TECHNISCHE UNIVERSITÄT DRESDEN

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SEMINAR LOGIC-BASED KNOWLEDGE REPRESENTATION

Logic Recap

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Outline

We review some key concepts of classical logic:

- Syntax and Semantics of Propositional Logic
- Model Theory vs. Proof Theory
- Soundness and Completeness
- Syntax and Semantics of First-Order Logic

Current Seminar Plan

Date	Topic	Presenter
25.04.	Modal Logic – Semantics	Hesham Morgan
02.05.	Modal Logic – Proof Theory	Anna Schuldt
09.05.	Temporal Reasoning	Florian Emmrich
16.05.	Epistemic Logics	Rajab Aghamov
23.05.	NMR Introduction	Puneetha Jangir Lok Ram Jangir
30.05.	Default Logic	Diksha Chawla
13.06.	Autoepistemic Logic	Avtar Singh
20.06.	NMR Recap	Syed Muhammad Mahmudul Haque
27.06.	Theorem Proving	Alexej Popovic
04.07.	SAT	Tom Friese
11.07.	ASP	Rutuja Mohekar

Note: Register seminar with examination office.

Motivation

In **Knowledge Representation and Reasoning** we want to ...

... formally represent a collection of **propositions** believed by some **agent**,

and to **derive** new information from these propositions by applying reasoning techniques.

Logic allows us to ...

... **formally** represent information in various logical systems,

and to **draw logical inferences** from given information.

Quiz: Logic Basics

Quiz: For each of the following statements, decide whether it holds.

1. In propositional logic (PL), $P \to Q$ is equivalent to $P \lor \neg Q$.
2. In PL, if φ is a tautology and $\mathcal{I}$ is an interpretation, then $\varphi^{\mathcal{I}} = \mathit{true}$.
3. In PL, if Δ and Γ are sets of formulas and φ is a formula, then $\Delta \models \varphi$ implies $\Delta \cup \Gamma \models \varphi$.
4. In first-order logic, $\neg\forall x.(P(x) \lor Q(x))$ is equivalent to $(\exists x.\neg P(x)) \land (\exists x.\neg Q(x))$.
5. In first-order logic, there is a proof system with a derivation relation $\vdash$ that coincides with the entailment relation $\models$.

Propositional Logic

Propositional Logic – Overview

- It is one of the simplest logics
- It can be used to write simple representations of a domain
- There exist reasoning algorithms that exhibit excellent performance in practice
- (Most of) you are already familiar with it.

Syntax: Propositional Alphabet

1. Propositional variables (**PL**):

   basic statements that can be true or false

2. The symbols $\top$ ("truth") and $\bot$ ("falsehood")

3. Propositional connectives:

   $\neg$ negation (not)
   $\wedge$ conjunction (and)
   $\vee$ disjunction (or)
   $\rightarrow$ implication (if ... then)
   $\leftrightarrow$ bi-directional implication (if and only if)

4. Punctuation symbols "(" and ")" can be used to avoid ambiguity

Semantics: Interpretations

Definition 1.1 (Interpretation):
An **interpretation** $\mathcal{I}$ assigns truth values to propositional variables:

$$\mathcal{I} : \mathbf{PL} \to \{true, false\}$$

An interpretation for a (set of) formulas X interprets the propositional variables occurring in X.

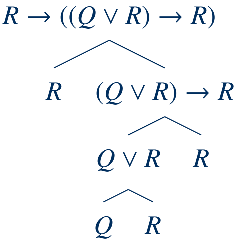
Example: An interpretation $\mathcal{I}$ for the formula $R \to ((Q \vee R) \to R)$:

$$\begin{aligned} R^{\mathcal{I}} &= true \\ Q^{\mathcal{I}} &= false \end{aligned}$$

A formula with n propositional variables has 2^n interpretations.

Semantics of Formulas

The truth value of the propositional variables in a formula α determines the truth value of α.



$$\begin{aligned} R^{\mathcal{I}} &= \mathit{true} \\ Q^{\mathcal{I}} &= \mathit{false} \\ (Q \lor R)^{\mathcal{I}} &= \mathit{true} \\ ((Q \lor R) \to R)^{\mathcal{I}} &= \mathit{true} \\ (R \to ((Q \lor R) \to R))^{\mathcal{I}} &= \mathit{true} \end{aligned}$$

Definition 1.2 (Model): We say that $\mathcal{I}$ is a **model** of α iff $\mathcal{I}$ makes α true.

Using Propositional Logic for KR

Propositional Logic provides a simple KR language.

To write down a representation of our domain do the following:

1. Identify the relevant propositions:

Benign	The tumour is benign
$\mathit{Metastasis}$	The tumour has metastasis
$\mathit{Stage}4$	The tumour is in Stage 4
$\ldots$	

2. Express our knowledge using a set of formulas (knowledge base):

$$\begin{aligned} \mathit{Benign} \\ \mathit{Benign} \leftrightarrow \neg \mathit{Metastasis} \\ \mathit{Stage}4 \rightarrow \mathit{Metastasis} \\ \ldots \end{aligned}$$

Reasoning with a Knowledge Base

Knowledge Base $\mathcal{K}_1$:

$$\begin{aligned} \mathit{Benign} \land \mathit{Stage}4 \\ \mathit{Benign} \leftrightarrow \neg \mathit{Metastasis} \\ \mathit{Stage}4 \to \mathit{Metastasis} \\ \ldots \end{aligned}$$

Knowledge Base $\mathcal{K}_2$:

$$\begin{aligned} \mathit{Benign} \\ \mathit{Benign} \leftrightarrow \neg \mathit{Metastasis} \\ \mathit{Stage}4 \to \mathit{Metastasis} \\ \ldots \end{aligned}$$

We would like to answer the following questions:

1. Do our KBs make sense?
   - $\mathcal{K}_1$ seems contradictory
2. What is the implicit knowledge we can derive from our KBs?
   - $\mathcal{K}_2$ seems to imply the formula $\neg \mathit{Stage}4$

Model Theory – Reasoning

Definition 1.3 (Semantic Consequence): Let Γ be a set of formulas and α a formula.
We write $\Gamma \models \alpha$ if and only if every model of Γ is also a model of α.

Definition 1.4 (Tautology): Let α be some formula.
We write $\models \alpha$ if and only if α is true in every interpretation.

Example from $\mathcal{K}_2$:

$$\{B, B \leftrightarrow \neg M, S4 \rightarrow M\} \models \neg S4$$

- Let $\mathcal{I}$ be a model of $\{B, B \leftrightarrow \neg M, S4 \rightarrow M\}$.
- Then $(B)^{\mathcal{I}} = \mathit{true}$, $(M)^{\mathcal{I}} = \mathit{false}$.
- Since $(S4 \rightarrow M)^{\mathcal{I}} = \mathit{true}$ and $(M)^{\mathcal{I}} = \mathit{false}$, it must hold that $(S4)^{\mathcal{I}} = \mathit{false}$.
- Thus $(\neg S4)^{\mathcal{I}} = \mathit{true}$.

Proof Theory

In proof theory:

- We do not consider the semantical interpretation of logical formulas.
- Rather, we are concerned with a syntactic description of our logical system ...
- that allows to put our logic on an axiomatic foundations and to
- syntactically derive formulas from a set of axioms and inference rules.
- Some well-known systems include: Hilbert Systems, Natural Deduction and Tableau Calculi.

Definition 1.5 (Syntactic Consequence): Let Γ be a set of formulas and α a formula. We write $\Gamma \vdash \alpha$ if and only if there is a derivation with conclusion α from Γ.

Definition 1.6 (Theorem): If $\Gamma = \emptyset$, we write $\vdash \alpha$ and we say that α is a **theorem**.

Proof Theory – Hilbert System

Axioms:

Axiom 1. $\phi \rightarrow (\psi \rightarrow \phi)$
Axiom 2. $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
Axiom 3. $(\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$

Inference Rule: (Modus Ponens): From $\phi \rightarrow \psi$ and ϕ, infer ψ.

Example: Show $\phi \rightarrow \psi, \psi \rightarrow \chi \vdash \phi \rightarrow \chi$

$$
\begin{aligned}
(\psi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)) &\qquad Ax1[\phi/\psi \rightarrow \chi; \psi/\phi] \\
(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) &\qquad Ax2 \\
\psi \rightarrow \chi &\qquad Premise \\
\phi \rightarrow (\psi \rightarrow \chi) &\qquad MP(1,3) \\
(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi) &\qquad MP(2,4) \\
\phi \rightarrow \psi &\qquad Premise \\
\phi \rightarrow \chi &\qquad MP(5,6)
\end{aligned}
$$

Soundness and Completeness

For propositional logic (and other logical systems) we can show that the **semantic** and **syntactic entailment** coincide. That is: the relations "$\models$" and "$\vdash$" coincide.

We distinguish both directions:

Theorem 1.7 (Soundness): $\Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha$

Theorem 1.8 (Completeness): $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$

[For proofs, see for instance: Dirk van Dalen, Logic and Structure (2008)]

Monotonicity

What does it mean for a logic to be monotone? Monotonicity is a property of the consequence relation:

Definition 1.9 (Monotonicity): Let Σ and Δ be sets of formulas and H be a formula. If $\Sigma \models H$ and $\Sigma \subseteq \Delta$ then $\Delta \models H$.

Example: Let $\Sigma = \{p, q\}, H = p, \Delta = \{p, q, r\}$. What if $\Delta = \{p, q, \neg p\}$?

What would we have to do to show that some entailment relation is non-monotonic? Find an example where:

- $\Sigma \models H$
- $\Sigma \subseteq \Delta$
- **But**: $\Delta \not\models H$

Limitations of Propositional Logic

Consider the following argument:

$$\begin{array}{rl} & \text{All men are mortal} \\ & \text{Socrates is a man} \\ \hline \therefore & \text{Socrates is mortal} \end{array}$$

The argument seems to be valid.

However, in propositional logic:

$$\begin{array}{rl} & p \\ & q \\ \hline \not\therefore & r \end{array}$$

First-Order Logic

FOL Syntax: Symbols

A first-order alphabet consists of

- Predicate Symbols, each with a fixed arity

Arthritis	Unary Predicate
Affects	Binary Predicate

- Function symbols, each with a fixed arity

ssnOf	Unary Function Symbol

- Constants: JohnSmith, MaryJones, JRA
- Variables: x, y, z
- Propositional connectives $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$
- Symbols $\top$ and $\bot$
- The universal and existential quantifiers: $\forall$, $\exists$

FOL Syntax: Terms

Terms stand for specific objects:

- Variables are terms
- Constants are terms
- The application of a function symbol to terms leads to a term

$\mathit{JohnSmith}$	stands for	the person named John Smith
$\mathit{ssnOf}(\mathit{JohnSmith})$	stands for	the ssn number of John Smith
x	stands for	some object (undetermined)
$\mathit{ssnOf}(x)$	stands for	some ssn number (undetermined)

FOL Syntax: Formulas

An atomic formula (atom) is of the form

$P(t_1, \ldots, t_n)$ P is an n-ary predicate, t_i are terms

Examples:

$\mathit{Child}(\mathit{JohnSmith})$	John Smith is a child
$\mathit{JuvenileArthritis}(\mathit{JRA})$	JRA is a juvenile arthritis
$\mathit{Affects}(\mathit{JRA}, \mathit{JohnSmith})$	John Smith is affected by JRA

An atom represents a simple statement:

- similar to atoms in propositional logic,
- but first-order atoms have finer-grained structure.

FOL Syntax: Formulas

Complex formulas:

- Every atom is a formula

$$Child(JohnSmith), Affects(x, JohnSmith)$$

- $\top$ and $\bot$ are formulas
- If α is a formula, then $\neg\alpha$ is a formula

$$\neg Affects(JRA, JohnSmith), \quad \neg Child(y)$$

- If α, β are formulas, $(\alpha \circ \beta)$ is a formula for $\{\circ \in \wedge, \vee, \rightarrow, \leftrightarrow\}$

$$Affects(JRA, y) \rightarrow Child(y) \vee Teenager(y)$$

- If α a formula and x a variable, $(\forall x.\alpha)$, $(\exists x.\alpha)$ are formulas

$$\forall y.(Affects(JRA, y)) \rightarrow Child(y) \vee Teenager(y))$$

$$\neg(\exists x.\exists y(JuvArthritis(x) \wedge Affects(x, y) \wedge Adult(y)))$$

FOL Syntax: Formulas

Intuitively, a free variable occurrence in a formula is one that does not appear in the scope of a quantifier:

$$Affects(JRA, \underline{y}) \rightarrow Child(\underline{y}) \vee Teenager(\underline{y})$$

$$\exists x.\big(JuvArthritis(x) \wedge Affects(x, \underline{y}) \wedge Adult(\underline{y})\big)$$

$$\exists x.\big(JuvArthritis(x)\big) \wedge Affects(\underline{x}, \underline{y}) \wedge Adult(\underline{y})$$

A variable occurrence is bound if it is not free.

A sentence is a formula with no free variable occurrences.

Example FOL Sentences

A juvenile disease affects only children or teenagers:

$$\forall x.\forall y.((\mathit{JuvDisease}(x) \land \mathit{Affects}(x, y)) \to \mathit{Child}(y) \lor \mathit{Teenager}(y))$$

Children and teenagers are not adults:

$$\forall x.((\mathit{Child}(x) \lor \mathit{Teenager}(x)) \to \neg \mathit{Adult}(x))$$

FOL Interpretations

As in PL, the meaning of sentences is given by interpretations.

An interpretation is a pair $\mathcal{I} = \langle D, \cdot^{\mathcal{I}} \rangle$ where:

- D is a non-empty set, called the interpretation domain.

$$D = \{u, v, w, s\}$$

- $\cdot^{\mathcal{I}}$ is the interpretation function and it associates:
  - With each constant c an object $c^{\mathcal{I}} \in D$.

$$\mathit{JohnSmith}^{\mathcal{I}} = u \quad \mathit{MaryWilliams}^{\mathcal{I}} = v \quad \mathit{JRA}^{\mathcal{I}} = w \quad \ldots$$

  - With each n-ary function symbol f, a function $f^{\mathcal{I}} : D^n \to D$.

$$\mathit{ssnOf}^{\mathcal{I}} = \{u \mapsto s, \ldots\}$$

  - With each n-ary predicate symbol P, a relation $P^{\mathcal{I}} \subseteq D^n$.

$$\mathit{Child}^{\mathcal{I}} = \{u, v\} \quad \mathit{Adult}^{\mathcal{I}} = \emptyset \quad \mathit{Affects}^{\mathcal{I}} = \{\langle w, u \rangle, \ldots\}$$

Evaluation of Terms

Terms are interpreted as elements of the interpretation domain.

We have already seen how to interpret constants

$$JohnSmith^{\mathcal{I}} = u \quad MaryWilliams^{\mathcal{I}} = v \quad JRA^{\mathcal{I}} = w \quad \ldots$$

To interpret terms, we need to interpret (free) variables by means of a mapping from variables to domain elements (an assignment)

Given $\mathcal{I}$ and assignment **a**, we can interpret any term.
Let $\mathcal{I}$ be as before and **a** map x to u:

$$\begin{aligned} JohnSmith^{\mathcal{I},\mathbf{a}} &= u \\ x^{\mathcal{I},\mathbf{a}} &= u \\ (ssnOf(x))^{\mathcal{I},\mathbf{a}} &= ssnOf^{\mathcal{I}}(u) = s \end{aligned}$$

Evaluation of Formulas

Given $\mathcal{I}$ and **a**, a formula is interpreted as either **true** or **false**.

Atomic formulas:

$$P(t_i, \dots, t_n)^{\mathcal{I},\mathbf{a}} = \textbf{true} \quad \text{iff} \quad \langle t_i^{\mathcal{I},\mathbf{a}}, \dots, t_n^{\mathcal{I},\mathbf{a}} \rangle \in P^{\mathcal{I}}$$

Examples:

$$Child(JohnSmith)^{\mathcal{I},\mathbf{a}} = \textbf{true} \quad \text{since} \quad JohnSmith^{\mathcal{I},\mathbf{a}} = u \text{ and } Child^{\mathcal{I}} = \{u, v\}$$

$$Affects(JRA, x)^{\mathcal{I},\mathbf{a}} = \textbf{true} \quad \text{since} \quad JRA^{\mathcal{I},\mathbf{a}} = w, \quad x^{\mathcal{I},\mathbf{a}} = u \text{ and } Affects^{\mathcal{I}} = \{\langle w, u \rangle\}$$

Propositional connectives are interpreted as usual:

$$\begin{aligned}
(\neg Child(JohnSmith))^{\mathcal{I},\mathbf{a}} &= \textbf{false} \\
(Affects(JRA, x) \land Child(JohnSmith))^{\mathcal{I},\mathbf{a}} &= \textbf{true} \\
(Child(JohnSmith) \rightarrow \neg Child(JohnSmith))^{\mathcal{I},\mathbf{a}} &= \textbf{false}
\end{aligned}$$

Evaluation of Formulas

Given $\mathcal{I}$ and **a**, a formula is interpreted as either **true** or **false.**

Existential quantifiers:

$$(\exists x.\mathit{Affects}(\mathit{JRA}, x))^{\mathcal{I},\mathbf{a}_\emptyset} = \textbf{true}$$

since there exists an assignment **a** extending $\mathbf{a}_\emptyset$ such that $\mathit{Affects}(\mathit{JRA}, x)^{\mathcal{I},\mathbf{a}} = \textbf{true}$

Universal quantifiers:

$$(\forall x.\mathit{Affects}(\mathit{JRA}, x))^{\mathcal{I},\mathbf{a}_\emptyset} = \textbf{false}$$

since it is not true that, for any assignment **a** extending $\mathbf{a}_\emptyset$, $\mathit{Affects}(\mathit{JRA}, x)^{\mathcal{I},\mathbf{a}} = \textbf{true}$.

Evaluation of Sentences

For interpreting a sentence φ under $\mathcal{I}$, **a**, the top-level assignment **a** is irrelevant.

Theorem 1.10: For any sentence φ and assignments **a**, **a**′, we have $\varphi^{\mathcal{I},\mathbf{a}} = \varphi^{\mathcal{I},\mathbf{a}'}$.

Example: Consider the sentence

$$\forall x \forall y.((JuvDisease(x) \land Affects(x, y)) \to (Child(y) \lor Teenager(y)))$$

Assume the interpretation $\mathcal{I}$ with $\mathbf{D} = \{u, v, w\}$ given as follows:

$$JuvDisease^{\mathcal{I}} = \{u\} \qquad Child^{\mathcal{I}} = \{w\} \qquad Teenager^{\mathcal{I}} = \emptyset \qquad Affects^{\mathcal{I}} = \{\langle u, w\rangle\}$$

φ without quantifiers must evaluate to true in $\mathcal{I}$ for all valuations $\mathbf{a} : \{x, y\} \to \mathbf{D}$.
Example for $\mathbf{a}_1 = \{x \mapsto u, y \mapsto v\}$:

$$(JuvDisease(x)^{\mathcal{I},\mathbf{a}_1} \land Affects(x, y)^{\mathcal{I},\mathbf{a}_1}) \to (Child(y)^{\mathcal{I},\mathbf{a}_1} \lor Teenager(y)^{\mathcal{I},\mathbf{a}_1})$$

$$(\mathbf{true} \land \mathbf{false}) \to (\mathbf{true} \lor \mathbf{false})$$

$$\mathbf{true}$$

Propositional vs. FOL Interpretations

More complicated to give meaning to FOL than to PL formulas:

$$JuvDisease \rightarrow AffectsChild \lor AffectsTeenager \quad (PL)$$

$$\forall x.\forall y.((JuvDisease(x) \land Affects(x, y)) \rightarrow (Child(y) \lor Teenager(y))) \quad (FOL)$$

PL Interpretations

- Assigns truth values to atoms
- The truth value of complex formulas determined by induction

Example formula has 8 possible interpretations and 7 models

FOL interpretations

- Specify the domain for quantifiers to quantify over
- Interpret constants, predicates, functions
- Assign objects to variables

Example formula has ∞ possible interpretations and ∞ models

Summary and Outlook

We reviewed syntax and semantics of PL and FOL.

Logical systems can be described from two points of view:

- model theory
- proof theory

For PL, FOL, and many other logics these points of view coincide (soundness and completeness).

PL, FOL, and many other logics are monotonic.

Open questions:

- How can we define systems other than PL and FOL? (Next session)
- What do non-monotonic logics look like? (In a few weeks)