TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

# COMPLEXITY THEORY

## Lecture 7: NP Completeness

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**Knowledge-Based Systems**

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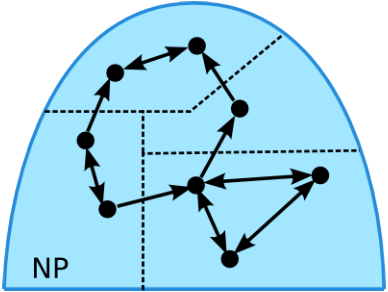
More recent versions of this slide deck might be available.
For the most current version of this course, see
`https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en`

# Are NP Problems Hard?

# The Structure of NP

Idea: polynomial many-one reductions define an order on problems



# NP-Hardness and NP-Completeness

**Definition 7.1:**

(1) A language **H** is NP-hard, if **L** $\leq_p$ **H** for every language **L** $\in$ NP.

(2) A language **C** is NP-complete, if **C** is NP-hard and **C** $\in$ NP.

## NP-Completeness

- NP-complete problems are the hardest problems in NP.
- They constitute the maximal class (wrt. $\leq_p$) of problems within NP.
- They are all equally difficult – an efficient solution to one would solve them all.

**Theorem 7.2:** If **L** is NP-hard and **L** $\leq_p$ **L**′, then **L**′ is NP-hard as well.

# Proving NP-Completeness

## How to show NP-completeness

To show that **L** is NP-complete, we must show that every language in NP can be reduced to **L** in polynomial time.

## Alternative approach

Given an NP-complete language **C**, we can show that another language **L** is NP-complete just by showing that

- **C** $\leq_p$ **L**
- **L** $\in$ NP

However: Is there any NP-complete problem at all?

Yes, thousands of them!

# The Cook-Levin Theorem

# The Cook-Levin Theorem

**Theorem 7.3 (Cook 1970, Levin 1973): SAT** is NP-complete.

**Proof:**

(1) SAT $\in$ NP

Take satisfying assignments as polynomial certificates for the satisfiability of a formula.

(2) SAT is hard for NP

Proof by reduction from any word problem of some polynomially time-bounded NTM.

□

# Proving the Cook-Levin Theorem: Main Objective

**Given:**

- a polynomial $p$
- a $p$-time bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$
- a word $w$

**Intended reduction:** Define a propositional logic formula $\varphi_{p,\mathcal{M},w}$ such that

(1) $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts $w$ in time $p(|w|)$

(2) $\varphi_{p,\mathcal{M},w}$ is polynomial with respect to $|w|$

# Proving the Cook-Levin Theorem: Rationale

**Given:** polynomial $p$, NTM $\mathcal{M}$, word $w$

**Intended reduction:** Define a propositional logic formula $\varphi_{p,\mathcal{M},w}$ such that

(1) $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts $w$ in time $p(|w|)$

(2) $\varphi_{p,\mathcal{M},w}$ is polynomial with respect to $|w|$

Why does this prove NP-hardness of **Sat**?

Because it leads to a reduction $\mathbf{L} \leq_p$ **Sat** for every language $\mathbf{L} \in$ NP:

- If $\mathbf{L} \in$ NP, then there is an NTM $\mathcal{M}$ that is time-bounded by some polynomial $p$, such that $\mathbf{L}(\mathcal{M}) = \mathbf{L}$.
- The function $f_{\mathcal{M},p} : w \mapsto \varphi_{p,\mathcal{M},w}$ shows $\mathbf{L} \leq_p$ **Sat**:
  - $f$ is a many-one reduction due to item (1) above
  - $f$ is polynomial due to item (2) above

**Note:** We do not claim the transformation $\langle p, \mathcal{M}, w\rangle \mapsto \varphi_{p,\mathcal{M},w}$ to be polynomial in the size of $p$, $\mathcal{M}$, and $w$. Indeed, this would not hold true under reasonable encodings of $p$. But being (multi-)exponential in $p$ is not a concern since the many-one reductions $f_{\mathcal{M},p}$ each use a fixed $p$ and only care about the asymptotic complexity as $w$ grows.

# Proving Cook-Levin: Encoding Configurations

**Idea:** Use logic to describe a run of $\mathcal{M}$ on input $w$ by a formula.

**Note:** On input $w$ of length $n := |w|$, every computation path of $\mathcal{M}$ is of length $\leq p(n)$ and uses $\leq p(n)$ tape cells.

**Use propositional variables for describing configurations:**

$Q_q$ for each $q \in Q$ means "$\mathcal{M}$ is in state $q \in Q$"

$P_i$ for each $0 \leq i < p(n)$ means "the head is at Position $i$"

$S_{a,i}$ for each $a \in \Gamma$ and $0 \leq i < p(n)$ means "tape cell $i$ contains Symbol $a$"

**Represent configuration** $(q, p, a_0 \ldots a_{p(n)})$ by truth assignments to variables from the set

$$\overline{C} := \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

using the truth assignment $\beta$ defined as

$$\beta(Q_s) := \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases} \qquad \beta(P_i) := \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \qquad \beta(S_{a,i}) := \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

# Proving Cook-Levin: Validating Configurations

We define a formula $\mathsf{Conf}(\overline{C})$ for a set of configuration variables

$$\overline{C} = \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

as follows:

$\mathsf{Conf}(\overline{C}) :=$ "the assignment is a valid configuration":

$$\bigvee_{q \in Q} \Big( Q_q \wedge \bigwedge_{q' \neq q} \neg Q_{q'} \Big)$$ "TM in exactly one state $q \in Q$"

$$\wedge \bigvee_{p < p(n)} \Big( P_p \wedge \bigwedge_{p' \neq p} \neg P_{p'} \Big)$$ "head in exactly one position $p \leq p(n)$"

$$\wedge \bigwedge_{0 \leq i < p(n)} \bigvee_{a \in \Gamma} \Big( S_{a,i} \wedge \bigwedge_{b \neq a \in \Gamma} \neg S_{b,i} \Big)$$ "exactly one $a \in \Gamma$ in each cell"

# Proving Cook-Levin: Validating Configurations

For an assignment $\beta$ defined on variables in $\overline{C}$ define

$$\mathsf{conf}(\overline{C},\beta) := \left\{ (q,p,w_0\ldots w_{p(n)}) \;\middle|\; \begin{array}{l} \beta(Q_q) = 1, \\ \beta(P_p) = 1, \\ \beta(S_{w_i,i}) = 1 \text{ for all } 0 \leq i < p(n) \end{array} \right\}$$

**Note:** $\beta$ may be defined on other variables besides those in $\overline{C}$.

**Lemma 7.4:** If $\beta$ satisfies $\mathsf{Conf}(\overline{C})$ then $|\mathsf{conf}(\overline{C},\beta)| = 1$.
We can therefore write $\mathsf{conf}(\overline{C},\beta) = (q,p,w)$ to simplify notation.

**Observations:**

- $\mathsf{conf}(\overline{C},\beta)$ is a potential configuration of $\mathcal{M}$, but it may not be reachable from the start configuration of $\mathcal{M}$ on input $w$.
- Conversely, every configuration $(q,p,w_1\ldots w_{p(n)})$ induces a satisfying assignment $\beta$ or which $\mathsf{conf}(\overline{C},\beta) = (q,p,w_1\ldots w_{p(n)})$.

# Proving Cook-Levin: Transitions Between Configurations

Consider the following formula $\mathsf{Next}(\overline{C}, \overline{C}')$ defined as

$$\mathsf{Conf}(\overline{C}) \wedge \mathsf{Conf}(\overline{C}') \wedge \mathsf{NoChange}(\overline{C}, \overline{C}') \wedge \mathsf{Change}(\overline{C}, \overline{C}').$$

$$\mathsf{NoChange} := \bigvee_{0 \le p < p(n)} \Big( P_p \wedge \bigwedge_{i \neq p, a \in \Gamma} (S_{a,i} \to S'_{a,i}) \Big)$$

$$\mathsf{Change} := \bigvee_{0 \le p < p(n)} \Big( P_p \wedge \bigvee_{\substack{q \in Q \\ a \in \Gamma}} (Q_q \wedge S_{a,p} \wedge \bigvee_{(q',b,D) \in \delta(q,a)} (Q'_{q'} \wedge S'_{b,p} \wedge P'_{D(p)})) \Big)$$

where $D(p)$ is the position reached by moving in direction $D$ from $p$.

**Lemma 7.5:** For any assignment $\beta$ defined on $\overline{C} \cup \overline{C}'$:

$\beta$ satisfies $\mathsf{Next}(\overline{C}, \overline{C}')$ if and only if $\mathsf{conf}(\overline{C}, \beta) \vdash_{\mathcal{M}} \mathsf{conf}(\overline{C}', \beta)$

# Proving Cook-Levin: Start and End

**Defined so far:**

- Conf($\overline{C}$): $\overline{C}$ describes a potential configuration
- Next($\overline{C}, \overline{C}'$): conf($\overline{C}, \beta$) $\vdash_{\mathcal{M}}$ conf($\overline{C}', \beta$)

**Start configuration:** For an input word $w = w_0 \cdots w_{n-1} \in \Sigma^*$, we define:

$$\text{Start}_{\mathcal{M},w}(\overline{C}) := \text{Conf}(\overline{C}) \wedge Q_{q_0} \ \wedge P_0 \ \wedge \bigwedge_{i=0}^{n-1} S_{w_i,i} \wedge \bigwedge_{i=n}^{p(n)-1} S_{\llcorner\!\lrcorner,i}$$

Then an assignment $\beta$ satisfies Start$_{\mathcal{M},w}(\overline{C})$ if and only if $\overline{C}$ represents the start configuration of $\mathcal{M}$ on input $w$.

**Accepting stop configuration:**

$$\text{Acc-Conf}(\overline{C}) := \text{Conf}(\overline{C}) \wedge Q_{q_{\text{accept}}}$$

Then an assignment $\beta$ satisfies Acc-Conf($\overline{C}$) if and only if $\overline{C}$ represents an accepting configuration of $\mathcal{M}$.

# Proving Cook-Levin: Adding Time

Since $\mathcal{M}$ is $p$-time bounded, each run may contain up to $p(n)$ steps
$\rightsquigarrow$ we need one set of configuration variables for each

**Propositional variables:**

$Q_{q,t}$ for all $q \in Q$, $0 \leq t \leq p(n)$ means "at time $t$, $\mathcal{M}$ is in state $q \in Q$"

$P_{i,t}$ for all $0 \leq i, t \leq p(n)$ means "at time $t$, the head is at position $i$"

$S_{a,i,t}$ for all $a \in \Gamma$ and $0 \leq i, t \leq p(n)$ means "at time $t$, tape cell $i$ contains symbol $a$"

**Notation:**

$$\overline{C}_t := \{Q_{q,t},\ P_{i,t},\ S_{a,i,t} \mid \quad q \in Q, 0 \leq i \leq p(n), \quad a \in \Gamma\}$$

# Proving Cook-Levin: The Formula

**Given:**

- a polynomial $p$
- a $p$-time bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$
- a word $w$

We define the formula $\varphi_{p,\mathcal{M},w}$ as follows:

$$\varphi_{p,\mathcal{M},w} := \text{Start}_{\mathcal{M},w}(\overline{C}_0) \wedge \bigvee_{0 \leq t \leq p(n)} \left( \text{Acc-Conf}(\overline{C}_t) \wedge \bigwedge_{0 \leq i < t} \text{Next}(\overline{C}_i, \overline{C}_{i+1}) \right)$$

"$C_0$ encodes the start configuration" and, for some polynomial time $t$:
"$\mathcal{M}$ accepts after $t$ steps" and "$\overline{C}_0, \ldots, \overline{C}_t$ encode a computation path"

**Lemma 7.6:** $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts $w$ in time $p(|w|)$.

Note that an accepting or rejecting stop configuration has no successor.

**Lemma 7.7:** The size of $\varphi_{p,\mathcal{M},w}$ is polynomial in $|w|$.

# The Cook-Levin Theorem

**Theorem 7.3 (Cook 1970, Levin 1973):** **Sat** is NP-complete.

**Proof:**

(1) **Sat** $\in$ NP

Take satisfying assignments as polynomial certificates for the satisfiability of a formula.

(2) **Sat** is hard for NP

Proof by reduction from any word problem of some polynomially time-bounded NTM.

□

# Further NP-complete Problems

# Towards More NP-Complete Problems

Starting with **Sat**, one can readily show more problems **P** to be NP-complete, each time performing two steps:

(1) Show that **P** $\in$ NP

(2) Find a known NP-complete problem **P′** and reduce **P′** $\leq_p$ **P**

Thousands of problems have now been shown to be NP-complete.
(See Garey and Johnson for an early survey)

## In this course:

| | | | | |
|---|---|---|---|---|
| | $\leq_p$ | **Clique** | $\leq_p$ | **Independent Set** |
| **Sat** | $\leq_p$ | **3-Sat** | $\leq_p$ | **Dir. Hamiltonian Path** |
| | $\leq_p$ | **Subset Sum** | $\leq_p$ | **Knapsack** |

# NP-Completeness of **Clique**

**Theorem 7.8: Clique** is NP-complete.

**Clique**: Given $G, k$, does $G$ contain a clique of order $k$?

**Proof:**

(1) **Clique** $\in$ NP

Take the vertex set of a clique of order $k$ as a certificate.

(2) **Clique** is NP-hard

We show **Sat** $\leq_p$ **Clique**

To every CNF-formula $\varphi$ assign a graph $G_\varphi$ and a number $k_\varphi$ such that

$$\varphi \text{ satisfiable} \iff G_\varphi \text{ contains clique of order } k_\varphi$$

# Sat $\leq_p$ Clique

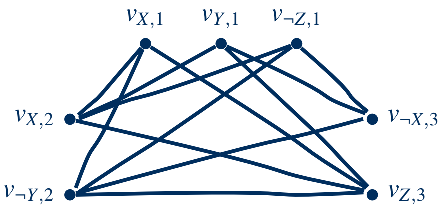
To every CNF-formula $\varphi$ assign a graph $G_\varphi$ and a number $k_\varphi$ such that

$\varphi$ satisfiable if and only if $G_\varphi$ contains clique of order $k_\varphi$

Given $\varphi = C_1 \wedge \cdots \wedge C_k$:

- Set $k_\varphi := k$
- For each clause $C_j$ and literal $L \in C_j$ add a vertex $v_{L,j}$
- Add edge $\{v_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

**Example 7.9:**
$\underbrace{(X \vee Y \vee \neg Z)}_{C_1} \wedge \underbrace{(X \vee \neg Y)}_{C_2} \wedge \underbrace{(\neg X \vee Z)}_{C_3}$



# SAT $\leq_p$ CLIQUE

To every CNF-formula $\varphi$ assign a graph $G_\varphi$ and a number $k_\varphi$ such that

$\varphi$ satisfiable if and only if $G_\varphi$ contains clique of order $k_\varphi$

Given $\varphi = C_1 \wedge \cdots \wedge C_k$:

- Set $k_\varphi := k$
- For each clause $C_j$ and literal $L \in C_j$ add a vertex $v_{L,j}$
- Add edge $\{v_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

## Correctness:

$G_\varphi$ has clique of order $k$ iff $\varphi$ is satisfiable.

## Complexity:

The reduction is clearly computable in polynomial time.

# NP-Completeness of Independent Set

**Independent Set**

Input: An undirected graph $G$ and a natural number $k$

Problem: Does $G$ contain $k$ vertices that share no edges (independent set)?

**Theorem 7.10: Independent Set** is NP-complete.

**Proof:** Hardness by reduction **Clique** $\leq_p$ **Independent Set**:

- Given $G := (V, E)$ construct $\overline{G} := (V, \{\{u, v\} \mid \{u, v\} \notin E \text{ and } u \neq v\})$
- A set $X \subseteq V$ induces a clique in $G$ iff $X$ induces an independent set in $\overline{G}$.
- Reduction: $G$ has a clique of order $k$ iff $\overline{G}$ has an independent set of order $k$.

□

# Summary and Outlook

NP-complete problems are the hardest in NP

Polynomial runs of NTMs can be described in propositional logic (Cook-Levin)

**Clique** and **Independent Set** are also NP-complete

**What's next?**

- More examples of problems
- The limits of NP
- Space complexities