supremum $[(z,\gamma)] \sqsupseteq [(y,\beta)]$. As noted in the proof of Corollary 3.2, A is of the form $\{[(z,\rho)] \mid \rho \in \Lambda\}$ with $\Lambda \subseteq \Gamma^\partial$. By Corollary 3.2, $\gamma \geq^\partial \beta$ is the least upper bound of Λ. But then there is $\delta \in \Lambda$ with $\alpha \leq^\partial \delta$. As before, we deduce that $[(x,\alpha)] = [(z,\alpha)] \sqsubseteq [(z,\delta)] \in A$.

Claim (ii) follows immediately from (i), since compactness is defined via $\ll$ and $d(x,x) \leq \alpha$ for any $\alpha \in \Gamma$. $\square$

The following lemma will be useful to treat certain pathological cases that can occur when dealing with the metric topology of gums.

Lemma 3.7 If the set $\Gamma^\partial \backslash \{\perp\}$ contains maximal elements, then the topology of open balls of X is discrete. In particular this is the case if $\perp$ is a compact element in Γ^∂.

Proof. Clearly, if there is some maximal element $\nu \in \Gamma^\partial \backslash \{\perp\}$, then singleton sets $\{x\}$ are open balls of the form $B_\nu(x)$. Hence, the topology is discrete.

Now assume $\perp$ is a compact element in Γ^∂. Every non-empty chain $\Sigma \subseteq \Gamma^\partial \backslash \{\perp\}$ has an upper bound in $\Gamma^\partial \backslash \{\perp\}$. To see this, note that otherwise $\perp$ would be the only and therefore least upper bound of Σ, which contradicts the assumption that $\perp$ is compact. Applying Zorn's Lemma, we find that $\Gamma^\partial \backslash \{\perp\}$ has a maximal element. $\square$

In what follows, we will look at the relations between bases of $\mathbf{B}X$, dense subsets of X, and bases of Γ^∂. Only at the very end of this section will we be able to compile all the results of these considerations into Theorem 3.17.

Proposition 3.8 Let D be a dense subset of X and let Δ be a base of Γ^∂. Then $(D \times \Delta)|_\approx = \{[(y,\beta)] \mid (y,\beta) \in (D \times \Delta)\}$ is a base of $\mathbf{B}X$.

Proof. Consider an element $[(x,\alpha)] \in \mathbf{B}X$. Since Δ is a base of Γ^∂, we find a set $\Lambda \subseteq \Delta \cap \mathop{\downarrow\!\!\downarrow}\alpha$ that is directed in Γ^∂ such that $\bigvee^\partial \Lambda = \alpha$. Using Proposition 3.1, we define a directed set $A = \pi_x(\Lambda)$ in $\mathbf{B}X$ with $\bigsqcup A = [(x,\alpha)]$. By Lemma 3.6, $A \subseteq \mathop{\downarrow\!\!\downarrow}[(x,\alpha)]$.

To show that $A \subseteq (D \times \Delta)|_\approx$, consider any element $[(x,\beta)] \in A$. We distinguish two cases. First suppose $\beta \neq \perp$. By density of D, there is $y \in D$ such that $d(x,y) < \beta$ and therefore $[(x,\beta)] = [(y,\beta)] \in (D \times \Delta)|_\approx$.

For the case $\beta = \perp$, we find that $\alpha = \perp$ and that $\perp \ll \perp$, i.e. $\perp$ is a compact element in Γ^∂. Hence, by Lemma 3.7, every subset of X is open. Consequently, the closure of the dense set D is just $D = X$. But this shows that $[(x,\perp)] \in (D \times \Delta)|_\approx$. $\square$

Proposition 3.9 Let B be a base of $\mathbf{B}X$. Then $\Delta = \{\beta \mid [(y,\beta)] \in B\}$ is a base of Γ^∂.

Proof. Consider some arbitrary $x \in X$. For any element $\alpha \in \Gamma^\partial$, $[(x,\alpha)]$ can be obtained as a least upper bound of a directed set $A \subseteq \twoheaddownarrow[(x,\alpha)] \cap B$. Corollary 3.2 yields that α is the least upper bound of $\Lambda = \{\beta \mid [(x,\beta)] \in A\}$ with respect to Γ^∂. Clearly $\Lambda \subseteq \Delta$. Finally, we derive $\Lambda \subseteq \twoheaddownarrow\alpha$ from Lemma 3.6. $\square$

Evidently, this result is not the full converse of Proposition 3.8, since we do not obtain a dense subset of X. Indeed, it is not clear how this should be done in general. A naïve approach for constructing a dense subset D of X from a base B of $\mathbf{B}X$, would be to define $D = \{x \in X \mid [(x,\beta)] \in B\}$. However, a little reflection shows that this definition will result in D being equal to X, which is clearly not what we wanted. A more elaborate attempt would be to *choose* one representative point from each element of B. However, the set of all chosen points can only be dense in X for a restricted class of gums.

Lemma 3.10 Let $\mathbf{B}X$ be a continuous dcpo. The following are equivalent:

(i) For every open ball $B_\alpha(x)$ there is some $y \in B_\alpha(x)$ and $\beta \in \Gamma^\partial$, such that $\beta \ll \bot$ and $\overline{B}_\beta(y) \subseteq B_\alpha(x)$.

(ii) For any base B of $\mathbf{B}X$ and any choice function $f : B \to X$ with $f[(x,\alpha)] \in \overline{B}_\alpha(x)$, the set $f(B)$ meets every open ball of X.

Proof. To see that (i) implies (ii), consider any open ball $B_\alpha(x)$. By the assumption, we find a closed ball $\overline{B}_\beta(y) \subseteq B_\alpha(x)$. The set $\twoheaduparrow[(y,\beta)]$ is Scott-open in $\mathbf{B}X$ by Proposition 2.15. In addition, using the fact that $\beta \ll \bot$, Lemma 3.6 implies that this set contains $[(y,\bot)]$. Now let B be any base of $\mathbf{B}X$. Proposition 2.15 implies that $\twoheaduparrow[(y,\beta)]$ is the union of all Scott-open filters of the form $\twoheaduparrow[(z,\gamma)]$, with $[(z,\gamma)] \in B \cap \twoheaduparrow[(y,\beta)]$. Especially, there is some $[(z,\gamma)] \in B \cap \twoheaduparrow[(y,\beta)]$ such that $[(y,\bot)] \in \twoheaduparrow[(z,\gamma)]$ and hence $\gamma \ll \bot$ by Lemma 3.6. For any choice function f in the above sense, $f[(z,\gamma)] \in B_\alpha(x)$. This is a consequence of the fact that, for any $v \in \overline{B}_\gamma(z)$, we find $[(v,\bot)] \in \twoheaduparrow[(z,\gamma)]$, again by Lemma 3.6 and the fact that $\gamma \ll \bot$, and thus $[(v,\bot)] \in \twoheaduparrow[(y,\beta)]$ by the definition of $[(z,\gamma)]$. But then $v \in \overline{B}_\beta(y) \subseteq B_\alpha(x)$. Hence, for any base B and any choice function f, the set $f(B)$ meets every open ball of X.

Now assume that condition (ii) holds. For a contradiction, suppose that there is an open ball $B_\alpha(x)$ such that for every $y \in B_\alpha(x)$ and $\beta \ll \bot$, $\overline{B}_\beta(y) \nsubseteq B_\alpha(x)$. Since $\mathbf{B}X$ is continuous, Γ^∂ is also continuous, by Proposition

3.9. Lemma 2.7 shows that $\downarrow\!\!\perp$ is a base of Γ^∂ and Proposition 3.8 states that $B = (X \times \downarrow\!\!\perp)|_\approx$ is a base of $\mathbf{B}X$.

Using the Axiom of Choice, we know that there exists a function $f : B \to X$ that chooses $f[(y, \beta)]$ to be some element in $\overline{B}_\beta(y) \backslash B_\alpha(x)$. Such a point always exists by the above assumptions. However, $f(B)$ does not meet the open ball $B_\alpha(x)$. □

Note that the previous lemma also yields a dense subset of the metric topology, as long as the open balls constitute a base. Unfortunately, this is not true in general. Below, we will impose stronger conditions than the ones in Lemma 3.10, which will be sufficient to obtain a base of open balls. Yet, Lemma 3.10 has been included, since it gives a precise characterization of the minimal requirements needed for constructing a dense subset of X from a base of $\mathbf{B}X$.

3.2 The Scott-topology on $\mathbf{B}X$

Our next aim will be to embed the open ball topology of X into $\max \mathbf{B}X$, as a subspace of the Scott-topology on $\mathbf{B}X$, thus obtaining a *model* for the metric topology of X:

Definition 3.11 A *model* of a topological space X is a continuous dcpo D and a homeomorphism $\iota : X \to \max D$ from X onto the maximal elements of D in their relative Scott-topology.

The immediate candidate for such an embedding is $\iota : X \to \max \mathbf{B}X$ with $\iota x = [(x, \perp)]$, which is clearly bijective. First let us note the following lemma:

Lemma 3.12 Consider $x \in X$ and $\alpha \in \Gamma$. The closed ball $\overline{B}_\alpha(x)$ is a (possibly infinite) union of open balls of X, and hence open in the metric topology, if $\alpha \neq \perp$ or $\alpha \ll \perp$ in Γ^∂.

Proof. Assume $\alpha \neq \perp$. Consider any $y \in \overline{B}_\alpha(x)$. For any $z \in B_\alpha(y)$, by the strong triangle inequality, $d(y, z) < \alpha$ and $d(x, y) \leq \alpha$ imply $d(x, z) \leq \alpha$, i.e. $z \in \overline{B}_\alpha(x)$. Thus $B_\alpha(y) \subseteq \overline{B}_\alpha(x)$. Clearly, $\overline{B}_\alpha(x) = \bigcup_{d(x,y) \leq \alpha} B_\alpha(x)$ is open.

If $\alpha = \perp$ then $\alpha \ll \perp$. Hence, by Lemma 3.7, every subset of X is a union of open balls. □

From this statement, we can easily obtain another important property of the metric topology:

Lemma 3.13 Every closed ball of a gum is also topologically closed.

Proof. For the proof, we employ the standard fact that the topological closure of a set S equals the set of all *adherent points* of S, where x is adherent to S if every open set O with $x \in O$ meets S.

Consider an arbitrary closed ball $\overline{B}_r(z)$. For a contradiction, we will assume that $\overline{B}_r(z)$ is not closed, i.e. there is a point $x \notin \overline{B}_r(z)$ that is adherent to $\overline{B}_r(z)$. We distinguish two cases.

First, assume that $r = \bot$. To see that x is not an adherent point, we show that $\overline{B}_r(z) \cap B_{d(x,z)}(x) = \emptyset$. Since $\overline{B}_r(z) = \{z\}$, this follows immediately from $z \notin B_{d(x,z)}(x)$.

For the other case, suppose that $r \neq \bot$. By Lemma 3.12, the set $\overline{B}_r(x)$ is open and it suffices to show that $\overline{B}_r(z) \cap \overline{B}_r(x) = \emptyset$. To see this, assume that there is some $y \in \overline{B}_r(z) \cap \overline{B}_r(x)$, i.e. we have $d(x,y) \leq r$ and $d(z,y) \leq r$. Then, by the strong triangle inequality, we find $d(x,z) \leq r$ and hence $x \in \overline{B}_r(z)$. This finishes our contradiction argument. $\square$

Now we can show that ι is continuous.

Proposition 3.14 For every Scott-open set $O \subseteq \mathbf{B}X$, $\iota^{-1}(O)$ is a (possibly infinite) union of open balls of X, and hence open in the metric topology.

Proof. First suppose that there is $[(x,\bot)] \in O$ such that there is no $[(y,\beta)] \in O$ with $[(y,\beta)] \sqsubset [(x,\bot)]$. We show that $[(x,\bot)]$ is compact. Indeed, for any directed set $A \subseteq \mathbf{B}X$ with $\bigsqcup A = [(x,\bot)]$ we have $A \cap O \neq \emptyset$ by Scott-openness of O. Since O does not contain any element strictly below $[(x,\bot)]$ we conclude $[(x,\bot)] \in A$.

If $[(x,\bot)]$ is compact, then $\bot$ is compact in Γ^∂ by Lemma 3.6. By Lemma 3.7, the metric topology of X is discrete and every subset of X, especially $\iota^{-1}(O)$, is a union of open balls.

Next, define the set $O^- = O \setminus \max \mathbf{B}X$ and assume that, for every $[(x,\bot)] \in O$, there is some $[(y,\beta)] \in O^-$ such that $[(y,\beta)] \sqsubset [(x,\bot)]$. Using this assumption and the fact that O is an upper set, we obtain that $O = \bigcup_{a \in O^-} \uparrow a$. Clearly, $\iota^{-1}(O) = \iota^{-1}\left(\bigcup_{a \in O^-} \uparrow a\right) = \bigcup_{a \in O^-} \iota^{-1}(\uparrow a)$. For this to be a union of open balls, it suffices to show that the sets $\iota^{-1}(\uparrow a)$ are unions of open balls.

Therefore, consider an element $a = [(y,\beta)] \in O^-$. We find that $\iota^{-1}(\uparrow [(y,\beta)]) = \overline{B}_\beta(y)$ by the definitions of ι and $\sqsubseteq$. To finish the proof, we simply employ Lemma 3.12 showing that $\overline{B}_\beta(y)$ is a union of open balls. $\square$

It turns out that the converse of this result is equivalent to various other conditions.

Theorem 3.15 Let X be chain-spherically complete and let Γ^∂ be a continuous dcpo. The following are equivalent:

(i) For every open ball $B_\alpha(x)$ and every $y \in B_\alpha(x)$, there is $\beta \in \Gamma^\partial$, with $\beta \ll \perp$ and $\overline{B}_\beta(y) \subseteq B_\alpha(x)$.

(ii) $\mathbf{B}X$ is a model for the metric topology of X, where the required homeomorphism is given by ι.

(iii) For every dense subset D of X and every base $\Delta \subseteq \mathord{\downarrow\!\!\downarrow}\perp$ of Γ^∂, $\{\overline{B}_\beta(y) \mid y \in D,\ \beta \in \Delta\}$ is a base for the metric topology of X.

Furthermore, under these conditions, the open balls form a base for the metric topology of X, and the relative Scott- and Lawson-topologies on $\max \mathbf{B}X$ coincide.

Proof. To show that (i) implies (ii), consider any open ball $B_\alpha(x)$. For any point $y \in B_\alpha(x)$, condition (i) yields a radius $\beta_y \ll \perp$, such that $\overline{B}_{\beta_y}(y) \subseteq B_\alpha(x)$. Using Corollary 3.4 and Proposition 3.8, we obtain that $\mathbf{B}X$ is a continuous dcpo. This implies that the set $\mathord{\uparrow\!\!\uparrow}[(y,\beta_y)] \subseteq \mathbf{B}X$ is Scott-open (see Proposition 2.15).

We show that, for any $\beta_y \ll \perp$, $\iota^{-1}(\mathord{\uparrow\!\!\uparrow}[(y,\beta_y)]) = \overline{B}_{\beta_y}(y)$. Indeed, for all $z \in \overline{B}_{\beta_y}(y)$, $d(y,z) \leq \beta_y$ and $\beta_y \ll \perp$ imply $[(z,\perp)] \in \mathord{\uparrow\!\!\uparrow}[(y,\beta_y)]$ by Lemma 3.6. Conversely, for any $[(z,\perp)] \in \mathord{\uparrow\!\!\uparrow}[(y,\beta_y)]$, we have $d(z,y) \leq \beta_y$ and hence $z \in \overline{B}_{\beta_y}(y)$.

Now obviously $\iota\,(B_\alpha(x)) = \iota\left(\bigcup_{d(x,y)<\alpha} \overline{B}_{\beta_y}(y)\right) = \bigcup_{d(x,y)<\alpha} \iota\left(\overline{B}_{\beta_y}(y)\right) = \bigcup_{d(x,y)<\alpha} (\mathord{\uparrow\!\!\uparrow}[(y,\beta_y)] \cap \max \mathbf{B}X)$ is open in the subspace topology on $\max \mathbf{B}X$. Since the open balls form a subbase for the metric topology, and since the bijection ι is compatible with unions and intersections, every open set in this topology is mapped to an open set of the relative Scott-topology on $\max \mathbf{B}X$, i.e. ι^{-1} is continuous. By Proposition 3.14, ι is also continuous and hence ι is a homeomorphism.

Now we show that (ii) implies (iii). Consider any open set $O \subseteq X$ in the metric topology. Then $\iota(O)$ is open in the relative Scott-topology on $\max \mathbf{B}X$. This implies that there is some Scott-open set $S \subseteq \mathbf{B}X$, such that $\iota(O) = S \cap \max \mathbf{B}X$. By Proposition 3.8, $B = \{[(y,\beta)] \mid y \in D,\ \beta \in \Delta\}$ is a base for $\mathbf{B}X$ and $S = \bigcup_{[(y,\beta)] \in S \cap B} \mathord{\uparrow\!\!\uparrow}[(y,\beta)]$, by Proposition 2.15. But then $O = \iota^{-1}\iota(O) = \iota^{-1}\left(\bigcup_{[(y,\beta)] \in S \cap B} \mathord{\uparrow\!\!\uparrow}[(y,\beta)] \cap \max \mathbf{B}X\right) = \bigcup_{[(y,\beta)] \in S \cap B} \iota^{-1}\left(\mathord{\uparrow\!\!\uparrow}[(y,\beta)] \cap \max \mathbf{B}X\right) = \bigcup_{[(y,\beta)] \in S \cap B} \overline{B}_\beta(y)$. The last equality is just another application of the fact that $\iota^{-1}(\mathord{\uparrow\!\!\uparrow}[(y,\beta)]) = \overline{B}_\beta(y)$, for all $\beta \ll \perp$. Thus O is a union of sets from $\{\overline{B}_\beta(y) \mid y \in D,\ \beta \in \Delta\}$.

Conversely, to see that any union of such sets is open, we can apply Lemma 3.12, showing that every closed ball with a radius $\beta \ll \perp$ is open in the metric topology.

To show that (iii) implies (i), we use the fact that every open ball $B_\alpha(x)$ is a union of basic open sets. We can choose X as a dense set and $\Delta = \downarrow\!\!\downarrow\perp$ as a base for Γ^∂, where the later is a consequence of Lemma 2.7. Consequently, every $y \in B_\alpha(x)$ is contained in some closed ball $\overline{B}_\beta(z) \subseteq B_\alpha(x)$, with $z \in D$ and $\beta \ll \perp$. From the basic fact that every point inside a closed ball is also its center, we conclude that $\overline{B}_\beta(z) = \overline{B}_\beta(y)$, which finishes the proof.

Now it is also easy to see that the open balls constitute a base for the metric topology. Indeed, for any open set O of the metric topology, $\iota(O)$ is Scott-open in $\mathbf{B}X$ by item (ii) above. But then using Proposition 3.14 we find that $\iota^{-1}\iota(O) = O$ is a union of open balls. In effect, every open set of the metric topology is a union of open balls.

Finally, we demonstrate that the relative Scott- and Lawson-topologies coincide. We only have to check that the additional open sets in $\max \mathbf{B}X$ that are induced by the basic open sets from Definition 2.14 are also open in the relative Scott-topology. Thus, consider any Scott-open set S and any finite set $F \subseteq \mathbf{B}X$. It is easy to see that $\iota^{-1}(\uparrow F)$ is closed in the metric topology, because it is a finite union of closed balls of the form $\iota^{-1}\uparrow[(y,\beta)] = \overline{B}_\beta(y)$, $[(y,\beta)] \in F$, and these balls are closed by Lemma 3.13. Hence, the finite intersection of open sets $O = \iota^{-1}(S) \cap (X \backslash \iota^{-1}(\uparrow F)) = \iota^{-1}(S \backslash \uparrow F)$ is open in X. But then, by the assumption, there is a Scott-open set $S' \subseteq \mathbf{B}X$ such that $\iota^{-1}(S') = O$. Consequently, S' and $S \backslash \uparrow F$ coincide on $\max \mathbf{B}X$, showing that the later is open in the relative Scott-topology. $\square$

There are also more common conditions that are sufficient to obtain the above properties:

Proposition 3.16 Let X be chain-spherically complete and let Γ^∂ be a continuous dcpo. $\mathbf{B}X$ is a model for the metric topology of X if, for every $\gamma \in \Gamma^\partial \backslash \{\perp\}$, $\gamma \ll \perp$. Especially this is the case if Γ^∂ is a linear dcpo.

Proof. Assume that there are maximal elements in $\Gamma^\partial \backslash \{\perp\}$. By Lemma 3.7, the metric topology of X is discrete. To show that the relative Scott-topology on $\max \mathbf{B}X$ is also discrete, we prove that $\perp$ is compact in Γ^∂. For a contradiction assume that there is a directed set $\Lambda \subseteq \Gamma^\partial$ with supremum $\perp$ and such that $\perp \notin \Lambda$. Consider some maximal element $\beta \in \Gamma^\partial$. Since $\beta \ll \perp$, we find some $\gamma \in \Lambda$ with $\beta \leq^\partial \gamma$. It is easy to see that this yields $\gamma = \beta$, i.e. that γ is maximal in $\Gamma^\partial \backslash \{\perp\}$. By directedness of Λ, γ is an upper bound of Λ, contradicting the assumption that $\perp$ is the least upper bound. Thus $\perp$ must be compact.

By Lemma 3.6, for every $x \in X$, $[(x, \perp)]$ is compact in $\mathbf{B}X$ and Proposition 2.15 implies that $\uparrow[(x, \perp)] = \{[(x, \perp)]\}$ is Scott-open. Therefore, the relative Scott-topology on $\max \mathbf{B}X$ is discrete as well and ι is the required homeomorphism.

Now suppose that there are no maximal elements in $\Gamma^\partial \backslash \{\perp\}$. For any open ball $B_\alpha(x)$ with radius α, we find some radius β such that $\alpha <^\partial \beta <^\partial \perp$. Thus, for all $y \in B_\alpha(x)$, $\overline{B}_\beta(y) \subseteq B_\alpha(x)$. Since in addition $\beta \ll \perp$, the gum satisfies condition (i) of Theorem 3.15. By the same theorem, the metric topology and the relative Scott-topology are homeomorphic.

Finally, suppose that Γ is linear. Consider any $\gamma \in \Gamma^\partial \backslash \{\perp\}$ and any directed set Λ with supremum $\perp$. There is some $\beta \in \Lambda$ with $\gamma < \beta$, since otherwise linearity of Γ would cause γ to be an upper bound of Λ, which is a contradiction. Thus $\gamma \ll \perp$, for every $\gamma \in \Gamma^\partial \backslash \{\perp\}$. □

Now that we found some conditions for getting a reasonably well-behaved metric topology with a base of open balls, we can use Lemma 3.10 to find a dense subset of the metric topology. The following theorem sums up our results on the relationships between dense subsets of X and bases of Γ^∂ on one side, and bases of $\mathbf{B}X$ on the other side.

Theorem 3.17 The space of formal balls $\mathbf{B}X$ is continuous (algebraic) iff Γ^∂ is continuous (algebraic). If the properties of Theorem 3.15 hold, then $\mathbf{B}X$ is ω-continuous (ω-algebraic) iff Γ^∂ is ω-continuous (ω-algebraic) and X is separable.

Proof. The first part of the claim follows from Proposition 3.8 and Proposition 3.9. The constructions for bases given in these propositions together with Lemma 3.6 imply algebraicity.

For the second part of the proof, note that, using Proposition 3.8 one can clearly obtain a countable base for $\mathbf{B}X$ from a countable base of Γ^∂ and a countable dense subset of X. For the converse, we use Proposition 3.9 to find a countable base of Γ^∂. Algebraicity is again immediate from Lemma 3.6. By Lemma 3.10, one can find a subset $D \subseteq X$ that meets every open ball. Theorem 3.15 states that open balls constitute a base for the metric topology, and hence D is a countable dense subset of X. □

Finally, we give an example of a gum that does not satisfy the properties of Theorem 3.15 or Lemma 3.10, although its metric topology has a base of open balls.

Example 3.18 The gum that we will employ for this example will be based on the classical ultrametric of infinite words, where distances are computed based on the length of the initial segments on which two words agree. We will modify this metric, since we need a non-linear set of distances.

Accordingly, define $X = \{a, b\}^\omega$ as the set of infinite words over the alphabet $\{a, b\}$. The set of distances Γ will be constructed as a Cartesian product of two posets. Let $(\mathbb{N}, \leq_{\mathbb{N}})$ be the set of natural number with its natural order. We define an ordered set $(\mathbf{N}, \leq)$ by setting $\mathbf{N} = \mathbb{N} \cup \{\omega\}$ and $\alpha < \beta$ iff either $\alpha \neq \omega = \beta$ or α, $\beta \in \mathbb{N}$ with $\alpha <_{\mathbb{N}} \beta$. In addition, let $\mathbf{2}$ be the two element set $\{0, 1\} \subseteq \mathbb{N}$ with its natural order.

Now we can define a set of distances by setting $\Gamma^\partial = \mathbf{N} \times \mathbf{2}$. Thus, the poset Γ has $(\omega, 1)$ as its least element $\perp$. Furthermore, Γ is a continuous dcpo, since both $\mathbf{N}$ and $\mathbf{2}$ are continuous dcpos (see [AJ94]). Using Lemma 2.7, a base Δ for Γ^∂ can easily be obtained by setting $\Delta = \mathord{\downarrow\!\!\!\downarrow}\perp = \Gamma^\partial \backslash \{(\omega, 0), (\omega, 1)\}$.

Now consider words x, $y \in \{a, b\}^\omega$, $x \neq y$, and a natural number n. The prefix of length n of x is denoted by $x|_n$ and we write $d'(x, y)$ for the greatest natural number n, such that $x|_n = y|_n$. We define a distance function $d : X \times X \to \Gamma$ as follows:

$$d(x, y) = \begin{cases} (\omega, 1) & \text{if } x = y \\ (d'(x, y), 1) & \text{if } x \neq y \end{cases}$$

We show that (X, d, Γ) is a gum. Properties (U1), (U2), and (U3) of Definition 2.1 are easily verified. For property (U4), consider points x, y, $z \in X$ and some distance $\alpha \in \Gamma$. Assume that $d(x, y) \leq \alpha$ and $d(y, z) \leq \alpha$. If $x = z$ then $d(x, z) \leq \alpha$ follows immediately. For the case $x = y$ we obtain $d(x, z) = d(y, z) \leq \alpha$. The case $y = z$ is treated similarly. Now consider the situation that x, y, and z are mutually distinct. Note that by the definition of d', we find that $d'(x, z) = \min_{\mathbb{N}}(d'(x, y), d'(y, z))$. This implies that $d(x, y) = \max_\Gamma(d(x, y), d(y, z))$ and thus $d(x, z) \leq \alpha$.

The gum (X, d, Γ) does not satisfy condition (i) of Lemma 3.10. Indeed, for any word x, the open ball $B_{(\omega,0)}(x)$ is just the singleton $\{x\}$. Yet every closed ball $\overline{B}_\alpha(x)$, with $\alpha \in \mathord{\downarrow\!\!\!\downarrow}\perp = \Delta$, is an infinite set. This also gives a counterexample for property (i) of Theorem 3.15.

Now for the base $B = (X \times \Delta)$ of $\mathbf{B}X$, one can define a choice function $f : B \to X$ by setting $f[(x, (n, m))] = x|_n a^\omega$. Here $x|_n a^\omega$ denotes the concatenation of the prefix $x|_n$ with the infinite word that consists only of letter a. Using Lemma 3.7, we find that the metric topology of X is discrete. But $f(B) \neq X$ and hence $f(B)$ is not dense in X. It is also easy to see that the relative Scott-topology on $\max \mathbf{B}X$ is not discrete and therefore is not homeomorphic to the metric topology.

4 Categories of gums

In this section, we investigate the relation between gums and their formal ball spaces in the framework of category theory. Our goal is to reconstruct gums from appropriate partially ordered sets. For such a construction to be possible, it will turn out to be necessary to equip gums with a designated point. Hence, for a gum (X, d, Γ) and $p \in X$, we will call a structure of the form $\big((X, d, \Gamma), p\big)$, or just (X, p), a *pointed gum*. In a similar but more restrictive way, we will define *pointed posets*[1].

Definition 4.1 Let $(P, \sqsubseteq)$ be a poset, consider $p \in \max P$, and let $(\iota_x : \downarrow p \to \downarrow x)_{x \in \max P}$ be a family of mappings. We say that $\big(P, p, (\iota_x)\big)$ is a *pointed poset* provided that the following hold:

(P1) $P = \downarrow \max P$,

(P2) the mappings (ι_x) are order-isomorphisms such that $\iota_p = \mathrm{id}_{\downarrow p}$ and, for all x, $y \in \max P$ and $a \in (\downarrow x \cap \downarrow y)$, $\iota_y \circ \iota_x^{-1} a = a$,

(P3) for all x, $y \in \max P$, the greatest lower bound $x \sqcap y$ exists.

To simplify notation, we define $\iota_{xy} = \iota_y \circ \iota_x^{-1}$.

The reasons for this definition will become apparent soon. Note that condition (P2) also implies $\iota_{yz} \circ \iota_{xy} = \iota_{xz}$, $\iota_{xx} = \mathrm{id}_{\downarrow x}$, $\iota_{xz} a = \iota_{yz} a$, and $\iota_{xy}^{-1} = \iota_{yx}$.

We can easily extend the definition of $\mathbf{B}$ to pointed gums by setting $\mathbf{B}(X, p) = \big(\mathbf{B}X, [(p, \perp)], (\iota_{[(x,\perp)]})\big)$, where the order-isomorphisms $(\iota_{[(x,\perp)]})$ are defined by setting $\iota_{[(x,\perp)]} = \pi_x \circ \pi_p^{-1}$, and π_x, π_p are the mappings defined in Proposition 3.1.

Now to obtain categories, the classes of pointed gums and pointed posets have to be equipped with suitable morphisms. Naturally, a morphism of gums will be a morphism of sets of points, i.e. some function, together with a morphism of posets with least element, where both morphisms are required to interact in an appropriate way. In addition, designated points have to be preserved.

Definition 4.2 Let $\big((X, d, \Gamma), p\big)$ and $\big((Y, e, \Delta), q\big)$ be pointed gums. A morphism $(f, \varphi) : (X, p) \to (Y, q)$ is a pair of mappings $f : X \to Y$ and $\varphi : \Gamma^\partial \to \Delta^\partial$, having the following properties:

(gm1) $\varphi(\perp_\Gamma) = \perp_\Delta$,

[1] Note that this term is sometimes used for posets with a least element, which is not what we have in mind here.

(gm2) φ is monotonic,

(gm3) $fp = q$,

(gm4) $e(fx, fy) \leq \varphi(d(x, y))$ for all x, $y \in \Gamma$.

The induced category of pointed gums will be denoted by **Gum**.

Note that **Gum** is indeed a category, where $(g, \psi) \circ (f, \varphi) = (g \circ f, \psi \circ \varphi)$ and $\mathrm{id}_{((X,d,\Gamma),p)} = (\mathrm{id}_X, \mathrm{id}_\Gamma)$. To see this, we just have to check the associativity and identity conditions in Definition 2.16. In addition, one has to verify that the composition of morphisms preserves the above properties. This is straightforward for (gm1) to (gm3). To show (gm4) for a composition $(g, \psi) \circ (f, \varphi)$, we observe that (gm2) and (gm4) imply $d''(gfx, gfy) \leq \psi(d'(fx, fy)) \leq \psi\varphi(d(x, y))$, where d, d', and d'' denote the respective distance functions in the involved gums.

Part of the above definition is inspired by the setting in [EH98]. There, in the context of real numbers as distance set, Lipschitz constants c (respectively their induced linear mappings $\varphi(x) = cx$) were used to give a bound for the expansion of a mapping f on the set of points.

We can now extend the definition of $\mathbf{B}$ to morphisms of gums. For a morphism $(f, \varphi) : \big((X, d, \Gamma), p\big) \to \big((X', d', \Gamma'), p'\big)$, we define $g = \mathbf{B}(f, \varphi)$ by setting $g[(x, \alpha)] = [(fx, \varphi\alpha)]$. To see that g is well-defined, consider x, $y \in X$ and $\alpha \in \Gamma$, such that $d(x, y) \leq \alpha$, i.e. $[(x, \alpha)] = [(y, \alpha)]$. Then $d'(fx, fy) \leq \varphi(d(x, y)) \leq \varphi(\alpha)$, follows from conditions (gm4) and (gm2), respectively. But this just says that $\mathbf{B}(f, \varphi)[(x, \alpha)] = \mathbf{B}(f, \varphi)[(y, \alpha)]$.

It is obvious that $\mathbf{B}$ meets the requirements of functoriality from Definition 2.17. Indeed, for all $[(x, \alpha)] \in \mathbf{B}X$, $(f, \varphi) : (X, p) \to (X', p')$ and $(f', \varphi') : (X', p') \to (X'', p'')$,

$$\begin{aligned}
\mathbf{B}\big((f', \varphi') \circ (f, \varphi)\big)[(x, \alpha)] &= \mathbf{B}(f' \circ f, \varphi' \circ \varphi)[(x, \alpha)] \\
&= [(f'(fx), \varphi'(\varphi\alpha))] \\
&= \mathbf{B}(f', \varphi')[(fx, \varphi\alpha)] \\
&= \big(\mathbf{B}(f', \varphi') \circ \mathbf{B}(f', \varphi')\big)[(x, \alpha)]
\end{aligned}$$

and $\mathbf{B}\,\mathrm{id}_{(X,p)}[(x, \alpha)] = [(x, \alpha)] = \mathrm{id}_{\mathbf{B}(X,p)}[(x, \alpha)]$. However, in order to speak of a functor, we also have to specify the category which $\mathbf{B}$ maps to. For this purpose, the following definition gives appropriate morphisms of pointed posets.

Definition 4.3 Let $\big(P, p, (\iota_x^P)\big)$ and $\big(Q, q, (\iota_x^Q)\big)$ be pointed posets. A morphism $g : P \to Q$ is a mapping with the following properties:

(pm1) for all $x \in \max P$, we have $gx \in \max Q$,

(pm2) g is monotonic,

(pm3) $gp = q$,

(pm4) for all $x \in \max P$ and $a \in \downarrow p$, $g(\iota_x^P a) = \iota_{gx}^Q(ga)$.

The induced category of pointed posets will be denoted by **Ball**.

The categorical properties of **Ball** are obviously satisfied, since composition of morphisms is just the usual composition of functions. The fact that composition preserves the properties (pm1) to (pm4) can be verified easily.

Using the above notation, we will often abbreviate $\big(P, p, (\iota_x^P)\big)$ as P. In what follows, we will demonstrate that the above definitions are indeed suitable to give a characterization of $\mathbf{B}X$ for a gum X.

Proposition 4.4 **B** is a functor from **Gum** to **Ball**.

Proof. Since we already have checked the conditions of functoriality from Definition 2.17, it only remains to show that **B** maps to objects and morphisms that belong to **Ball** according to the definitions 4.1 and 4.3.

Consider some pointed gum $\big((X, d, \Gamma), p\big)$. We want to show that $\mathbf{B}(X, p)$ is a pointed poset. Properties (P1) and (P2) of Definition 4.1 are obvious. For (P3) note that, for any x, $y \in X$, $[(x, d(x, y))] = [(y, d(x, y))]$ is a lower bound of $[(x, \perp)]$ and $[(y, \perp)]$. It is the greatest lower bound, since any other lower bound has to be of the form $[(x, \gamma)]$ with $d(x, y) \leq \gamma$.

Now let $(f, \varphi) : \big((X, d, \Gamma), p\big) \to \big((Y, e, \Delta), q\big)$ be a morphism of **Gum**. We show that $g = \mathbf{B}(f, \varphi)$ is a morphism of pointed posets. Property (pm1) of Definition 4.3 follows immediately from (gm1), i.e. from $\varphi(\perp_\Gamma) = \perp_\Delta$. To see that g is monotonic, consider $[(x, \alpha)], [(y, \beta)] \in \mathbf{B}X$ with $[(x, \alpha)] \sqsubseteq [(y, \beta)]$. By monotonicity of φ, $\beta \leq \alpha$ implies $\varphi\beta \leq \varphi\alpha$. In addition, $d(x, y) \leq \alpha$ yields $e(fx, fy) \leq \varphi(d(x, y)) \leq \varphi\alpha$. Thus $[(fx, \varphi\alpha)] \sqsubseteq [(fy, \varphi\beta)]$. Property (pm3) is again clear from the properties (gm1) and (gm3). For (pm4), consider some element $[(x, \perp_\Gamma)] \in \max \mathbf{B}X$ and some element $[(p, \alpha)] \in \downarrow[(p, \perp_\Gamma)]$. Denoting the order-isomorphisms of $\mathbf{B}(X, p)$ and $\mathbf{B}(Y, q)$ by $\iota^X_{[(x,\perp_\Gamma)]} = \pi_x \circ \pi_p^{-1}$ and $\iota^Y_{[(y,\perp_\Delta)]} = \pi'_y \circ \pi'^{-1}_q$, respectively, we obtain

$$\begin{array}{rcl} g\left(\iota^X_{[(x,\perp_\Gamma)]}[(p,\alpha)]\right) & = & g\left(\pi_x\pi_p^{-1}[(p,\alpha)]\right) \;=\; g\left(\pi_x\alpha\right) \\ & = & g[(x,\alpha)] \;=\; [(fx,\varphi\alpha)] \\ & = & \pi'_{fx}(\varphi\alpha) \;=\; \pi'_{fx}\pi'^{-1}_q[(q,\varphi\alpha)] \\ & = & \iota^Y_{[(fx,\perp_\Delta)]}[(q,\varphi\alpha)] \;=\; \iota^Y_{g[(x,\perp_\Gamma)]}\big(g[(p,\alpha)]\big) \end{array}$$

by the definitions of g, ι^X, and ι^Y. $\square$

In order to show that **Ball** contains exactly those pointed posets that can – up to isomorphism – be obtained as orders of formal balls, we specify a mapping from pointed posets to pointed gums explicitly.

Proposition 4.5 The following definition yields a functor $\mathbf{G} : \mathbf{Ball} \to \mathbf{Gum}$.

For a pointed poset $\big(P, p, (\iota_x)\big)$, define $\mathbf{G}P = \big((X, d, \Gamma), p\big)$, where $X = \max P$ and $\Gamma = (\downarrow p)^\partial$. For any x, $y \in \max P$, let $d(x, y)$ be given by $\iota^P_{xp}(x \sqcap y) \in \Gamma$.

For a morphism $g : P \to Q$, set $\mathbf{G}g = (f, \varphi)$, with $f : \max P \to \max Q : x \mapsto gx$ and $\varphi : \downarrow p \to \downarrow q : \gamma \mapsto g\gamma$.

Proof. To see that $\mathbf{G}$ is indeed well-defined, first note that the supremum required for the definition of d will always exist by (P3) of Definition 4.1. By Definition 4.1 (P2), we find that $\iota_x^{-1}(x \sqcap y) = \iota_y^{-1}(x \sqcap y)$, and hence that $\iota^P_{xp}(x \sqcap y) = \iota_x^{-1}(x \sqcap y) = \iota_y^{-1}(x \sqcap y) = \iota^P_{yp}(x \sqcap y)$. Furthermore, consider the mappings f and φ as defined above. Since g satisfies (pm1) of Definition 4.3 and $fx = gx$, for all $x \in \max P$, f surely maps $\max P$ to $\max Q$. For any element $\gamma \in \downarrow p$, $\varphi\gamma = g\gamma$ is an element of $\downarrow q$, because $gp = q$ and $\gamma \sqsubseteq p$ implies $g\gamma \sqsubseteq gp$ by (pm3) and (pm2).

The definition of $\mathbf{G}g$ immediately implies that $\mathbf{G}$ satisfies the conditions of Definition 2.17.

We prove that $\mathbf{G}P = \big((X, d, \Gamma), p\big)$ is a pointed gum. Clearly, Γ has a least element $\perp = p$. Now consider x, y, $z \in X$ and $\gamma \in \Gamma$. Assume $d(x, y) = \perp$, then $x \sqcap y$ is maximal in P and thus $x = y$. Conversely, $d(x, x) = \iota^P_{xp}(x \sqcap x) = \iota^P_{xp} x = p = \perp$. Symmetry of d follows immediately from symmetry of $\sqcap$ and property (P2) of Definition 4.1. For the strong triangle inequality, assume that $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$. Thus $x \sqcap y \sqsupseteq \iota^P_{py}\gamma$ and $y \sqcap z \sqsupseteq \iota^P_{py}\gamma$, especially $\iota^P_{py}\gamma \sqsubseteq x$ and $\iota^P_{py}\gamma \sqsubseteq z$. But then $\iota^P_{py}\gamma \sqsubseteq x \sqcap z$ and hence $\gamma = \iota^P_{xp}\iota^P_{py}\gamma \geq \iota^P_{xp}(x \sqcap z) = d(x, z)$.

Finally, we show that any morphism $g : (P, \sqsubseteq_P) \to (Q, \sqsubseteq_Q)$ of pointed posets is mapped to a morphism $\mathbf{G}g = (f, \varphi)$ of pointed gums. Note that φ is monotonic and preserves $\perp$ (i.e. the designated point p) by the properties (pm1), (pm2), and (pm3) of g. Next, let $\mathbf{G}P = \big((X, d, \Gamma), p\big)$ and $\mathbf{G}Q = \big((Y, e, \Delta), q\big)$ and consider any x, $y \in X$. From monotonicity of g one obtains $g(x \sqcap_P y) \sqsubseteq_Q gx$ and $g(x \sqcap_P y) \sqsubseteq_Q gy$. This implies $g(x \sqcap_P y) \sqsubseteq_Q (gx \sqcap_Q gy)$. But this just says that $\varphi(d(x, y)) \geq e(fx, fy)$ in Δ, by the definitions of d, e, and $\leq$. $\square$

Now we can state the main result of this section:

$$
\begin{tikzcd}
(X,p) \arrow[r, "\eta_{(X,p)}"] \arrow[d, "(f,\varphi)"'] & \mathbf{GB}(X,p) \arrow[d, "\mathbf{GB}(f,\varphi)"] \\
(Y,q) \arrow[r, "\eta_{(Y,q)}"] & \mathbf{GB}(Y,q)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{BG}P \arrow[r, "\epsilon_{\mathbf{BG}P}"] \arrow[d, "\mathbf{BG}g"'] & P \arrow[d, "g"] \\
\mathbf{BG}Q \arrow[r, "\epsilon_{\mathbf{BG}Q}"] & Q
\end{tikzcd}
$$

Figure 2: The natural isomorphisms η and ϵ for the proof of Theorem 4.6.

Theorem 4.6 The functor $\mathbf{B}$ is an equivalence of categories, whose left adjoint is given by the functor $\mathbf{G}$.

Proof. By Definition 2.19, we have to show that there exist natural isomorphisms $\eta : \mathrm{id}_{\mathbf{Gum}} \Rightarrow \mathbf{GB}$ and $\epsilon : \mathbf{BG} \Rightarrow \mathrm{id}_{\mathbf{Ball}}$.

For a pointed gum $\big((X, d, \Gamma), p\big)$, we define $\eta_{(X,p)} = \big(f_{(X,p)}, \varphi_{(X,p)}\big) : (X, p) \to \mathbf{GB}(X, p)$ by setting $f_{(X,p)}x = [(x, \bot)]$ and $\varphi_{(X,p)}\alpha = [(p, \alpha)]$. We have to check the properties of Definition 4.2. Evidently, $f_{(X,p)}$ is bijective and $\varphi_{(X,p)}$ is an order-isomorphism. This implies that $\varphi_{(X,p)}$ and its inverse preserve $\bot$ (gm1) and are monotonic (gm2). In addition, $f_{(X,p)}$ and its inverse clearly preserve the designated points, as required by (gm3). Denoting the distance mapping on $\mathbf{GB}(X, p)$ by e and using ι_{xp} to abbreviate the order-isomorphism $\iota^{\mathbf{B}(X,p)}_{[(x,\bot)][(p,\bot)]}$, we can state

$$
\begin{aligned}
e([(x, \bot)], [(y, \bot)]) &= \iota_{xp}([(x, \bot)] \sqcap [(y, \bot)]) \\
&= \iota_{xp}[(x, d(x, y))] \\
&= [(p, d(x, y))] \\
&= \varphi_{(X,p)}(d(x, y)).
\end{aligned}
$$

By application of $\varphi^{-1}_{(X,p)}$, one obtains $d(x, y) = \varphi^{-1}_{(X,p)}(e([(x, \bot)], [(y, \bot)]))$. Thus both $f_{(X,p)}$ and its inverse satisfy (gm4). Hence, $\eta_{(X,p)}$ is an isomorphism in $\mathbf{Gum}$.

To show that η is natural, consider any morphism $(f, \varphi) : \big((X, d, \Gamma), p\big) \to \big((Y, e, \Delta), q\big)$. We have to show that $\eta_{(Y,q)} \circ (f, \varphi) = \mathbf{GB}(f, \varphi) \circ \eta_{(X,p)}$, i.e. that the diagram on the left hand side of Figure 2 commutes. We denote $\mathbf{GB}(f, \varphi)$ by $(f_{\mathbf{GB}}, \varphi_{\mathbf{GB}})$. For any $x \in X$, $(f_{(Y,q)} \circ f)x = f_{(Y,q)}(fx) = [(fx, \bot_\Delta)] = [(fx, \varphi\bot_\Gamma)] = f_{\mathbf{GB}}[(x, \bot_\Gamma)] = (f_{\mathbf{GB}} \circ f_{(X,p)})x$. In addition, for any $\alpha \in \Gamma$, $(\varphi_{(Y,q)} \circ \varphi)\alpha = \varphi_{(Y,q)}(\varphi\alpha) = [(q, \varphi\alpha)] = [(fp, \varphi\alpha)] = \varphi_{\mathbf{GB}}[(p, \alpha)] = (\varphi_{\mathbf{GB}} \circ \varphi_{(X,p)})\alpha$. Thus η is a natural isomorphism.

Next, we define $\epsilon_{\mathbf{BG}P} : \mathbf{BG}P \to P$ by $\epsilon_{\mathbf{BG}P}[(x, \alpha)] = \iota^P_x \alpha$, for $[(x, \alpha)] \in \mathbf{BG}P$. By requirement (P2) of Definition 4.1, the result of this operation is independent of the choice of the representative x and ϵ is well-defined. Note that the distance $\perp$ in $\mathbf{G}P$ is just the designated point p of P and thus all maximal elements of $\mathbf{BG}P$ are of the form $[(x,p)]$, $x \in \max P$.

We have to check the properties of Definition 4.3. Elements $[(x,p)] \in \max \mathbf{BG}P$ are mapped to $\iota^P_x(p) \in \max P$, which is what (pm1) requires. The preservation of designated points (pm3) follows from the fact that $\iota^P_p(p) = p$.

To show monotonicity (pm2), consider $[(x,\alpha)], [(y,\beta)] \in \mathbf{BG}P$ such that $[(x,\alpha)] \sqsubseteq [(y,\beta)]$. As noted before, this implies that $[(x,\alpha)] = [(y,\alpha)]$. Hence, using $\alpha \geq \beta$ and monotonicity of ι^P_y, one obtains $\iota^P_x\alpha = \iota^P_y\alpha \sqsubseteq \iota^P_y\beta$, with respect to the order of P.

Now for property (pm4), consider $[(x,p)] \in \max \mathbf{BG}P$ and $[(p,\alpha)] \in \downarrow[(p,p)]$. Then

$$\begin{array}{rclcl} \epsilon_{\mathbf{BG}P}\left(\iota^{\mathbf{BG}P}_{[(x,p)]}[(p,\alpha)]\right) & = & \epsilon_{\mathbf{BG}P}[(x,\alpha)] & & \\ & = & \iota^P_x\alpha & = & \iota^P_{\epsilon_{\mathbf{BG}P}[(x,p)]}\big(\epsilon_{\mathbf{BG}P}[(p,\alpha)]\big), \end{array}$$

where the final equality follows from the facts that $x = \iota^P_x p = \epsilon_{\mathbf{BG}P}[(x,p)]$ and $\alpha = \iota^P_p\alpha = \epsilon_{\mathbf{BG}P}[(p,\alpha)]$.

To see that $\epsilon_{\mathbf{BG}P}$ is an isomorphism, consider an element $a \in P$. We define $\kappa_{\mathbf{BG}P}(a) = [(x, \iota^P_{xp}a)]$, for any $x \in \max P$ with $a \sqsubseteq x$. Property (P1) of Definition 4.1 implies that such an x exists. Assume there is another element $y \in \max P$ with $a \sqsubseteq y$. Property (P2) implies $\iota^P_{xp}a = \iota^P_{yp}a$. Since $d(x,y)$ in $\mathbf{G}P$ is defined to be isomorphic to the greatest lower bound $x \sqcap y$ in P, $\iota^P_{xp}a \sqsubseteq d(x,y)$ and therefore $\iota^P_{xp}a \geq d(x,y)$. Hence we deduce that $[(x,\iota^P_{xp}a)] = [(y,\iota^P_{yp}a)]$ and that κ is well-defined.

Furthermore, ϵ and κ are inverse to each other, since $\epsilon_{\mathbf{BG}P}\kappa_{\mathbf{BG}P}a = \epsilon_{\mathbf{BG}P}[(x,\iota^P_{xp}a)] = \iota^P_x\iota^P_{xp}a = a$ and $\kappa_{\mathbf{BG}P}\epsilon_{\mathbf{BG}P}[(x,\alpha)] = \kappa_{\mathbf{BG}P}\iota^P_x\alpha = [(x,\iota^P_{xp}\iota^P_x\alpha)] = [(x,\alpha)]$.

We also have to check the properties (pm1) to (pm4) for κ. As before, it is easy to see that (pm1) and (pm3) hold. Property (pm2) follows from monotonicity of ϵ and the fact that κ is its inverse. For (pm4), let $x \in \max P$ and $a \in \downarrow p$. Using the abbreviation $\theta = \kappa_{\mathbf{BG}P}$, we find

$$\begin{array}{rclcl} \theta(\iota^P_x a) & = & [(x,\iota^P_{xp}\iota^P_x a)] & = & [(x,a)] \\ & = & \iota^{\mathbf{BG}P}_{[(x,p)]}[(p,a)] & = & \iota^{\mathbf{BG}P}_{\theta(x)}(\theta a), \end{array}$$

where the final equality follows from $[(x,p)] = \iota^P_{xp}[(x,x)] = \theta x$ and $[(p,a)] = \iota^P_{pp}[(p,a)] = \theta a$.

Naturality of ϵ is again shown via a straightforward calculation (compare the right diagram of Figure 2). Consider some morphism $g : P \to Q$

and let $[(x,\alpha)] \in \mathbf{BG}P$. Then $(g \circ \epsilon_{\mathbf{BG}P})[(x,\alpha)] = g(\iota_x^P \alpha) = \iota_{g(x)}^Q g\alpha = \epsilon_{\mathbf{BG}Q}[(gx, g\alpha)] = (\epsilon_{\mathbf{BG}Q} \circ \mathbf{BG}g)[(x,\alpha)]$. This finishes the proof. $\square$

In the rest of this section, we consider various subcategories of **Gum** and **Ball**. $\mathbf{Gum_{dcpo*}}$ is the full subcategory of **Gum** consisting of pointed gums (X, d, Γ), where X is chain-spherically complete and Γ^∂ is a dcpo. The subcategory of $\mathbf{Gum_{dcpo*}}$ obtained by restricting to morphisms (f, φ) for which φ is Scott-continuous will be called $\mathbf{Gum_{dcpo}}$. Note that Scott-continuity refers to the dual orders of distances by the definition of φ. To see that this is indeed a subcategory, one just has to check that the composition law of **Gum** preserves this additional property.

The complementing categories of pointed posets are denoted $\mathbf{Ball_{dcpo*}}$ and $\mathbf{Ball_{dcpo}}$. $\mathbf{Ball_{dcpo*}}$ is the full subcategory consisting just of directed complete pointed posets, called *pointed dcpos*, and $\mathbf{Ball_{dcpo}}$ is the subcategory of $\mathbf{Ball_{dcpo*}}$ where the morphisms additionally are Scott-continuous.

Theorem 4.7 The functors **B** and **G** restrict to an equivalence of the categories $\mathbf{Gum_{dcpo*}}$ ($\mathbf{Gum_{dcpo}}$) and $\mathbf{Ball_{dcpo*}}$ ($\mathbf{Ball_{dcpo}}$).

Proof. By Corollary 3.4, it is clear that objects from $\mathbf{Gum_{dcpo*}}$ are indeed mapped to $\mathbf{Ball_{dcpo*}}$. For the converse, consider a pointed dcpo (P, p). By Theorem 4.6, $\mathbf{BG}(P, p)$ is isomorphic to (P, p). But this implies that $\mathbf{BG}(P, p)$ is a pointed dcpo and we can again use Corollary 3.4 to show that $\mathbf{G}(P, p)$ is an object of $\mathbf{Gum_{dcpo*}}$. This already shows that the functors **B** and **G** restrict to the categories $\mathbf{Gum_{dcpo*}}$ and $\mathbf{Ball_{dcpo*}}$. For $\mathbf{Gum_{dcpo}}$ and $\mathbf{Ball_{dcpo}}$ we still have to consider morphisms.

We show that morphisms of $\mathbf{Gum_{dcpo}}$ are mapped to morphisms of $\mathbf{Ball_{dcpo}}$, i.e. that the additional requirement of Scott-continuity is satisfied. Consider a morphism $g = \mathbf{B}(f, \varphi)$, where $(f, \varphi) : (X, p) \to (Y, q)$ is a morphism of $\mathbf{Gum_{dcpo}}$, and a directed subset $A \subseteq \mathbf{B}X$ with $\bigsqcup A = [(x, \alpha)]$. For any $[(y, \beta)] \in A$, $g[(y, \beta)] \sqsubseteq_{\mathbf{B}Y} g[(x, \alpha)]$, i.e. $g[(x, \alpha)]$ is an upper bound of $g(A)$. By Corollary 3.2, α is the least upper bound of the directed set $\Lambda = \{\beta \mid [(y, \beta)] \in A\}$ within Γ^∂, the dual poset of distances of X. Scott-continuity of φ with respect to Γ^∂ yields that $\bigvee^\partial \varphi(\Lambda) = \varphi(\alpha)$. Thus $g[(x, \alpha)]$ is the least upper bound of $g(A)$, again by Corollary 3.2.

To see that a morphism g of $\mathbf{Ball_{dcpo}}$ is also mapped to a morphism of $\mathbf{Gum_{dcpo}}$, just note that the mapping φ in $\mathbf{G}g = (f, \varphi)$ simply is the restriction of g to $\downarrow p$ and consequently inherits Scott-continuity. Therefore the functors **B** and **G** restrict to the categories $\mathbf{Gum_{dcpo}}$ and $\mathbf{Ball_{dcpo}}$.

The claimed equivalence of categories now follows from the proof of Theorem 4.6 together with the observation that the required natural isomorphisms are just the restrictions of the above definitions of η and ϵ to the respective subcategories. To see that these restrictions are also morphisms in $\mathbf{Gum_{dcpo}}$ and $\mathbf{Ball_{dcpo}}$, one just has to note that order-isomorphisms are always Scott-continuous. $\square$

It is easy to see that similar results could be shown for categories that impose further restrictions on the objects. Especially, Theorem 3.17 suggests that one could include (ω-) continuity as well. Proving that $\mathbf{B}$ and $\mathbf{G}$ restrict to these classes of objects is done by a completely similar reasoning as in the first part of the above proof. For the class of morphisms one can freely choose whether Scott-continuity should be required or not. In any case, no additional verifications are needed to establish the desired categorical equivalences.

5 Fixed point theorems

In the following, we give a domain theoretic proof for a variant of the Prieß-Crampe and Ribenboim theorem (see [PCR99]), where we restrict ourselves to gums from the category $\mathbf{Gum_{dcpo*}}$. For more special situations, we can even prove a theorem that can be compared with the Banach fixed point theorem, in the sense that it obtains the desired fixed point from a *countable* chain of closed balls. Since we do not need all of the categorical results from the previous section here, we will give the necessary preconditions explicitly, dropping some of the structure that was introduced for **Gum**. The proof follows the ones of [EH98, Theorem 18] and [HS03, p.16].

Theorem 5.1 Let (X, d, Γ) be a gum, where Γ^∂ is a dcpo and X is chain-spherically complete. Consider mappings $f : X \to X$ and $\varphi : \Gamma^\partial \to \Gamma^\partial$, such that, for all $\alpha \in \Gamma \backslash \{\perp\}$, $\varphi\alpha < \alpha$ and, for all $x, y \in X$, $d(fx, fy) \leq \varphi(d(x, y))$. Then the following hold:

(i) If φ is monotonic, then f has a unique fixed point on X.

(ii) If φ is Scott-continuous, then the unique fixed point of f is the only element of the singleton set $\bigcap_{n\in\mathbb{N}} \overline{B}_{\varphi^n d(x,fx)}(f^n x)$, for arbitrary $x \in X$.

Proof. Although we ignore some of the categorical structure introduced above, we can still define $\mathbf{B}(f, \varphi)$ as before.

We want to find an arbitrary fixed point of $\mathbf{B}(f, \varphi)$ on $\mathbf{B}X$. Consider some point $x \in X$ and set $\alpha = d(x, fx)$. Assume without loss of generality that x is not a fixed point of f. For all $[(y, \beta)] \sqsupseteq [(x, \alpha)]$, we have $\varphi\beta < \beta \leq \alpha$ and $d(fx, fy) \leq \varphi(d(x, y)) < d(x, y) \leq \alpha$. Using the strong triangle inequality on $d(x, fx) = \alpha$ and $d(fx, fy) \leq \alpha$, one gets $d(x, fy) \leq \alpha$ and consequently $[(fy, \varphi\beta)] \sqsupseteq [(x, \alpha)]$. Thus $\mathbf{B}(f, \varphi)$ maps $\uparrow[(x, \alpha)]$ to itself.

Since $\uparrow[(x, \alpha)]$ by Corollary 3.4 is a cpo with least element $[(x, \alpha)]$, we can apply the fixed point theorems stated in Section 2. Note that $\mathbf{B}(f, \varphi)$ is monotonic, as shown in Proposition 4.4. Thus, by Proposition 2.11, $\mathbf{B}(f, \varphi)$ has a (least) fixed point $[(z, \gamma)]$ on $\uparrow[(x, \alpha)]$. Furthermore, if φ is Scott-continuous, Theorem 4.7 asserts that $\mathbf{B}(f, \varphi)$ is also Scott-continuous and hence $[(z, \gamma)] = \bigsqcup^{\uparrow}_{n\in\mathbb{N}}[(f^n x, \varphi^n\alpha)]$ by Proposition 2.10.

Now $\mathbf{B}(f, \varphi)[(z, \gamma)] = [(z, \gamma)]$ implies that $\varphi\gamma = \gamma$ and thus $\gamma = \perp$. However, formal balls of the form $[(z, \perp)]$ are equivalence classes with only one representative and thus $fz = z$, i.e. z is a fixed point of f. To show the uniqueness of z, suppose for a contradiction that there is $z' \neq z$ such that $fz' = z'$. Then $d(z, z') = d(fz, fz') \leq \varphi(d(z, z')) < d(z, z')$.

For the Scott-continuous case, we already observed that $[(f^n x, \varphi^n\alpha)]_{n\in\mathbb{N}}$ is a chain in $\mathbf{B}X$. By the definition of $\sqsubseteq$, $\left(\overline{B}_{\varphi^n\alpha}(f^n x)\right)_{n\in\mathbb{N}}$ is a chain of closed balls with $z \in \bigcap_{n\in\mathbb{N}} \overline{B}_{\varphi^n\alpha}(f^n x)$. To see that this intersection is indeed a singleton set, assume that there is a $z' \neq z$ such that $z' \in \bigcap_{n\in\mathbb{N}} \overline{B}_{\varphi^n\alpha}(f^n x)$. Then $d(f^n x, z') \leq \varphi^n\alpha$ for every $n \in \mathbb{N}$ and hence $[(z', 0)]$ is an upper bound of $[(f^n x, \varphi^n\alpha)]_{n\in\mathbb{N}}$. This contradicts the assumption that $[(z, 0)]$ is the least such upper bound. □

We can compare part (i) of this theorem with [PCR99, 5.3 (2)]. It is clear that our preconditions are strictly stronger than those required in [PCR99][2], although the obtained result is not. This deserves some discussion.

First of all, we have to verify that the preconditions are indeed stronger than those in the original theorem. Instead of assuming that $d(fx, fy) < d(x, y)$, i.e. that f is *strictly contracting*, we require the existence of a mapping φ that gives a uniform bound for the contraction of f. As the following example will show, this is a strictly stronger assumption.

Example 5.2 This example will again be a variation of the classical ultrametric of infinite words. We use the notation of Example 3.18. Let $X = \{a, b\}^\omega$ be a set of points and define a set of distances by $\Gamma = \{0, 1, \omega\} \cup \big(\{n \in \mathbb{N} \mid n >_{\mathbb{N}} 1\} \times \{a, b\}\big)$. Hence Γ consists of two copies of $\mathbb{N} \cup \{\omega\}$, where 0, 1, and ω are identified. We define the order $\leq$ on Γ by setting $x < y$ iff one of the following holds:

(i) $x = \omega$ and $y \neq \omega$,

(ii) $x \neq 0$ and $y = 0$,

(iii) $x \neq 0$, $x \neq 1$ and $y = 1$,

(iv) $x = (n, l)$ and $y = (m, l)$, for n, $m \in \mathbb{N}$ and $l \in \{a, b\}$, where $n > m$.

The distance function d is given by:

$$d(x, y) = \begin{cases} \omega & \text{if } x = y \\ 0 & \text{if } x \neq y \text{ and } d'(x, y) = 0 \\ 1 & \text{if } x \neq y \text{ and } d'(x, y) = 1 \\ (n, l) & \text{if } x \neq y,\ d'(x, y) = n > 1 \text{ and } x|_{n-1}l = x|_n,\ l \in \{a, b\} \end{cases}$$

Thus we define the distance between words as usual, but we additionally classify distances according to the last letter l of the segments that coincide.

[2]Strictly speaking, our condition of X being chain-spherically complete is weaker than the requirement of spherical completeness in [PCR99]. However, their proof can be modified to use the weaker assumption as well.

One can readily check the properties (U1) to (U4) to see that (X, d, Γ) is a gum.

To define a function $f : X \to X$, we set $f(x) = ax$, i.e. f prepends the letter a to each word x. Clearly, for two different words this increases the length of the equal initial segments without changing the final letter of this segments. Thus f is strictly contracting with fixed point a^ω.

However, we cannot define a strictly increasing (on Γ^∂) function φ that gives a bound for f. For a contradiction, assume that there is an appropriate mapping φ and consider $[(a^\omega, 1)] \in \mathbf{B}X$. Clearly, $1 > \varphi(1) > \omega$. Assume $\varphi(1) = (n, a)$ for some $n \in \mathbb{N}$. We use the notation $w \ldots$ to denote an arbitrary but fixed infinite word with the finite word w as its initial segment. We find $d(ba \ldots, bb \ldots) = 1$ and $d(f(ba \ldots), f(bb \ldots)) = (2, b)$. Since we require $d(fx, fy) \leq \varphi(d(x, y))$, we find $(2, b) \leq \varphi(1) = (n, a)$. A contradiction. Assuming $\varphi(1) = (n, b)$ leads to a similar result.

The basic problem here is that, for every $\beta \in \Gamma$, the set $\{d(fx, fy) \mid x, y \in X,\ d(x, y) \leq \beta\}$ needs to have an upper bound α in Γ, where $\alpha < \beta$. While this is not true in general, there are cases where such an α can be found if f is strictly contracting. For instance, when defining $\Gamma = \{\delta \mid \delta \leq \gamma\}^\partial$, for some ordinal γ, one can take α to be the successor of β. This has been done in [HS03].

Now, even if we can find a strictly increasing mapping φ meeting the above conditions, it is still possible that φ fails to be monotonic. For an example, we modify the gum from Example 5.2, such that 1 is not identified, i.e. we have distances $(1, a)$ and $(1, b)$. Then a mapping φ to bound f can be defined by $\varphi(0) = (1, a)$, $\varphi(n, a) = (n + 1, a)$, $\varphi(n, b) = (n + 1, b)$, and $\varphi(\omega) = \omega$. However, there is no such mapping that is monotonic, because this would force $\varphi(1, a) \leq \varphi(0)$ and $\varphi(1, b) \leq \varphi(0)$, which is obviously impossible.

Another strong assumption in the above theorem is directed completeness of Γ^∂ which is necessary in order to apply fixed point theorems to $\mathbf{B}X$. The merit of these requirements is that, although we obtain a similar result as in the original theorem, our proof gives explicit instructions how to obtain the required fixpoint. Indeed, the proof for Proposition 2.11 as given in [DP02] uses a construction that does not rely on the Axiom of Choice (AC). Corollary 3.4 still uses AC to conclude that $\mathbf{B}X$ is a dcpo when X is chain-spherically complete. However, as shown in Proposition 3.5, we can overcome this problem by requiring X to be directed-spherically complete. Thus we can easily modify Theorem 5.1 (i) in order to avoid the use of the Axiom of Choice. In contrast, the original proof of [PCR99] requires the existence of maximal chains ("Kuratowski's Lemma"), which is equivalent to AC.

Part (ii) of Theorem 5.1 should rather be compared to the Banach fixed point theorem for classical metric spaces. In the classical case, one uses the cpo of real numbers as a set of distances. Mappings f on the set of points are bounded by linear functions $\varphi(\alpha) = c\alpha$, for $c < 1$. Such functions are also Scott-continuous and strictly increasing. On the other hand, we require gums to be chain-spherically complete, which is a strictly stronger precondition than the completeness needed in the classical setting (for details, we refer to [Hit01, Section 1.3]). Another additional constraint is of course the strong triangle inequality, which one cannot avoid when dealing with arbitrary posets of distances. Finally, we remark that we do not need AC here either. In fact, since we consider only the supremum of a chain, the proof of the employed fixed point theorem (Proposition 2.10) remains valid as long as chains have a least upper bound in $\mathbf{B}X$. For this it suffices to assume that X is chain-spherically complete, as demonstrated in Proposition 3.3.

Summing up, one may argue that, in order to find a result as strong as Theorem 5.1 (ii), one needs to keep up many strong restrictions known from classical metric spaces. The additional requirements on (spherical) completeness and the triangle inequality account for the broader class of possible distance sets one obtains when considering gums.

6 Summary and conclusion

Taking up a technique from [EH98] that was suggested for the study of generalized ultrametric spaces in [HS03], we have investigated the relation between gums and their spaces of formal balls. In Section 3, it was shown that there are close connections between domain theoretic properties of the space of formal balls $\mathbf{B}X$ and the dually ordered set of distances of a gum Γ^∂. Especially, certain completeness conditions on the ultrametric and its set of distances were found to have equivalent completeness properties for $\mathbf{B}X$. In addition, the metric topology of a gum was studied and conditions were introduced for which the domain $\mathbf{B}X$ yields a computational model for this topology. It was argued that similar restrictions should be imposed on the very general notion of a gum in order to obtain a reasonably well-behaved metric topology. After all, it remains an open question, in which way a topology on a gum should be defined. Our results give evidence that various possible definitions may coincide when using appropriate conditions.

In Section 4, the connections between a gum and its space of formal balls were studied in the setting of category theory. For this purpose, appropriate categories of gums and of partial orders were introduced and the functor $\mathbf{B}$ was extended to the morphisms of these categories. By demonstrating that $\mathbf{B}$ is indeed the left adjoint of a categorical equivalence, it could be shown that the spaces of formal balls actually form a very restricted subcategory of all posets. This observation raises doubts concerning the use of $\mathbf{B}$ as a tool for establishing a connection between the theory of ultrametric spaces and domain theory.

Yet, in Section 5, the space of formal balls could be employed to obtain a modified version of the Prieß-Crampe and Ribenboim Theorem, which establishes the original result without the use of the Axiom of Choice. However, the possibility to describe the required fixed point instead of just stating its mere existence comes at the price of stronger preconditions. A further strengthening of the assumptions even led to a result that may be compared to the Banach fixed point theorem for classical metric spaces, since a fixed point is described as the intersection of an ω-chain of closed balls.

From our considerations, one may draw various conclusions concerning the study of the space of formal balls and of generalized ultrametric spaces. First of all, the rather peculiar characteristics of the category of formal balls suggests that this approach is not appropriate to link gums to the area of domain theory. Still this result settles the conjecture that this method could be used for this purpose. One might also presume that a similar characterization is possible in the classical case as well. Furthermore, it has been shown that the space of formal balls can effectively contribute to investigate properties of gums. In fact, it played a vital role in the investigation of the metric topology as well as in the validation of two fixed point theorems. Finally, the treatment of gums in their full generality lead to the discovery of various special conditions on the set of distances, which gave accurate characterizations of certain desired situations. In the light of these findings, stronger restrictions, such as linearity or directedness of the distance set, might be relaxed in future works on this topic.

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