TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 13: Space Hierarchy and Gaps

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Knowledge-Based Systems

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Review

Review: Time Hierarchy Theorems

Time Hierarchy Theorem 12.12 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g \cdot \log g \in o(f)$, then

$$\mathsf{DTime}_*(g) \subsetneq \mathsf{DTime}_*(f)$$

Nondeterministic Time Hierarchy Theorem 12.14 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g(n+1) \in o(f(n))$, then

$$\mathsf{NTime}_*(g) \subsetneq \mathsf{NTime}_*(f)$$

In particular, we find that P $\neq$ ExpTime and NP $\neq$ NExpTime:

$$\neq$$

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace} \subseteq \mathsf{ExpTime} \subseteq \mathsf{NExpTime} \subseteq \mathsf{ExpSpace}$$

$$\neq$$

A Hierarchy for Space

Space Hierarchy

For space, we can always assume a single working tape:

- Tape reduction leads to a constant-factor increase in space
- Constant factors can be eliminated by space compression

Therefore, $\mathrm{DSpace}_k(f) = \mathrm{DSpace}_1(f)$.

Space turns out to be easier to separate – we get:

Space Hierarchy Theorem 13.1: If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is space-constructible, and $g \in o(f)$, then

$$\mathrm{DSpace}(g) \subsetneq \mathrm{DSpace}(f)$$

Challenge: TMs can run forever even within bounded space.

Proving the Space Hierarchy Theorem (1)

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Proof: Again, we construct a diagonalisation machine $\mathcal{D}$. We define a multi-tape TM $\mathcal{D}$ for inputs of the form $\langle \mathcal{M}, w \rangle$ (other cases do not matter), assuming that $|\langle \mathcal{M}, w \rangle| = n$

- Compute $f(n)$ in unary to mark the available space on the working tape
- Initialise a separate countdown tape with the largest binary number that can be written in $f(n)$ space
- Simulate $\mathcal{M}$ on $\langle \mathcal{M}, w \rangle$, making sure that only previously marked tape cells are used
- Time-bound the simulation using the content of the countdown tape by decrementing the counter in each simulated step
- If $\mathcal{M}$ rejects (in this space bound) or if the time bound is reached without $\mathcal{M}$ halting, then accept; otherwise, if $\mathcal{M}$ accepts or uses unmarked space, reject

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L$(\mathcal{D}) \in$ DSpace(f):

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There is w such that $\langle \mathcal{M}, w\rangle \in \mathbf{L}(\mathcal{D})$ iff $\langle \mathcal{M}, w\rangle \notin \mathbf{L}(\mathcal{M})$:

- As for time, we argue that some w is long enough to ensure that f is sufficiently larger than g, so $\mathcal{D}$'s simulation can finish.
- The countdown measures $2^{f(n)}$ steps. The number of possible distinct configurations of $\mathcal{M}$ on w is $|Q| \cdot n \cdot g(n) \cdot |\Gamma|^{g(n)} \in 2^{O(g(n)+\log n)}$, and due to $f(n) \geq \log n$ and $g \in o(f)$, this number is smaller than $2^{f(n)}$ for large enough n.
- If $\mathcal{M}$ has d tape symbols, then $\mathcal{D}$ can encode each in $\log d$ space, and due to $\mathcal{M}$'s space bound $\mathcal{D}$'s simulation needs at most $\log d \cdot g(n) \in o(f(n))$ cells.

Therefore, there is w for which $\mathcal{D}$ simulates $\mathcal{M}$ long enough to obtain (and flip) its output, or to detect that it is not terminating (and to accept, flipping again). □

Space Hierarchies

Like for time, we get some useful corollaries:

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Proof: As for time, but easier. $\square$

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Proof: Savitch tells us that NL $\subseteq$ DSpace$(\log^2 n)$. We can apply the Space Hierachy Theorem since $\log^2 n \in o(n)$. $\square$

Corollary 13.4: For all real numbers $0 < a < b$, we have DSpace$(n^a) \subsetneq$ DSpace(n^b).

In other words: The hierarchy of distinct space classes is very fine-grained.

The Gap Theorem

Why Constructibility?

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Yes. The following theorem shows why (for time):

Special Gap Theorem 13.5: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that DTime$(f(n))$ = DTime$(2^{f(n)})$.

This has been shown independently by Boris Trakhtenbrot (1964) and Allan Borodin (1972).

Reminder: For this we continue to use the strict definition of DTime(f) where no constant factors are included (no hiddden $O(f)$). This simplifes proofs; the factors are easy to add back.

Proving the Gap Theorem

Special Gap Theorem 13.8: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that DTime($f(n)$) = DTime($2^{f(n)}$).

Proof idea: We divide time into exponentially long intervals of the form:

$$[0, n], \quad [n+1, 2^n], \quad [2^n + 1, 2^{2^n}], \quad [2^{2^n} + 1, 2^{2^{2^n}}], \quad \cdots$$

(for some appropriate starting value n)

We are looking for gaps of time where no TM halts, since:

- for every finte set of TMs,
- and every finite set of inputs to these TMs,
- there is some interval of the above form $[m+1, 2^m]$

such none of the TMs halts in between $m+1$ and 2^m steps on any of the inputs.

The task of f is to find the start m of such a gap for a suitable set of TMs and words

Gaps in Time

We consider an (effectively computable) enumeration of all Turing machines:

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Definition 13.6: For arbitrary numbers $i, a, b \geq 0$ with $a \leq b$, we say that $\mathsf{Gap}_i(a, b)$ is true if:

- Given any TM $\mathcal{M}_j$ with $0 \leq j \leq i$,
- and any input string w for $\mathcal{M}_j$ of length $|w| = i$,

$\mathcal{M}_j$ on input w will halt in less than a steps, in more than b steps, or not at all.

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Lemma 13.7: Given $i, a, b \geq 0$ with $a \leq b$, it is decidable if $\mathsf{Gap}_i(a, b)$ holds.

Proof: We just need to ensure that none of the finitely many TMs $\mathcal{M}_0, \ldots, \mathcal{M}_i$ will halt after a to b steps on any of the finitely many inputs of length i. This can be checked by simulating TM runs for at most b steps. $\square$

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$$\text{in}(n) = |\Sigma_0|^n + \cdots + |\Sigma_n|^n \qquad \text{where } \Sigma_i \text{ is the input alphabet of } \mathcal{M}_i$$

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We recursively define a series of numbers $k_0, k_1, k_2, \ldots$ by setting $k_0 = 2n$ and $k_{i+1} = 2^{k_i}$ for $i \geq 0$, and we consider the following list of intervals:

$$\begin{array}{cccc} [k_0 + 1, k_1], & [k_1 + 1, k_2], & \cdots, & [k_{\mathsf{in}(n)} + 1, k_{\mathsf{in}(n)+1}] \\ \| & \| & & \| \\ [2n + 1, 2^{2n}], & [2^{2n} + 1, 2^{2^{2n}}], & \cdots, & [2^{\cdot^{\cdot^{2n}}} + 1, 2^{2^{\cdot^{\cdot^{2n}}}}] \end{array}$$

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[2n + 1, 2^{2n}], & [2^{2n} + 1, 2^{2^{2n}}], & \cdots, & [2^{\cdot^{\cdot^{2n}}} + 1, 2^{2^{\cdot^{\cdot^{2n}}}}]
\end{array}$$

Let $f(n)$ be the least number k_i with $0 \leq i \leq \mathsf{in}(n)$ such that $\mathsf{Gap}_n(k_i + 1, k_{i+1})$ is true.

Properties of f

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Claim: The function f is well-defined.

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Claim: The function f is computable.

Proof: We can compute $\text{in}(n)$ and k_i for any i, and we can decide $\text{Gap}_n(k_i + 1, k_{i+1})$. $\square$

Papadimitriou: "notice the fantastically fast growth, as well as the decidedly unnatural definition of this function."

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Therefore we have $\mathbf{L} \in \mathsf{DTime}(f(n))$. $\square$

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Could we still do a state-space extension as Papadimitriou suggested?

- It seems possible to do a multiplication of states to do the finite automaton detection of words $|w| < j$ alongside the normal operation of the TM
- However, we'll have to leave the movement of the heads to the original TM
- Due to the other requirements, it seems compulsory that the TM will always read the whole input in $f(n)$, so our superimposed finite automaton would get enough information to decide acceptance

(However, this argument has no connection to Papadimitriou's $2j$ bound)

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This leads to a generalised Gap Theorem:

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Example 13.9: There is a function f such that

$$\mathrm{DTime}(f(n)) = \mathrm{DTime}\left(\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{f(n)\ \text{times}} \right)$$

Moreover, the Gap Theorem can also be shown for space (and for other resources) in a similar fashion (space is a bit easier since the case of short words $|w| < j$ is easy to handle in very little space)

Discussion: Significance of the Gap Theorem

What have we learned?

[1] Allender, Loui, Reagan: Complexity Theory. In Computing Handbook, 3rd ed., CRC Press, 2014)

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Discussion: Significance of the Gap Theorem

What have we learned?

- More time (or space) does not always increase computational power
- However, this only works for extremely fast-growing, very unnatural functions

"Fortunately, the gap phenomenon cannot happen for time bounds t
that anyone would ever be interested in"[1]

Main insight: better stick to constructible functions

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Summary and Outlook

Hierarchy theorems tell us that more time/space leads to more power:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace} \subseteq \mathsf{ExpTime} \subseteq \mathsf{NExpTime} \subseteq \mathsf{ExpSpace}$$

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What's next?

- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation
- P vs. NP again