TECHNISCHE UNIVERSITÄT DRESDEN

Artificial Intelligence, Computational Logic

PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 5 Answer-Set Programming: Motivation and Introduction

*slides adapted from Torsten Schaub [Gebser et al.(2012)]

Lucia Gomez Alvarez

Dresden

DRESDEN concept Exzellenz aus Wissenschaft und Kultur

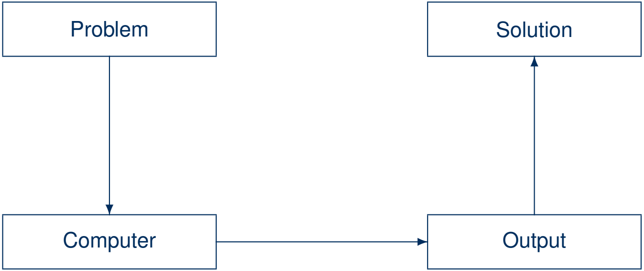
Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction (CSP)
7. Evolutionary Algorithms/ Genetic Algorithms
8. Structural Decomposition Techniques (Tree/Hypertree Decompositions)

Outline

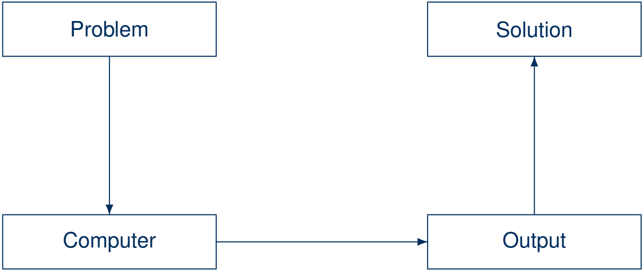
1. Motivation
   - Declarative Problem Solving
   - ASP in a Nutshell
   - ASP Paradigm
2. Introduction
   - Syntax
   - Semantics
   - Examples
   - Language Constructs and Extensions
   - Modelling

Informatics



Informatics

"What is the problem?" versus "How to solve the problem?"



Traditional (Imperative) programming

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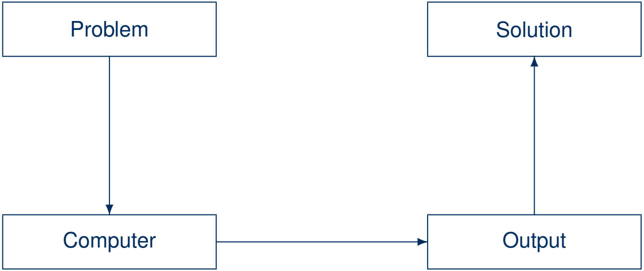

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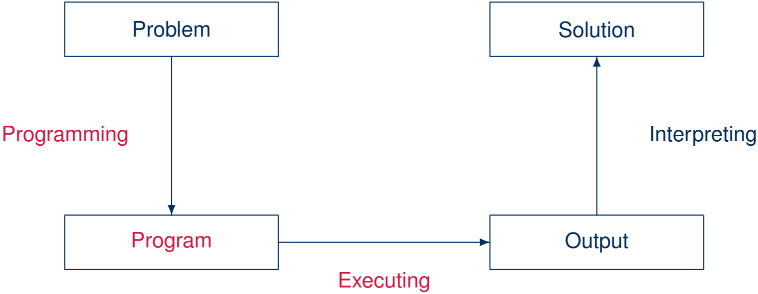

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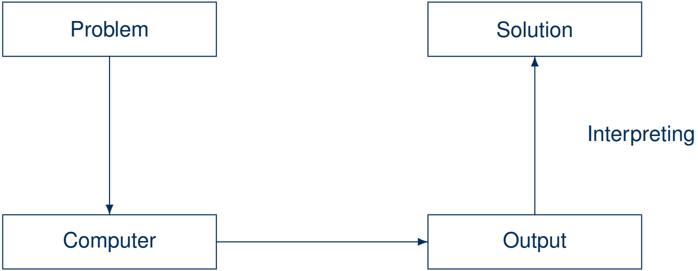

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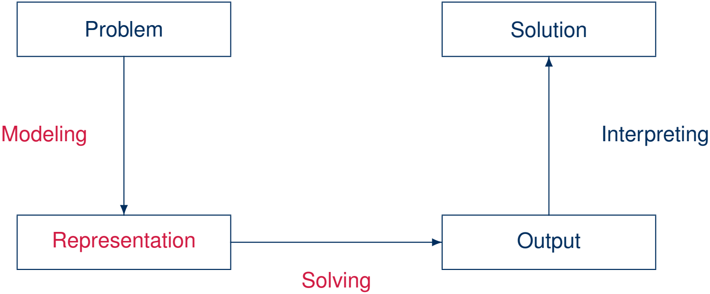

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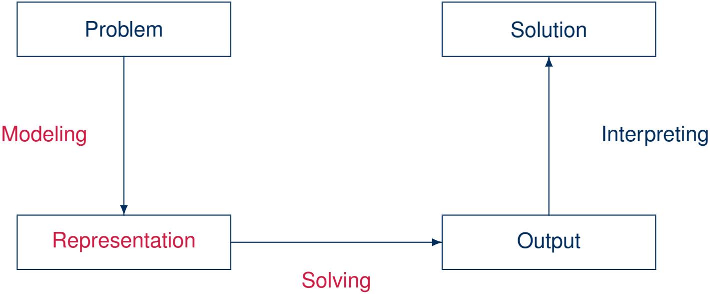

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in a Nutshell

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- ASP embraces many emerging application areas

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ASP = DB+LP+KR+SAT

KR's shift of paradigm

Theorem Proving based approach (eg. Prolog)

1. Provide a representation of the problem
2. A solution is given by a derivation of a query

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LP-style playing with blocks

Prolog program

```
on(a,b).
on(b,c).

above(X,Y) :- on(X,Y).
above(X,Y) :- on(X,Z), above(Z,Y).
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LP-style playing with blocks

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Prolog queries (testing entailment)

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Shuffled Prolog program

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Prolog queries (answered via fixed execution)

```
?- above(a,c).

Fatal Error: local stack overflow.
```

SAT-style playing with blocks

Formula

$$
\begin{array}{ll}
 & on(a,b) \\
\wedge & on(b,c) \\
\wedge & (on(X,Y) \rightarrow above(X,Y)) \\
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Herbrand model

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➥ Answer Set Programming (ASP)

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Stable Herbrand model

{ on(a, b), on(b, c), above(b, c), above(a, b), above(a, c) }

ASP-style playing with blocks

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Stable Herbrand model (and no others)

$\{$ on(a,b), on(b,c), above(b,c), above(a,b), above(a,c) $\}$

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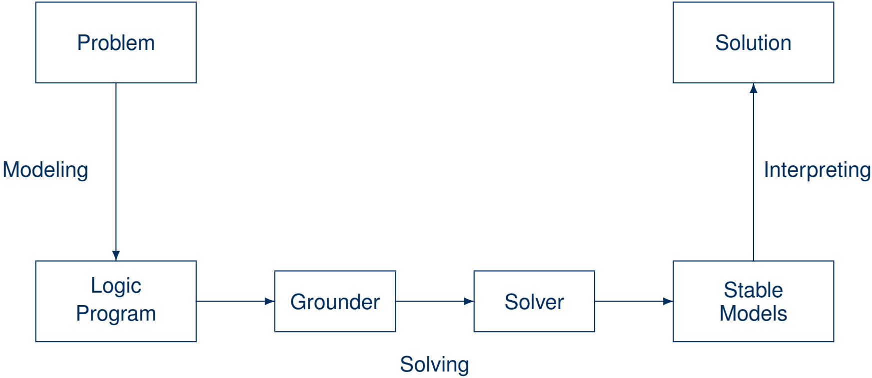
ASP versus Prolog

ASP	Prolog
Model generation	Query orientation
Bottom-up	Top-down
Modeling language	Programming language
Rule-based format	
Instantiation Flat terms	Unification Nested terms

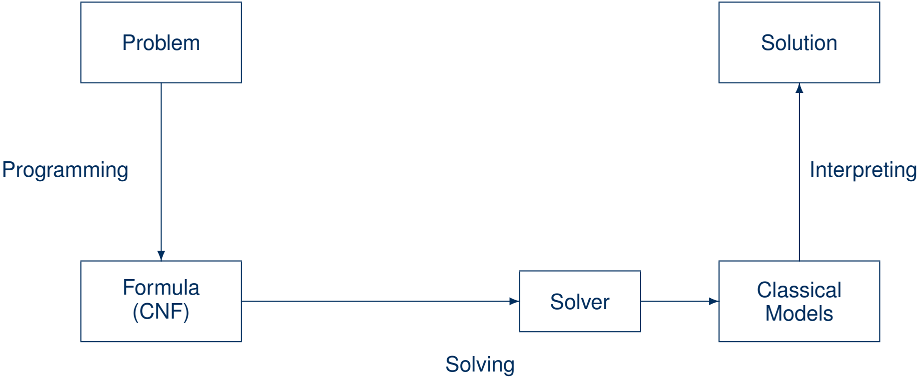
ASP versus SAT

ASP	SAT
Model generation	
Bottom-up	
Constructive Logic	Classical Logic
Closed (and open) world reasoning	Open world reasoning
Modeling language	—
Complex reasoning modes	Satisfiability Testing
Satisfiability	Satisfiability
Enumeration	Enumeration
Optimization	—
Intersection/Union	—
$NP^{(NP)}$	NP

ASP solving


Problem
Solution
Modeling
Interpreting
Logic Program
Grounder
Solver
Stable Models
Solving

SAT solving



Problem
Solution
Programming
Interpreting
Formula
(CNF)
Solver
Classical
Models
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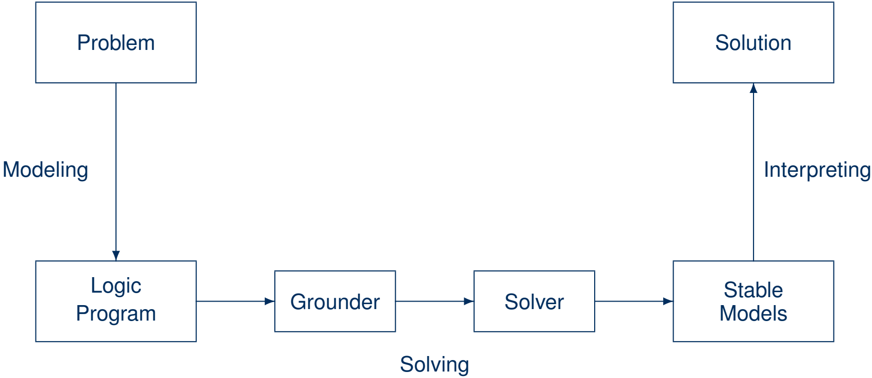
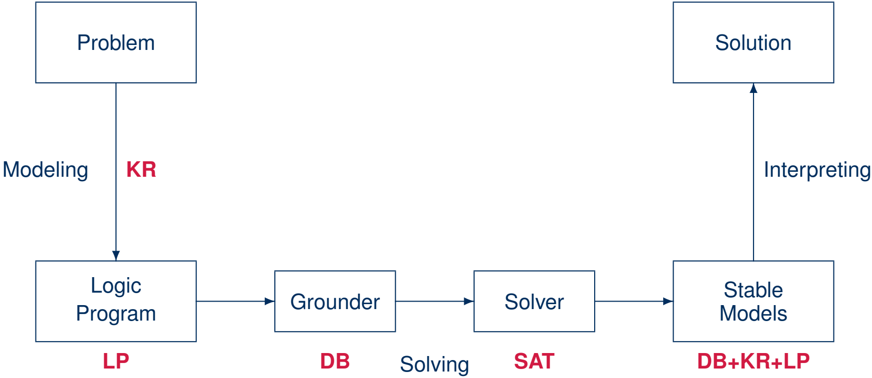
Rooting ASP solving



Rooting ASP solving



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  - Automated Planning
  - Code Optimization
  - Composition of Renaissance Music
  - Database Integration
  - Decision Support for NASA shuttle controllers
  - Model Checking
  - Product Configuration
  - Robotics
  - System Biology
  - System Synthesis
  - (industrial) Team-building
  - and many many more

What does ASP offer?

- Integration of DB, KR, and SAT techniques
- Succinct, elaboration-tolerant problem representations
  - Rapid application development tool
- Easy handling of dynamic, knowledge intensive applications
  - including: data, frame axioms, exceptions, defaults, closures, etc

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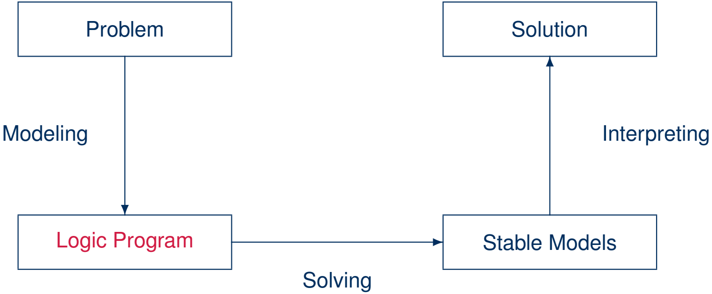
Problem solving in ASP: Syntax



Normal logic programs

- A (normal) logic program over a set $\mathcal{A}$ of atoms is a finite set of rules
- A (normal) rule, r, is of the form

$$a_0 \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n$$

where $0 \leq m \leq n$ and each $a_i \in \mathcal{A}$ is an atom for $0 \leq i \leq n$

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- Notation

$$\begin{array}{rcl} head(r) & = & a_0 \\ body(r) & = & \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\} \\ body(r)^+ & = & \{a_1, \ldots, a_m\} \\ body(r)^- & = & \{a_{m+1}, \ldots, a_n\} \end{array}$$

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- A program is called positive if $body(r)^- = \emptyset$ for all its rules

Rough notational convention

We sometimes use the following notation interchangeably
in order to stress the respective view:

	true, false	if	and	or	iff	default negation	classical negation
source code		`:-`	`,`	`|`		`not`	`-`
logic program		$\leftarrow$	$,$	$;$		*not*	$\neg$
formula	$\bot, \top$	$\rightarrow$	$\wedge$	$\vee$	$\leftrightarrow$	$\sim$	$\neg$

Default Vs Classical Negation

Example of default negation:

```
trainSchedulingInstance(rf654,Dresden,Berlin,9,30,am).
trainSchedulingInstance(rf654,Dresden,Berlin,10,30,am).

takeCar(P5,origin,dest,h,m,d) :- trip(origin,dest,h,m,d), not
trainSchedulingInstance(-,origin,dest,h,m,d).
```

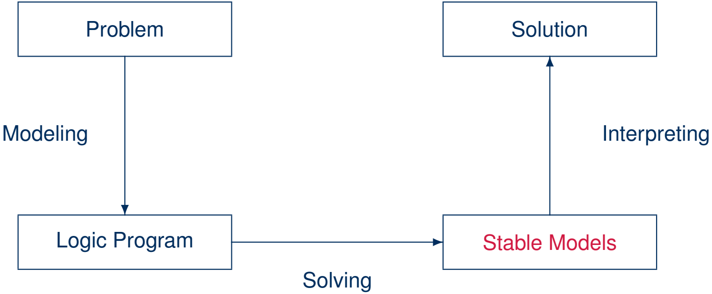
Example of classical negation

```
-domestic(giraffe).
strong(tiger).
wild(p) :- -domestic(p).
```

Combined example of default and classical negation

```
fly(X) :- bird(X), not -fly(X).
-fly(X) :- penguin(X).
bird(X) :- penguin(X). bird(X) :- eagle(X).
```

Problem solving in ASP: Semantics



Formal Definition

Stable models of positive programs

- A set of atoms X is closed under a positive program P iff for any $r \in P$, $head(r) \in X$ whenever $body(r)^{+} \subseteq X$
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- The smallest set of atoms which is closed under a positive program P is denoted by $Cn(P)$
  - $Cn(P)$ corresponds to the $\subseteq$-smallest model of P (ditto)
- The set $Cn(P)$ of atoms is the stable model of a positive program P

Some “logical” remarks

- Positive rules are also referred to as definite clauses
  - Definite clauses are disjunctions with exactly one positive atom:

$$a_0 \vee \neg a_1 \vee \cdots \vee \neg a_m$$

  - A set of definite clauses has a (unique) smallest model

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  - Non-definite Horn clauses can be regarded as integrity constraints
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- This smallest model is the intended semantics of such sets of clauses
  - Given a positive program P, $Cn(P)$ corresponds to the smallest model of the set of definite clauses corresponding to P

Basic idea

Consider the logical formula Φ and its three (classical) models:

$\{p, q\}, \{q, r\},$ and $\{p, q, r\}$

Φ $\boxed{q \;\wedge\; (q \wedge \neg r \rightarrow p)}$

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$\{p, q\}, \{q, r\},$ and $\{p, q, r\}$

p	$\mapsto$	1
q	$\mapsto$	1
r	$\mapsto$	0

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P_Φ

q	$\leftarrow$	
p	$\leftarrow$	$q,\ not\ r$

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Informally, a set X of atoms is a stable model of a logic program P

- if X is a (classical) model of P and
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(rooted in intuitionistic logics HT (Heyting, 1930) and G3 (Gödel, 1932))

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Formal Definition

Stable model of normal programs

- The Gelfond-Lifschitz Reduct[Gelfond and Lifschitz(1991)], P^X, of a program P relative to a set X of atoms is defined by

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- Note: $Cn(P^X)$ is the $\subseteq$–smallest (classical) model of P^X
- Note: Every atom in X is justified by an "applying rule from P"

A closer look at P^X

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- In other words, given a set X of atoms from P,

  P^X is obtained from P by deleting

  1. each rule having $not\ a$ in its body with $a \in X$ and then
  2. all negative atoms of the form $not\ a$ in the bodies of the remaining rules

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  P^X is obtained from P by deleting
  1. each rule having *not a* in its body with $a \in X$
     and then
  2. all negative atoms of the form *not a*
     in the bodies of the remaining rules

- Note: Only negative body literals are evaluated w.r.t. X

A first example

$$P = \{p \leftarrow p,\ q \leftarrow \mathit{not}\ p\}$$

A first example

$$P = \{p \leftarrow p,\ q \leftarrow not\ p\}$$

X		$Cn(P^X)$
$\emptyset$		
$\{p\}$		
$\{q\}$		
$\{p, q\}$		

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$\{p\}$	$p \leftarrow p$	$\emptyset$	✘
$\{q\}$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$	
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$\{p\}$	$p \leftarrow p$	$\emptyset$	✘
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$\{p, q\}$	$p \leftarrow p$	$\emptyset$	✘

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$\{p\}$		$\emptyset$

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Some properties

- A logic program may have zero, one, or multiple stable models!

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- A logic program may have zero, one, or multiple stable models!
- If X is an stable model of a logic program P, then X is a model of P (seen as a formula)
- If X and Y are stable models of a normal program P, then $X \not\subset Y$

Programs with Variables

Let P be a logic program

- Let $\mathcal{T}$ be a set of (variable-free) terms
- Let $\mathcal{A}$ be a set of (variable-free) atoms constructable from $\mathcal{T}$

Programs with Variables

Let P be a logic program

- Let $\mathcal{T}$ be a set of variable-free terms (also called Herbrand universe)
- Let $\mathcal{A}$ be a set of (variable-free) atoms constructable from $\mathcal{T}$ (also called alphabet or Herbrand base)

Programs with Variables

Let P be a logic program

- Let $\mathcal{T}$ be a set of (variable-free) terms
- Let $\mathcal{A}$ be a set of (variable-free) atoms constructable from $\mathcal{T}$

- Ground Instances of $r \in P$: Set of variable-free rules obtained by replacing all variables in r by elements from $\mathcal{T}$:

$$ground(r) = \{r\theta \mid \theta : var(r) \to \mathcal{T}, var(r\theta) = \emptyset\}$$

  where $var(r)$ stands for the set of all variables occurring in r;
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  θ is a (ground) substitution

- Ground Instantiation of P: $ground(P) = \bigcup_{r \in P} ground(r)$

An example

$$P = \{\ r(a,b) \leftarrow,\ r(b,c) \leftarrow,\ t(X,Y) \leftarrow r(X,Y)\ \}$$

$$\mathcal{T} = \{a,b,c\}$$

$$\mathcal{A} = \left\{ \begin{array}{c} r(a,a), r(a,b), r(a,c), r(b,a), r(b,b), r(b,c), r(c,a), r(c,b), r(c,c), \\ t(a,a), t(a,b), t(a,c), t(b,a), t(b,b), t(b,c), t(c,a), t(c,b), t(c,c) \end{array} \right\}$$

An example

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$$ground(P) = \left\{ \begin{array}{lllll} r(a,b) \leftarrow\ , & & & & \\ r(b,c) \leftarrow\ , & & & & \\ t(a,a) \leftarrow r(a,a), & t(b,a) \leftarrow r(b,a), & t(c,a) \leftarrow r(c,a), \\ t(a,b) \leftarrow r(a,b), & t(b,b) \leftarrow r(b,b), & t(c,b) \leftarrow r(c,b), \\ t(a,c) \leftarrow r(a,c), & t(b,c) \leftarrow r(b,c), & t(c,c) \leftarrow r(c,c) \end{array} \right\}$$

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- Intelligent Grounding aims at reducing the ground instantiation

Stable models of programs with Variables

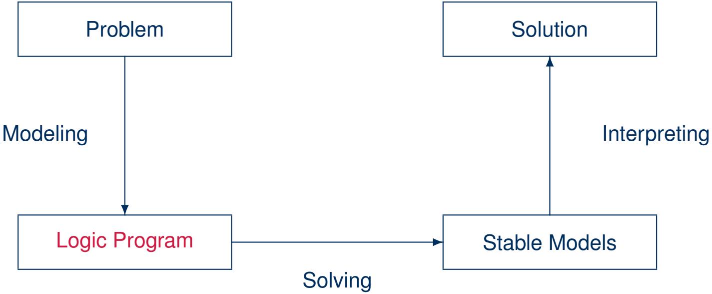
Let P be a normal logic program with variables

Stable models of programs with Variables

Let P be a normal logic program with variables

- A set X of (ground) atoms is a stable model of P, if $Cn(ground(P)^X) = X$

Problem solving in ASP: Extended Syntax



Language Constructs and Extensions

Language Constructs and Extensions

- Variables (over the Herbrand Universe)
  - `p(X) :- q(X)` over constants $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ stands for
    `p(a) :- q(a), p(b) :- q(b), p(c) :- q(c)`

Language Constructs and Extensions

- Conditional Literals
  - `p :- q(X) : r(X)` given `r(a), r(b), r(c)` stands for
    `p :- q(a), q(b), q(c)`

Language Constructs and Extensions

- Disjunction
  - `p(X) | q(X) :- r(X)`

Language Constructs and Extensions

- Integrity Constraints
  - `:- q(X), p(X)`

Language Constructs and Extensions

- Choice
  - `2 { p(X,Y) : q(X) } 7 :- r(Y)`

Language Constructs and Extensions

- Aggregates
  - `s(Y) :- r(Y), 2 #count { p(X,Y) : q(X) } 7`
  - also: `#sum`, `#avg`, `#min`, `#max`, `#even`, `#odd`

Language Constructs and Extensions

- Variables (over the Herbrand Universe)
  - `p(X) :- q(X)` over constants $\{a, b, c\}$ stands for
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  - `p :- q(X) : r(X)` given `r(a), r(b), r(c)` stands for
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  - `p(X) | q(X) :- r(X)`
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  - `:- q(X), p(X)`
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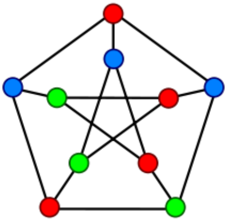
  - `2 { p(X,Y) : q(X) } 7 :- r(Y)`
- Aggregates
  - `s(Y) :- r(Y), 2 #count { p(X,Y) : q(X) } 7`
  - also: `#sum, #avg, #min, #max, #even, #odd`

Example 3-Colorability



- Vertices are represented with predicates $\mathrm{node}(X)$;
- Edges are represented with predicates $\mathrm{edge}(X, Y)$.

Question: Is there a valid assignment of three colors for an input graph G such that no two adjacent vertices have the same color?

Graph coloring

```
node(1..6).
```

Graph coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).
```

Graph coloring

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node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
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edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).
```

Graph coloring

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node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
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col(r).   col(b).   col(g).
```

} Problem instance

Graph coloring

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node(1..6).

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edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
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edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).


1 { color(X,C) : col(C) } 1 :- node(X).
```

Graph coloring

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node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
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col(r).   col(b).   col(g).


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:- edge(X,Y), color(X,C), color(Y,C).
```

Graph coloring

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} Problem encoding

Graph coloring

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col(r).   col(b).   col(g).
```

Problem instance

```
1 { color(X,C) : col(C) } 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

Problem encoding

color.lp

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
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edge(3,1).  edge(3,4).  edge(3,5).
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col(r).   col(b).   col(g).


1 { color(X,C) : col(C) } 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```
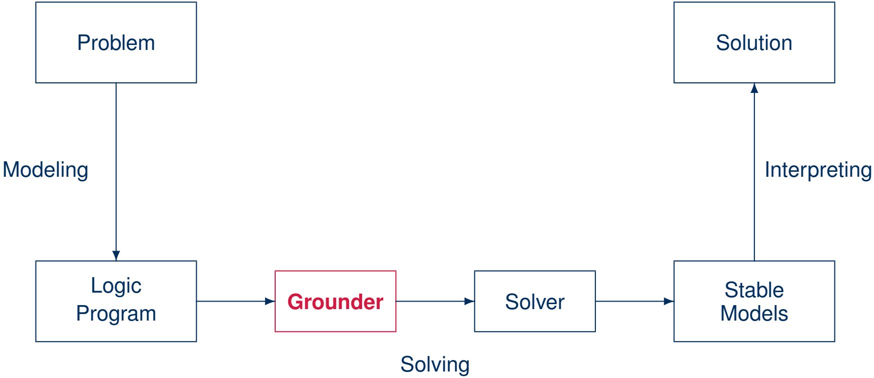

ASP solving process



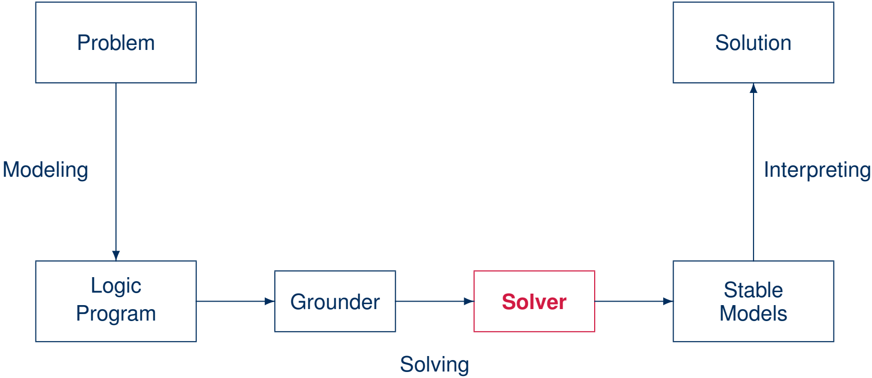
Graph coloring: Grounding

```
$ gringo --text color.lp
```

Graph coloring: Grounding

```
$ gringo --text color.lp
node(1).  node(2).  node(3).  node(4).  node(5).  node(6).

edge(1,2).  edge(1,3).  edge(1,4).  edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).  edge(4,1).  edge(4,2).  edge(5,3).
edge(5,4).  edge(5,6).  edge(6,2).  edge(6,3).  edge(6,5).

col(r).  col(b).  col(g).

1 {color(1,r), color(1,b), color(1,g)} 1.
1 {color(2,r), color(2,b), color(2,g)} 1.
1 {color(3,r), color(3,b), color(3,g)} 1.
1 {color(4,r), color(4,b), color(4,g)} 1.
1 {color(5,r), color(5,b), color(5,g)} 1.
1 {color(6,r), color(6,b), color(6,g)} 1.

 :- color(1,r), color(2,r).  :- color(2,g), color(5,g).  ...  :- color(6,r), color(2,r).
 :- color(1,b), color(2,b).  :- color(2,r), color(6,r).       :- color(6,b), color(2,b).
 :- color(1,g), color(2,g).  :- color(2,b), color(6,b).       :- color(6,g), color(2,g).
 :- color(1,r), color(3,r).  :- color(2,g), color(6,g).       :- color(6,r), color(3,r).
 :- color(1,b), color(3,b).  :- color(3,r), color(1,r).       :- color(6,b), color(3,b).
 :- color(1,g), color(3,g).  :- color(3,b), color(1,b).       :- color(6,g), color(3,g).
 :- color(1,r), color(4,r).  :- color(3,g), color(1,g).       :- color(6,r), color(5,r).
 :- color(1,b), color(4,b).  :- color(3,r), color(4,r).       :- color(6,b), color(5,b).
 :- color(1,g), color(4,g).  :- color(3,b), color(4,b).       :- color(6,g), color(5,g).
 :- color(2,r), color(4,r).  :- color(3,g), color(4,g).
 :- color(2,b), color(4,b).  :- color(3,r), color(5,r).
 :- color(2,g), color(4,g).  :- color(3,b), color(5,b).
 :- color(2,r), color(5,r).  :- color(3,g), color(5,g).
 :- color(2,b), color(5,b).  :- color(4,r), color(1,r).
```

ASP solving process



Graph coloring: Solving

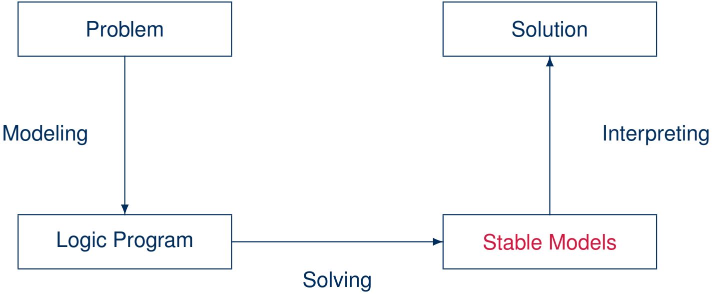
```
$ gringo color.lp | clasp 0
```

Graph coloring: Solving

```
$ gringo color.lp | clasp 0
```

```
clasp version 2.1.0
Reading from stdin
Solving...
Answer: 1
edge(1,2) ... col(r) ... node(1) ... color(6,b) color(5,g) color(4,b) color(3,r) color(2,r
Answer: 2
edge(1,2) ... col(r) ... node(1) ... color(6,r) color(5,g) color(4,r) color(3,b) color(2,b
Answer: 3
edge(1,2) ... col(r) ... node(1) ... color(6,g) color(5,b) color(4,g) color(3,r) color(2,r
Answer: 4
edge(1,2) ... col(r) ... node(1) ... color(6,r) color(5,b) color(4,r) color(3,g) color(2,g
Answer: 5
edge(1,2) ... col(r) ... node(1) ... color(6,g) color(5,r) color(4,g) color(3,b) color(2,b
Answer: 6
edge(1,2) ... col(r) ... node(1) ... color(6,b) color(5,r) color(4,b) color(3,g) color(2,g
SATISFIABLE

Models      : 6
Time        : 0.002s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
```

Problem solving in ASP: Reasoning Modes



Outline

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- See also: `http://potassco.sourceforge.net`