The guarded fragment of FO

$\cancel{\forall \vec{x}\ \varphi(\vec{x})}$

Only relational symbols!

$\forall \vec{x}\ \ R(\vec{x}) \to \varphi(\vec{x})$

$\exists \vec{x}\ \ R(\vec{x}) \land \varphi(\vec{x})$

all free variables must be GUARDED by some atom

Complete for Two exponential time

2ExpTime - complete

$Exp \subseteq NExp \subseteq 2Exp$

$GF^2 \to Exp$, $FO^2 \to NExp$, $GF \to 2Exp$

$\forall x\ \ x = x \to \forall y\ \ ----$

$\forall x\ \forall y\ \ ---$

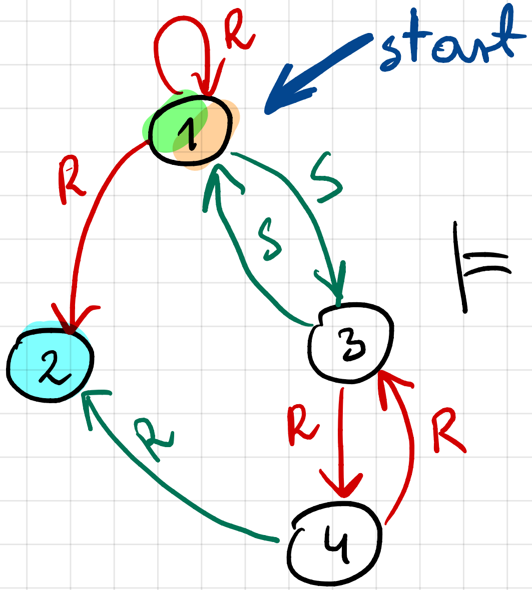
We will focus on GF^2 but the presented methods can be generalised to GF.

Lemma: For every GF^2 formula φ there exist an equi-sat formula ψ (computable in NP) that is a conjunction of:

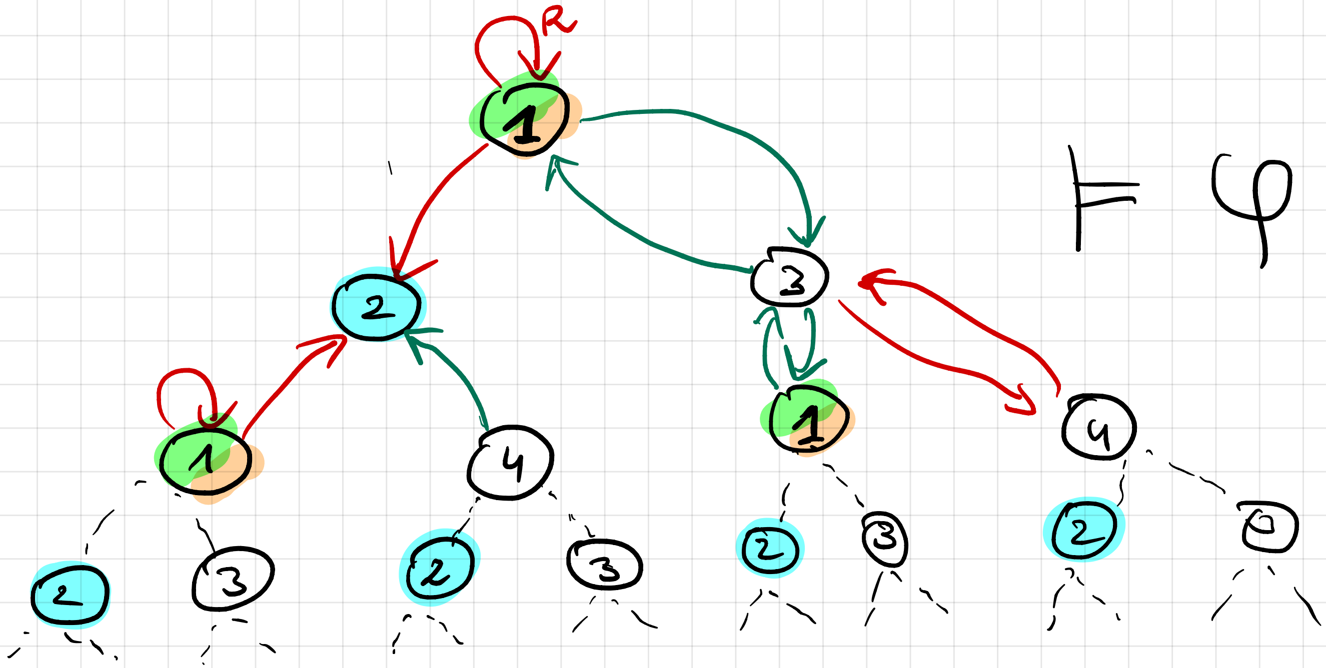
(1) $\exists x\ \ \alpha(x) \land \lambda(x)$ (2) $\forall x\ \ \alpha(x) \to \lambda(x)$

with λ quantifier-free, α, β are atomic

(3) $\forall x \underbrace{\ \forall y}_{x = x \to}\ \ \beta(x,y) \to \lambda(x,y)$ (4) $\forall x\ \ \alpha(x) \to (\exists y\,[\beta(x,y) \land \lambda(x,y)])$



$\models \varphi$



$\models \varphi$

Main idea: For a node v copy all its successors below v and preserve all the original connections between v and their copies

Lemma: For every GF^2 formula φ there exist an equi-sat formula ψ (computable in NP) that is a conjunction of:

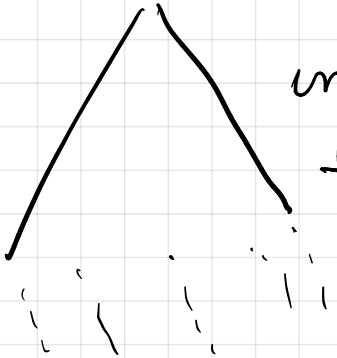
(1) $\exists x \; \alpha(x) \wedge \lambda(x)$ (2) $\forall x \; \alpha(x) \to \lambda(x)$ $\boxed{\text{with } \lambda \text{ quantifier-free, } \alpha, \beta \text{ are atomic}}$

(3) $\underbrace{\forall x}_{x=x \to} \forall y \; \beta(x,y) \to \lambda(x,y)$ (4) $\forall x \; \alpha(x) \to (\exists y [\beta(x,y) \wedge \lambda(x,y)])$

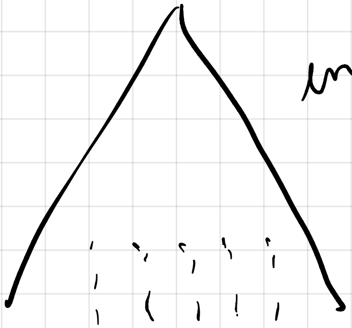
TFCAE :

- $\mathfrak{A} \models \varphi$

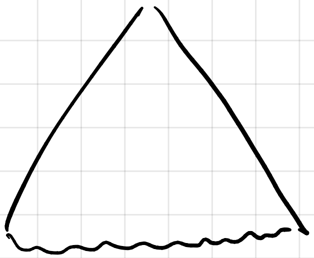
$\uparrow$ obvious $\downarrow$ unravelling

-  infinite tree-like $\models \varphi$

$\uparrow$ obvious $\downarrow$ pruning

-  infinite tree-like structure of small degree of every node $\models \varphi$

  $\leq \mathrm{poly}(|\varphi|)$

-  finite tree-like structure of depth $\approx \exp(|\varphi|)$ and small branchg $\tilde{\models} \ \varphi$

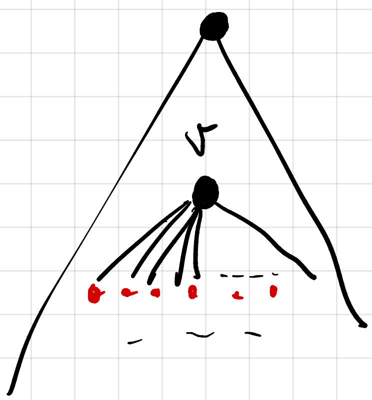
  satisfies φ except leaves

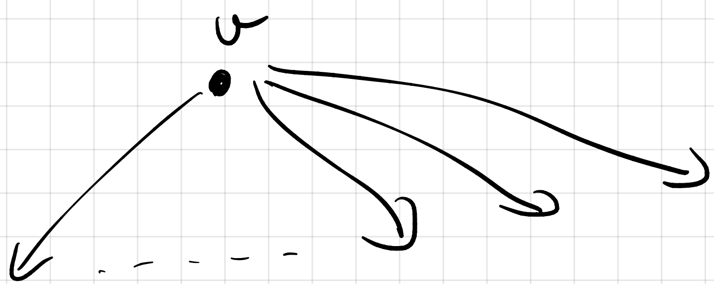
φ is a conjunction of :

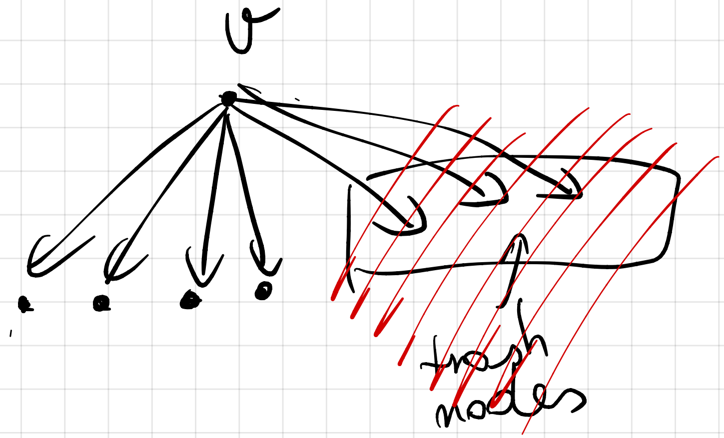
always set at the root

(1) $\exists x \ \alpha(x) \wedge \lambda(x)$ (2) $\forall x \ \alpha(x) \to \lambda(x)$ $\boxed{\text{with } \lambda \text{ quantifier-free, } \alpha, \beta \text{ are atomic}}$

(3) $\forall \underbrace{x \ \forall}_{x = x \to} y \ \beta(x,y) \to \lambda(x,y)$ (4) $\forall x \ \alpha(x) \to (\exists y [\beta(x,y) \wedge \lambda(x,y)])$


$\models \varphi$





I select the minimal number of chilren of v serving as witnesses for $\forall x \exists y$ conjuncts and remove all other children of v except the selected elements

v after surgery has small branching $\leq |\varphi|$ and the resulting structure is still a model of φ.

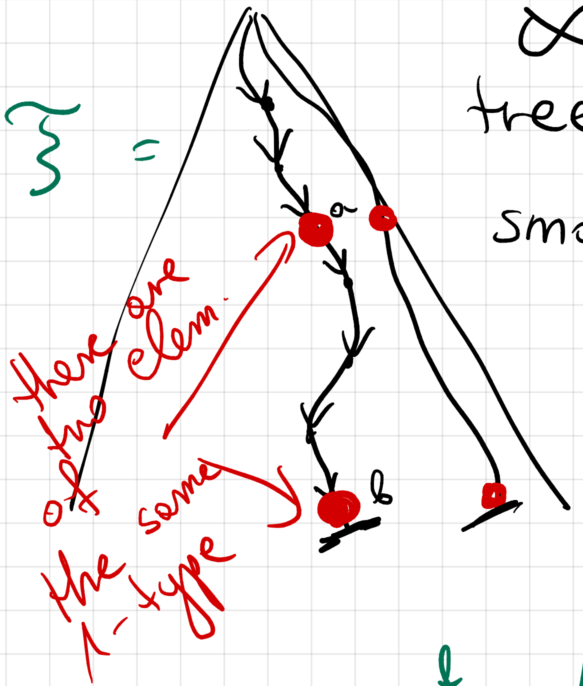
φ is a conjunction of:

always sat at the root

(1) $\exists x \ \alpha(x) \wedge \lambda(x)$ (2) $\forall x \ \alpha(x) \to \lambda(x)$

$\boxed{\text{with } \lambda \text{ quantifier-free, } \alpha, \beta \text{ are atomic}}$

(3) $\forall \underbrace{x \ \forall}_{x=x \to} y \ \beta(x,y) \to \lambda(x,y)$ (4) $\forall x \ \alpha(x) \to (\exists y [\beta(x,y) \wedge \lambda(x,y)])$

$\xi =$ 

∞ tree-like

small-branching

$\models \varphi$

$sig(\varphi) = \Sigma$

$\leq 2^{|\Sigma|}$ different 1-types

repeat as long as can

For every branch in ξ if it has a branch of size $> 2^{|\Sigma|}+1$ we select two element on such a branch with equal 1-types (pidgeon principle) and remove all the elements below the second selected element.

If $\varphi \in GF^2$ has a model then it has a tree-like "nearly model" of depth $= \underbrace{2^{|\varphi|}+1}$ and degree of every element $\leq |\varphi|$.

because we can always take a submodel in which every branch ends on an element having a repetition of its 1-type above

Goal: Design an algorithm that checks whether such a structure exists.

We employ APSpace = ExpTime

Polynomial space algorithm
* guessing poly-size things
* run things in parallel (run things on separate threads) polynomially many