TECHNISCHE
UNIVERSITÄT
DRESDEN

International Center
for Computational Logic

COMPLEXITY THEORY

Lecture 7: NP Completeness

Markus Krötzsch

Knowledge-Based Systems

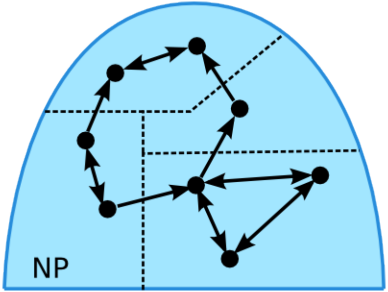
TU Dresden, 8th Nov 2022

More recent versions of this slide deck might be available.
For the most current version of this course, see
`https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en`

Are NP Problems Hard?

The Structure of NP

Idea: polynomial many-one reductions define an order on problems



NP-Hardness and NP-Completeness

Definition 7.1:

(1) A language $\mathbf{H}$ is NP-hard, if $\mathbf{L} \leq_p \mathbf{H}$ for every language $\mathbf{L} \in \mathrm{NP}$.

(2) A language $\mathbf{C}$ is NP-complete, if $\mathbf{C}$ is NP-hard and $\mathbf{C} \in \mathrm{NP}$.

NP-Completeness

- NP-complete problems are the hardest problems in NP.
- They constitute the maximal class (wrt. $\leq_p$) of problems within NP.
- They are all equally difficult – an efficient solution to one would solve them all.

Theorem 7.2: If $\mathbf{L}$ is NP-hard and $\mathbf{L} \leq_p \mathbf{L}'$, then $\mathbf{L}'$ is NP-hard as well.

Proving NP-Completeness

How to show NP-completeness

To show that **L** is NP-complete, we must show that every language in NP can be reduced to **L** in polynomial time.

Alternative approach

Given an NP-complete language **C**, we can show that another language **L** is NP-complete just by showing that

- **C** $\leq_p$ **L**
- **L** $\in$ NP

However: Is there any NP-complete problem at all?

Yes, thousands of them!

The Cook-Levin Theorem

The Cook-Levin Theorem

Theorem 7.3 (Cook 1970, Levin 1973): **Sat** is NP-complete.

Proof:

(1) **Sat** $\in$ NP

Take satisfying assignments as polynomial certificates for the satisfiability of a formula.

(2) **Sat** is hard for NP

Proof by reduction from any word problem of some polynomially time-bounded NTM.

$\square$

Proving the Cook-Levin Theorem: Main Objective

Given:

- a polynomial p
- a p-time bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$
- a word w

Intended reduction: Define a propositional logic formula $\varphi_{p,\mathcal{M},w}$ such that

(1) $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts w in time $p(|w|)$

(2) $\varphi_{p,\mathcal{M},w}$ is polynomial with respect to $|w|$

Proving the Cook-Levin Theorem: Rationale

Given: polynomial p, NTM $\mathcal{M}$, word w

Intended reduction: Define a propositional logic formula $\varphi_{p,\mathcal{M},w}$ such that

(1) $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts w in time $p(|w|)$

(2) $\varphi_{p,\mathcal{M},w}$ is polynomial with respect to $|w|$

Why does this prove NP-hardness of **Sat**?

Because it leads to a reduction **L** $\leq_p$ **Sat** for every language **L** $\in$ NP:

- If **L** $\in$ NP, then there is an NTM $\mathcal{M}$ that is time-bounded by some polynomial p, such that **L**$(\mathcal{M})$ = **L**.
- The function $f_{\mathcal{M},p} : w \mapsto \varphi_{p,\mathcal{M},w}$ shows **L** $\leq_p$ **Sat**:
  - f is a many-one reduction due to item (1) above
  - f is polynomial due to item (2) above

Note: We do not claim the transformation $\langle p, \mathcal{M}, w\rangle \mapsto \varphi_{p,\mathcal{M},w}$ to be polynomial in the size of p, $\mathcal{M}$, and w. Indeed, this would not hold true under reasonable encodings of p. But being (multi-)exponential in p is not a concern since the many-one reductions $f_{\mathcal{M},p}$ each use a fixed p and only care about the asymptotic complexity as w grows.

Proving Cook-Levin: Encoding Configurations

Idea: Use logic to describe a run of $\mathcal{M}$ on input w by a formula.

Note: On input w of length $n := |w|$, every computation path of $\mathcal{M}$ is of length $\leq p(n)$ and uses $\leq p(n)$ tape cells.

Use propositional variables for describing configurations:

Q_q for each $q \in Q$ means "$\mathcal{M}$ is in state $q \in Q$"

P_i for each $0 \leq i < p(n)$ means "the head is at Position i"

$S_{a,i}$ for each $a \in \Gamma$ and $0 \leq i < p(n)$ means "tape cell i contains Symbol a"

Represent configuration $(q, p, a_0 \ldots a_{p(n)})$ by truth assignments to variables from the set

$$\overline{C} := \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

using the truth assignment β defined as

$$\beta(Q_s) := \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases} \qquad \beta(P_i) := \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \qquad \beta(S_{a,i}) := \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

Proving Cook-Levin: Validating Configurations

We define a formula Conf($\overline{C}$) for a set of configuration variables

$$\overline{C} = \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

as follows:

Conf($\overline{C}$) := "the assignment is a valid configuration":

$$\bigvee_{q \in Q} \Big(Q_q \land \bigwedge_{q' \neq q} \neg Q_{q'}\Big)$$

"TM in exactly one state $q \in Q$"

$$\land \bigvee_{p < p(n)} \Big(P_p \land \bigwedge_{p' \neq p} \neg P_{p'}\Big)$$

"head in exactly one position $p \leq p(n)$"

$$\land \bigwedge_{0 \leq i < p(n)} \bigvee_{a \in \Gamma} \Big(S_{a,i} \land \bigwedge_{b \neq a \in \Gamma} \neg S_{b,i}\Big)$$

"exactly one $a \in \Gamma$ in each cell"

Proving Cook-Levin: Validating Configurations

For an assignment β defined on variables in $\overline{C}$ define

$$\text{conf}(\overline{C},\beta) := \left\{ (q,p,w_0\ldots w_{p(n)}) \;\middle|\; \begin{array}{l} \beta(Q_q) = 1, \\ \beta(P_p) = 1, \\ \beta(S_{w_i,i}) = 1 \text{ for all } 0 \leq i < p(n) \end{array} \right\}$$

Note: β may be defined on other variables besides those in $\overline{C}$.

Lemma 7.4: If β satisfies Conf($\overline{C}$) then $|\text{conf}(\overline{C},\beta)| = 1$.
We can therefore write $\text{conf}(\overline{C},\beta) = (q,p,w)$ to simplify notation.

Observations:

- $\text{conf}(\overline{C},\beta)$ is a potential configuration of $\mathcal{M}$, but it may not be reachable from the start configuration of $\mathcal{M}$ on input w.
- Conversely, every configuration $(q,p,w_1\ldots w_{p(n)})$ induces a satisfying assignment β or which $\text{conf}(\overline{C},\beta) = (q,p,w_1\ldots w_{p(n)})$.

Proving Cook-Levin: Transitions Between Configurations

Consider the following formula Next($\overline{C}, \overline{C}'$) defined as

$$\text{Conf}(\overline{C}) \wedge \text{Conf}(\overline{C}') \wedge \text{NoChange}(\overline{C}, \overline{C}') \wedge \text{Change}(\overline{C}, \overline{C}').$$

$$\text{NoChange} := \bigvee_{0 \leq p < p(n)} \Big(P_p \wedge \bigwedge_{i \neq p, a \in \Gamma} (S_{a,i} \to S'_{a,i})\Big)$$

$$\text{Change} := \bigvee_{0 \leq p < p(n)} \Big(P_p \wedge \bigvee_{\substack{q \in Q \\ a \in \Gamma}} (Q_q \wedge S_{a,p} \wedge \bigvee_{(q',b,D) \in \delta(q,a)} (Q'_{q'} \wedge S'_{b,p} \wedge P'_{D(p)}))\Big)$$

where $D(p)$ is the position reached by moving in direction D from p.

Lemma 7.5: For any assignment β defined on $\overline{C} \cup \overline{C}'$:

β satisfies Next($\overline{C}, \overline{C}'$) if and only if conf($\overline{C}, \beta$) $\vdash_{\mathcal{M}}$ conf($\overline{C}', \beta$)

Proving Cook-Levin: Start and End

Defined so far:

- Conf($\overline{C}$): $\overline{C}$ describes a potential configuration
- Next($\overline{C}, \overline{C}'$): conf($\overline{C}, \beta$) $\vdash_{\mathcal{M}}$ conf($\overline{C}', \beta$)

Start configuration: For an input word $w = w_0 \cdots w_{n-1} \in \Sigma^*$, we define:

$$\mathsf{Start}_{\mathcal{M},w}(\overline{C}) := \mathsf{Conf}(\overline{C}) \land Q_{q_0} \land P_0 \land \bigwedge_{i=0}^{n-1} S_{w_i,i} \land \bigwedge_{i=n}^{p(n)-1} S_{\llcorner\!\lrcorner,i}$$

Then an assignment β satisfies Start$_{\mathcal{M},w}(\overline{C})$ if and only if $\overline{C}$ represents the start configuration of $\mathcal{M}$ on input w.

Accepting stop configuration:

$$\mathsf{Acc\text{-}Conf}(\overline{C}) := \mathsf{Conf}(\overline{C}) \land Q_{q_{\mathrm{accept}}}$$

Then an assignment β satisfies Acc-Conf($\overline{C}$) if and only if $\overline{C}$ represents an accepting configuration of $\mathcal{M}$.

Proving Cook-Levin: Adding Time

Since $\mathcal{M}$ is p-time bounded, each run may contain up to $p(n)$ steps
$\leadsto$ we need one set of configuration variables for each

Propositional variables:

$Q_{q,t}$ for all $q \in Q$, $0 \leq t \leq p(n)$ means "at time t, $\mathcal{M}$ is in state $q \in Q$"

$P_{i,t}$ for all $0 \leq i, t \leq p(n)$ means "at time t, the head is at position i"

$S_{a,i,t}$ for all $a \in \Gamma$ and $0 \leq i, t \leq p(n)$ means "at time t, tape cell i contains symbol a"

Notation:

$$\overline{C}_t := \{Q_{q,t},\ P_{i,t},\ S_{a,i,t} \mid \quad q \in Q, 0 \leq i \leq p(n), \quad a \in \Gamma\}$$

Proving Cook-Levin: The Formula

Given:

- a polynomial p
- a p-time bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$
- a word w

We define the formula $\varphi_{p,\mathcal{M},w}$ as follows:

$$\varphi_{p,\mathcal{M},w} := \text{Start}_{\mathcal{M},w}(\overline{C}_0) \wedge \bigvee_{0 \leq t \leq p(n)} \left(\text{Acc-Conf}(\overline{C}_t) \wedge \bigwedge_{0 \leq i < t} \text{Next}(\overline{C}_i, \overline{C}_{i+1}) \right)$$

"C_0 encodes the start configuration" and, for some polynomial time t:

"$\mathcal{M}$ accepts after t steps" and "$\overline{C}_0, \ldots, \overline{C}_t$ encode a computation path"

Lemma 7.6: $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts w in time $p(|w|)$.

Note that an accepting or rejecting stop configuration has no successor.

Lemma 7.7: The size of $\varphi_{p,\mathcal{M},w}$ is polynomial in $|w|$.

The Cook-Levin Theorem

Theorem 7.3 (Cook 1970, Levin 1973): SAT is NP-complete.

Proof:

(1) **SAT** $\in$ NP

Take satisfying assignments as polynomial certificates for the satisfiability of a formula.

(2) **SAT** is hard for NP

Proof by reduction from any word problem of some polynomially time-bounded NTM.

□

Further NP-complete Problems

Towards More NP-Complete Problems

Starting with **Sat**, one can readily show more problems **P** to be NP-complete, each time performing two steps:

1. Show that **P** $\in$ NP
2. Find a known NP-complete problem **P′** and reduce **P′** $\leq_p$ **P**

Thousands of problem have now been shown to be NP-complete.
(See Garey and Johnson for an early survey)

In this course:

$\leq_p$ **Clique** $\leq_p$ **Independent Set**

Sat $\leq_p$ **3-Sat** $\leq_p$ **Dir. Hamiltonian Path**

$\leq_p$ **Subset Sum** $\leq_p$ **Knapsack**

NP-Completeness of **CLIQUE**

Theorem 7.8: CLIQUE is NP-complete.

CLIQUE: Given G, k, does G contain a clique of order $\geq k$?

Proof:

(1) **CLIQUE** $\in$ NP

Take the vertex set of a clique of order k as a certificate.

(2) **CLIQUE** is NP-hard

We show **SAT** $\leq_p$ **CLIQUE**

To every CNF-formula φ assign a graph G_φ and a number k_φ such that

$$\varphi \text{ satisfiable} \quad \Longleftrightarrow \quad G_\varphi \text{ contains clique of order } k_\varphi$$

SAT $\leq_p$ CLIQUE

To every CNF-formula φ assign a graph G_φ and a number k_φ such that

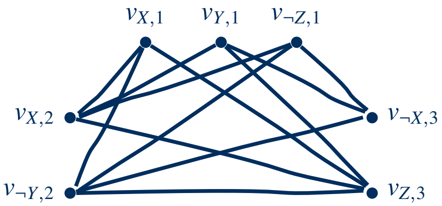
φ satisfiable if and only if G_φ contains clique of order k_φ

Given $\varphi = C_1 \wedge \cdots \wedge C_k$:

- Set $k_\varphi := k$
- For each clause C_j and literal $L \in C_j$ add a vertex $v_{L,j}$
- Add edge $\{v_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

Example 7.9:

$$\underbrace{(X \vee Y \vee \neg Z)}_{C_1} \wedge \underbrace{(X \vee \neg Y)}_{C_2} \wedge \underbrace{(\neg X \vee Z)}_{C_3}$$



SAT $\leq_p$ CLIQUE

To every CNF-formula φ assign a graph G_φ and a number k_φ such that

φ satisfiable if and only if G_φ contains clique of order k_φ

Given $\varphi = C_1 \wedge \cdots \wedge C_k$:

- Set $k_\varphi := k$
- For each clause C_j and literal $L \in C_j$ add a vertex $v_{L,j}$
- Add edge $\{u_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

Correctness:

G_φ has clique of order k iff φ is satisfiable.

Complexity:

The reduction is clearly computable in polynomial time.

NP-Completeness of INDEPENDENT SET

INDEPENDENT SET

Input: An undirected graph G and a natural number k

Problem: Does G contain k vertices that share no edges (independent set)?

Theorem 7.10: INDEPENDENT SET is NP-complete.

Proof: Hardness by reduction **CLIQUE** $\leq_p$ **INDEPENDENT SET**:

- Given $G := (V, E)$ construct $\overline{G} := (V, \{\{u, v\} \mid \{u, v\} \notin E \text{ and } u \neq v\})$
- A set $X \subseteq V$ induces a clique in G iff X induces an independent set in $\overline{G}$.
- Reduction: G has a clique of order k iff $\overline{G}$ has an independent set of order k.

□

Summary and Outlook

NP-complete problems are the hardest in NP

Polynomial runs of NTMs can be described in propositional logic (Cook-Levin)

Clique and **Independent Set** are also NP-complete

What's next?

- More examples of problems
- The limits of NP
- Space complexities