Introduction to Automatic Structures

Summer Semester 2016

Exercise Sheet 2 – Word Automatic Structures

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Exercise 2.1 Which of the following statements hold true?

(a) Any word automatic structure has a finite domain.

(b) For all relations R, both R and its convolution $c(R)$ have the same cardinality.

(c) If A is countably infinite, and $P \subseteq A$ is a unary relation, then the structure $(A; P)$ has an automatic presentation.

(d) If the structure $(A; R_1, \ldots, R_n)$ has an automatic presentation, then also $(A; R_1, \ldots, R_{n-1})$ is automata presentable.

(e) If $(A; R_1, \ldots, R_n)$ is an automatic structure, then it has an automatic presentation, too.

Exercise 2.2 For each of the following structures, check whether it is automatic, and if it has an automatic presentation.

(a) $(\{0, \ldots, n-1\}; +, \leq)$ where $+$ denotes the addition modulo n.

(b) $(\mathbb{N}; +, \leq)$

(c) $(\mathbb{R}; +, \leq)$

(d) $(\{a, b\}^*; R_1)$ where

$$R_1 \coloneqq \{\, (u, v) \in (\{a, b\}^*)^2 \mid \forall\, i\colon u(i) = a \Rightarrow v(i) = a \,\}.$$

(e) $(\{a\}^*; R_2, R_3)$ where

$$R_2 \coloneqq \{\, (u, v, w) \in (\{a\}^*)^3 \mid \mathsf{length}(u) > \mathsf{length}(v) > \mathsf{length}(w) \,\},$$
$$\text{and} \quad R_3 \coloneqq \{\, (u, v, w) \in (\{a\}^*)^3 \mid \mathsf{length}(u) > \mathsf{length}(v) + \mathsf{length}(w) \,\}.$$

(f) $(\Sigma^*; \leq_1, \ldots, \leq_n)$ where

- $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is an alphabet,
- $|w|_i$ denotes the number of occurrences of the symbol σ_i in w, and
- $u \leq_i v$ if $|u|_i \leq |v|_i$.

Exercise 2.3 Let R be a relation of arity n, and consider a permutation π on $\{1, \ldots, n\}$. Show that $\pi(R) \coloneqq \{\, (x_{\pi(1)}, \ldots, x_{\pi(n)}) \mid (x_1, \ldots, x_n) \in R \,\}$ is automatic if R is automatic.

Exercise 2.4 Let R be a relation of arity m, S a relation of arity n, and $\ell \leq m+1$. The *join* of R and S at position ℓ is defined as the relation

$$R \bowtie_\ell S := \{\, (x_1, \dots, x_{n+\ell-1}) \mid (x_1, \dots, x_m) \in R \text{ and } (x_\ell, \dots, x_{n+\ell-1}) \in S \,\}.$$

Show that $R \bowtie_\ell S$ is automatic if R and S are automatic.

Exercise 2.5 Let Σ be a finite alphabet, and R a binary relation on Σ^*, such that the structure $(\Sigma^*; R)$ is automatic. For each word w over Σ, we denote by $\overleftarrow{w}$ the word w written backwards. Define the relation

$$\overleftarrow{R} := \{\, (\overleftarrow{w_1}, \overleftarrow{w_2}) \mid (w_1, w_2) \in R \,\}.$$

Prove or disprove the following statements:

(a) The structure $(\Sigma^*; \overleftarrow{R}\,)$ is automatic.

(b) The structure $(\Sigma^*; \overleftarrow{R}\,)$ has an automatic presentation.

Exercise 2.6 Let $\mathcal{A} = (A; R_1, \dots, R_n)$ be an automatic structure, and consider a subset $X \subseteq A$. The *restriction* of $\mathcal{A}$ to X is defined as the structure $\mathcal{A}|_X := (X; R_1|_X, \dots, R_n|_X)$ where $R_i|_X := R_i \cap X^{\mathrm{arity}(R_i)}$. Prove or refute the following statements:

(a) If X is regular, then the restriction $\mathcal{A}|_X$ is automatic.

(b) The restriction $\mathcal{A}|_X$ is automatic.

Exercise 2.7 Let $\mathcal{A} = (A; R_1, \dots, R_n)$ be an automata presentable structure.

(a) Is there a structure with domain $\wp(A)$ that has an automatic presentation?

(b) Let $\mathcal{B}$ be another automata presentable structure. Is there a structure with domain B^A that has an automatic presentation?