Computing Local Unifiers in the Description Logic $\mathcal{EL}$ without the Top Concept

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Introduction

Unification in Description Logics (DLs) has been proposed in [7] as a novel inference service that can, for example, be used to detect redundancies in ontologies. For instance, assume that one knowledge engineer defines the concept of *professors that are mothers* as $\mathsf{Person} \sqcap \mathsf{Female} \sqcap \exists \mathsf{child}.\top \sqcap \exists \mathsf{job}.\mathsf{Professor}$, whereas another knowledge engineer represents this notion in a somewhat different way, e.g., by using the concept term $\mathsf{Mother} \sqcap \exists \mathsf{job}.(\mathsf{Teacher} \sqcap \mathsf{Researcher})$. These two concept terms are not equivalent, i.e., they are not interpreted by the same set of individuals in every interpretation, but they are nevertheless meant to represent the same concept. They can obviously be made equivalent by substituting the concept name $\mathsf{Professor}$ in the first term by the concept term $\mathsf{Teacher} \sqcap \mathsf{Researcher}$ and the concept name Mother in the second term by the concept term $\mathsf{Person} \sqcap \mathsf{Female} \sqcap \exists \mathsf{child}.\top$. We call a substitution that makes two concept terms equivalent a *unifier* of the two terms. Such a unifier proposes definitions for the concept names that are used as variables. In our example, we know that, if we define Mother as $\mathsf{Person} \sqcap \mathsf{Female} \sqcap \exists \mathsf{child}.\top$ and $\mathsf{Professor}$ as $\mathsf{Teacher} \sqcap \mathsf{Researcher}$, then the two concept terms from above are equivalent w.r.t. these definitions.

The concept terms of the above example are formulated in the DL $\mathcal{EL}$, which has the concept constructors conjunction ($\sqcap$), existential restriction ($\exists r.C$), and the top concept ($\top$). This DL has recently drawn considerable attention since, on the one hand, important inference problems such as the subsumption problem are polynomial in $\mathcal{EL}$ [1, 4]. On the other hand, though quite inexpressive, $\mathcal{EL}$ can be used to define biomedical ontologies. For example, the large medical ontology SNOMED CT[3] can be expressed in $\mathcal{EL}$. Unification in $\mathcal{EL}$ was first investigated in [5], where it was shown that the decision problem is NP-complete. Basically, the proof that one can check for the existence of an $\mathcal{EL}$-unifier within nondeterministic polynomial time given in [5] proceeds as follows. First, it is shown that any solvable $\mathcal{EL}$-unification problem has a *local* unifier, i.e., a unifier that is "built from atoms" of the input problem. Second, since the definition of locality implies that a local substitution can be guessed in polynomial time,

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[3] see http://www.ihtsdo.org/snomed-ct/

one can test for the existence of a local unifier within NP by guessing a local substitution and then checking whether it is indeed a unifier. In particular, this means that the results of [5] also show how to compute all local unifiers of a given $\mathcal{EL}$-unification problem. In [6] it was shown that one can employ a SAT solver to search for local $\mathcal{EL}$-unifiers.

Actually, if one takes a closer look at the concept definitions in SNOMED CT, then one sees that they do not use the top concept, i.e., SNOMED CT is not formulated in $\mathcal{EL}$, but rather in its sub-logic $\mathcal{EL}^{-\top}$, which differs from $\mathcal{EL}$ in that the use of the top concept is disallowed. If we employ $\mathcal{EL}$-unification to detect redundancies in (extensions of) SNOMED CT, then a unifier may introduce concept terms that contain the top concept, and thus propose definitions for concept names that are of a form that is not used in SNOMED CT. Apart from this practical motivation for investigating unification in $\mathcal{EL}^{-\top}$, we also found it interesting to see how such a small change in the logic influences the unification problem. Surprisingly, it turned out that the complexity of the problem increases considerably: we were able to show in [2] that deciding unifiability in $\mathcal{EL}^{-\top}$ is PSpace-complete. In [2], we restricted the attention to the decision problem, and did not address the problem of how to compute unifiers of solvable $\mathcal{EL}^{-\top}$-unification problems.

In the present paper we introduce a notion of locality for $\mathcal{EL}^{-\top}$-unifiers, and show that we can always compute a local unifier for a solvable $\mathcal{EL}^{-\top}$-unification problem. However, whereas any $\mathcal{EL}$-unification problem has only exponentially many local $\mathcal{EL}$-unifiers, each of which can be represented in polynomial space using structure sharing, a given $\mathcal{EL}^{-\top}$-unification problem can have infinitely many local $\mathcal{EL}^{-\top}$-unifiers. We show that a solvable $\mathcal{EL}^{-\top}$-unification problem always has a local $\mathcal{EL}^{-\top}$-unifier of at most exponential size, which can effectively be computed.

The Description Logics $\mathcal{EL}$ and $\mathcal{EL}^{-\top}$

Starting with a set N_C of concept names and a set N_R of role names, $\mathcal{EL}$*-concept terms* are built using the concept constructors *top-concept* ($\top$), *conjunction* ($C \sqcap D$), and *existential restriction* ($\exists r.C$ for every $r \in N_R$). The $\mathcal{EL}$-concept term C is an $\mathcal{EL}^{-\top}$*-concept term* if $\top$ does not occur in C. Since $\mathcal{EL}^{-\top}$-concept terms are special $\mathcal{EL}$-concept terms, most definitions transfer from $\mathcal{EL}$ to $\mathcal{EL}^{-\top}$, and thus we only formulate them for $\mathcal{EL}$.

The semantics of $\mathcal{EL}$ and $\mathcal{EL}^{-\top}$ is defined in the usual way, using the notion of an interpretation $\mathcal{I} = (\mathcal{D}_\mathcal{I}, \cdot^\mathcal{I})$, which consists of a nonempty domain $\mathcal{D}_\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$ that assigns binary relations on $\mathcal{D}_\mathcal{I}$ to role names and subsets of $\mathcal{D}_\mathcal{I}$ to concept terms, as shown in the semantics column of Table 1. The concept term C *is subsumed by* the concept term D (written $C \sqsubseteq D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for all interpretations $\mathcal{I}$. We say that C *is equivalent to* D (written $C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$, i.e., iff $C^\mathcal{I} = D^\mathcal{I}$ holds for all interpretations $\mathcal{I}$.

In order to define locality of unifiers in $\mathcal{EL}$, we need the notion of an atom. An $\mathcal{EL}$-concept term is called an *atom* iff it is a concept name $A \in N_C$ or an

Name	Syntax	Semantics	$\mathcal{EL}$	$\mathcal{EL}^{-\top}$
concept name	A	$A^{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}}$	×	×
role name	r	$r^{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}} \times \mathcal{D}_{\mathcal{I}}$	×	×
top-concept	$\top$	$\top^{\mathcal{I}} = \mathcal{D}_{\mathcal{I}}$	×	
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$	×	×
existential restriction	$\exists r.C$	$(\exists r.C)^{\mathcal{I}} = \{x \mid \exists y : (x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$	×	×
subsumption	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$	×	×
equivalence	$C \equiv D$	$C^{\mathcal{I}} = D^{\mathcal{I}}$	×	×

Table 1. Syntax and semantics of $\mathcal{EL}$ and $\mathcal{EL}^{-\top}$.

existential restriction $\exists r.D$. Concept names and existential restrictions $\exists r.D$, where D is a concept name or $\top$, are called *flat atoms*. The set $\mathrm{At}(C)$ of *atoms of an $\mathcal{EL}$-concept term C* consists of all the subterms of C that are atoms. For example, $C = A \sqcap \exists r.(B \sqcap \exists r.\top)$ has the atom set $\mathrm{At}(C) = \{A, \exists r.(B \sqcap \exists r.\top), B, \exists r.\top\}$. Obviously, any $\mathcal{EL}$-concept term C is a conjunction $C = C_1 \sqcap \ldots \sqcap C_n$ of atoms and $\top$. We call the atoms among $C_1, \ldots, C_n$ the *top-level atoms* of C. The $\mathcal{EL}$-concept term C is called *flat* if all its top-level atoms are flat.

The notion of a top-level atom allows for a simple *recursive characterization of subsumption* in $\mathcal{EL}$. We have $C \sqsubseteq D$ iff every top-level atom of D subsumes some top-level atom of C. In addition, the only atom subsumed by $A \in N_C$ is A itself, and all atoms subsumed by $\exists r.E$ are of the form $\exists r.E'$ with $E' \sqsubseteq E$.

In order to define locality of unifiers in $\mathcal{EL}^{-\top}$, we additionally need the notion of a particle: $\mathcal{EL}^{-\top}$-concept terms of the form $\exists r_1.\cdots\exists r_n.A$ for $n \geq 0$ role names $r_1, \ldots, r_n$ and a concept name A are called *particles* . The set $\mathrm{Part}(C)$ of all particles of a given $\mathcal{EL}^{-\top}$-concept term C is defined as

- $\mathrm{Part}(C) := \{C\}$ if C is a concept name,
- $\mathrm{Part}(C) := \{\exists r.E \mid E \in \mathrm{Part}(D)\}$ if $C = \exists r.D$,
- $\mathrm{Part}(C) := \mathrm{Part}(C_1) \cup \mathrm{Part}(C_2)$ if $C = C_1 \sqcap C_2$.

For example, the particles of $C = A \sqcap \exists r.(A \sqcap \exists r.B)$ are $A, \exists r.A, \exists r.\exists r.B$.

Unification in $\mathcal{EL}$ and $\mathcal{EL}^{-\top}$

To define unification in $\mathcal{EL}$ and $\mathcal{EL}^{-\top}$ simultaneously, let $\mathcal{L} \in \{\mathcal{EL}, \mathcal{EL}^{-\top}\}$. When defining unification in $\mathcal{L}$, we assume that the set of concept names is partitioned into a set N_v of concept variables (which may be replaced by substitutions) and a set N_c of concept constants (which must not be replaced by substitutions). An *$\mathcal{L}$-substitution* σ is a mapping from N_v into the set of all $\mathcal{L}$-concept terms. This mapping is extended to concept terms in the usual way, i.e., by replacing all occurrences of variables in the term by their σ-images. An $\mathcal{L}$-concept term is called *ground* if it contains no variables, and an $\mathcal{L}$-substitution σ is called *ground* if the concept terms $\sigma(X)$ are ground for all $X \in N_v$.

Unification tries to make concept terms equivalent by applying a substitution.

Definition 1. *An* $\mathcal{L}$-unification problem *is of the form* $\Gamma = \{C_1 \equiv^? D_1, \ldots, C_n \equiv^? D_n\}$*, where* $C_1, D_1, \ldots C_n, D_n$ *are* $\mathcal{L}$*-concept terms. The* $\mathcal{L}$*-substitution* σ *is an* $\mathcal{L}$-unifier *of* Γ *iff it solves all the equations* $C_i \equiv^? D_i$ *in* Γ*, i.e., iff* $\sigma(C_i) \equiv \sigma(D_i)$ *for* $i = 1, \ldots, n$*. In this case,* Γ *is called* $\mathcal{L}$-unifiable.

In the following, we will use the subsumption $C \sqsubseteq^? D$ as an abbreviation for the equation $C \sqcap D \equiv^? C$. Obviously, σ solves this equation iff $\sigma(C) \sqsubseteq \sigma(D)$.

Clearly, every $\mathcal{EL}^{-\top}$-unification problem Γ is also an $\mathcal{EL}$-unification problem. Whether Γ is $\mathcal{L}$-unifiable or not may depend, however, on whether $\mathcal{L} = \mathcal{EL}$ or $\mathcal{L} = \mathcal{EL}^{-\top}$. As an example, consider the problem $\Gamma := \{A \sqsubseteq^? X, B \sqsubseteq^? X\}$, where A, B are distinct concept constants and X is a concept variable. Obviously, the substitution that replaces X by $\top$ is an $\mathcal{EL}$-unifier of Γ. However, Γ does not have an $\mathcal{EL}^{-\top}$-unifier. In fact, for such a unifier σ, we would need to have $A \sqsubseteq \sigma(X)$ and $B \sqsubseteq \sigma(X)$, and it is easy to see that this is only possible if $\sigma(X) \equiv \top$.

As shown in [5], we may without loss of generality restrict our attention to ground unifiers of *flat* $\mathcal{L}$*-unification problems*, i.e., unification problems in which the left- and right-hand sides of equations are flat $\mathcal{L}$-concept terms. Given a flat $\mathcal{L}$-unification problem Γ, we denote by $\mathrm{At}(\Gamma)$ the set of all atoms of Γ, i.e., the union of all sets of atoms of the concept terms occurring in Γ. By $\mathrm{Var}(\Gamma)$ we denote the variables that occur in Γ, and by $\mathrm{NV}(\Gamma) := \mathrm{At}(\Gamma) \setminus \mathrm{Var}(\Gamma)$ the set of all *non-variable atoms* of Γ.

Local unifiers

In $\mathcal{EL}$, every solvable unification problem has a *local* $\mathcal{EL}$*-unifier*, i.e., an $\mathcal{EL}$-unifier γ such that, for every variable X, the top-level atoms of $\gamma(X)$ are of the form $\gamma(D)$ for $D \in \mathrm{NV}(\Gamma)$.

Example 1. Consider the flat $\mathcal{EL}$-unification problem Γ that consists of the three equations

$$X \equiv^? Y \sqcap A, \quad Y \sqcap \exists r.X \equiv^? \exists r.X, \quad Z \sqcap \exists r.X \equiv^? \exists r.X.$$

Then the substitutions $\sigma_0 := \{X \mapsto A, Y \mapsto \top, Z \mapsto \top\}$ and $\sigma_1 := \{X \mapsto A, Y \mapsto \top, Z \mapsto \exists r.A\}$ are the only local $\mathcal{EL}$-unifiers of Γ. In fact, we have $\mathrm{NV}(\Gamma) = \{A, \exists r.X\}$, and thus the only possible image for X in a local unifier σ is A (since $\sigma(\exists r.X) = \exists r.\sigma(X)$ obviously cannot be a conjunct of $\sigma(X)$). Since the first equation implies that $A = \sigma(X) \sqsubseteq \sigma(Y)$, we know that $\sigma(Y)$ can only be $\top$ or A. However, the second equation prevents the second possibility. Finally, the third equation ensures that $\sigma(Z)$ is $\top$ or $\exists r.A$.

Note that σ_0 and σ_1 both contain $\top$, and thus are not $\mathcal{EL}^{-\top}$-unifiers. This shows that Γ does not have an $\mathcal{EL}^{-\top}$-unifier that is local in the sense defined above. Nevertheless, Γ has an $\mathcal{EL}^{-\top}$-unifier. For example, the substitution $\gamma_1 := \{X \mapsto A \sqcap \exists r.A,\ Y \mapsto \exists r.A, Z \mapsto \exists r.\exists r.A\}$ is such a unifier. Except for the atom A, the top-level atoms of $\gamma_1(X), \gamma_1(Y), \gamma_1(Z)$ are not of the form $\gamma(D)$ for some $D \in \mathrm{NV}(\Gamma)$, but the ones different from A are all particles of $\gamma(D)$ for some $D \in \mathrm{NV}(\Gamma)$. This motivates the following definition.

Definition 2. *The $\mathcal{EL}^{-\top}$-unifier γ of Γ is a* local $\mathcal{EL}^{-\top}$-unifier *of Γ if, for every variable X, each top-level atom of $\gamma(X)$ is of the form $\gamma(D)$ for some $D \in \mathrm{NV}(\Gamma)$ or a particle of $\gamma(D)$ for some $D \in \mathrm{NV}(\Gamma)$.*

The unification problem of Example 1 can be used to demonstrate that a given $\mathcal{EL}^{-\top}$-unification problem can have infinitely many local $\mathcal{EL}^{-\top}$-unifiers. It is easy to see that the substitutions

$$\gamma_n := \{X \mapsto A \sqcap \exists r.A \sqcap \cdots \sqcap (\exists r.)^n A, Y \mapsto \exists r.A \sqcap \cdots \sqcap (\exists r.)^n A, Z \mapsto (\exists r.)^{n+1} A\}$$

are all local $\mathcal{EL}^{-\top}$-unifiers of Γ in the sense of Definition 2. Indeed, every top-level atom of $\gamma_n(X)$, $\gamma_n(Y)$, and $\gamma_n(Z)$ is either A or a particle of $\gamma_n(\exists r.X)$.

We are now ready to formulate the main result of this paper.

Theorem 1. *Given a solvable $\mathcal{EL}^{-\top}$-unification problem Γ, we can construct a local $\mathcal{EL}^{-\top}$-unifier of Γ of at most exponential size in time exponential in the size of Γ.*

We now provide a high-level description of the procedure for $\mathcal{EL}^{-\top}$-unification from [2, 3] and show how it can be adapted such that it produces a local $\mathcal{EL}^{-\top}$-unifier of size at most exponential in the size of Γ whenever there is an $\mathcal{EL}^{-\top}$-unifier.

Constructing local $\mathcal{EL}^{-\top}$-unifiers

The first step of the $\mathcal{EL}^{-\top}$-unification procedure reduces $\mathcal{EL}^{-\top}$-unifiability of Γ to solvability of a certain kind of linear language inclusions over the alphabet N_R. These inclusions are of the form $X_i \subseteq L_0 \cup L_1 X_1 \cup \cdots \cup L_n X_n$, where $X_1, \ldots, X_n$ are indeterminates, $i \in \{1, \ldots, n\}$, and each L_i ($i \in \{0, \ldots, n\}$) is a subset of $N_R \cup \{\epsilon\}$. For each variable $X \in N_v$ and each constant $A \in N_c$, there is one indeterminate X_A in these inclusions.

A *solution* θ of such an inclusion assigns sets $\theta(X_i) \subseteq N_R^*$ to the indeterminates such that $\theta(X_i) \subseteq L_0 \cup L_1\theta(X_1) \cup \cdots \cup L_n\theta(X_n)$. A solution to a set $\mathcal{I}$ of such inclusions is called *admissible* if, for every variable $X \in N_v$, there is a constant $A \in N_c$ such that $\theta(X_A)$ is nonempty. This condition will ensure that the constructed unifier of Γ is indeed an $\mathcal{EL}^{-\top}$-substitution, i.e., it does not contain $\top$. We are also only interested in *finite* solutions, i.e., solutions θ such that all the sets $\theta(X_i)$ are finite.

The problem of finding an $\mathcal{EL}^{-\top}$-unifier for Γ can be reduced to the problem of finding a finite, admissible solution to a certain set of such language inclusions. More precisely, there is a set $\mathfrak{F}_\Gamma$ of exponentially many sets $\mathcal{I}$ of language inclusions (of polynomial size) such that Γ is $\mathcal{EL}^{-\top}$-unifiable iff there is a finite, admissible solution for one $\mathcal{I} \in \mathfrak{F}_\Gamma$. This reduction uses nondeterministic polynomial time in the size of Γ since we can guess an element of $\mathfrak{F}_\Gamma$ in polynomial time.

Lemma 1. *The $\mathcal{EL}^{-\top}$-unification problem Γ has an $\mathcal{EL}^{-\top}$-unifier iff there is a set $\mathcal{I} \in \mathfrak{F}_\Gamma$ that has a finite, admissible solution.*

In this paper, we are further concerned with *local* solutions and their connection to local $\mathcal{EL}^{-\top}$-unifiers of Γ.

Definition 3. *Let $\mathcal{I}$ be a finite set of inclusions of the above form. A solution θ of $\mathcal{I}$ is called* local *if all words $w \in \theta(X) \setminus \{\epsilon\}$ for some indeterminate X occur on the right-hand side of some inclusion $X_i \subseteq L_0 \cup L_1 X_1 \cup \cdots \cup L_n X_n$ under θ, i.e., either $w \in L_0$ or $w \in (L_i \setminus \{\epsilon\})\theta(X_i)$ for some $i \in \{1, \ldots, n\}$.*

The next lemma states the close connection between the two notions of locality.

Lemma 2. *If there is a finite, local, admissible solution θ for one $\mathcal{I} \in \mathfrak{F}_\Gamma$, then one can construct a local $\mathcal{EL}^{-\top}$-unifier σ of Γ that is of size at most exponential in the size of Γ and polynomial in the size of θ.*

Example 2. One element of $\mathfrak{F}_\Gamma$ for the $\mathcal{EL}^{-\top}$-unification problem Γ from Example 1 consists of the inclusions

$$Y_A \subseteq X_A, \quad X_A \subseteq \{\epsilon\} \cup Y_A, \quad Y_A \subseteq \{r\}, \quad Z_A \subseteq \{r\}X_A.$$

For any $n \in \mathbb{N}$, the mapping $\{X_A \mapsto \{\epsilon, r, \ldots, r^n\}, Y_A \mapsto \{r, \ldots, r^n\}, Z_A \mapsto \{r^{n+1}\}\}$ is a finite, local, admissible solution of these inclusions, which corresponds to the local $\mathcal{EL}^{-\top}$-unifier γ_n of Γ (see Example 1).

This illustrates that there may be infinitely many such solutions for a given $\mathcal{I} \in \mathfrak{F}_\Gamma$. However, there always is one of size at most exponential in the size of Γ if there is one at all. To show this, we consider the remaining part of the $\mathcal{EL}^{-\top}$-unification algorithm. There we use the computational model of *alternating finite automata with ϵ-transitions (ϵ-AFA)*, which are a special case of two-way alternating finite automata. In order to decide the existence of a finite, admissible solution of $\mathcal{I}$, for each variable X_A an ϵ-AFA $\mathcal{A}(X, A)$ is constructed that has the following property.

Lemma 3. *The language accepted by $\mathcal{A}(X, A)$ is non-empty iff there is a finite solution θ of $\mathcal{I}$ such that $\theta(X_A) \neq \emptyset$.*

The emptiness test for such automata is a PSPACE-complete task [8]. Furthermore, if the language accepted by $\mathcal{A}(X, A)$ is non-empty, then one can construct a run of this automaton of size at most exponential in the size of Γ. This run can then be translated into a finite solution of $\mathcal{I}$ with the property that $\theta(X_A) \neq \emptyset$. Using a weak condition on the structure of runs of $\mathcal{A}(X, A)$, we can even construct a finite, local solution of $\mathcal{I}$ with this property.

Lemma 4. *If the language accepted by $\mathcal{A}(X, A)$ is non-empty, then one can construct a finite, local solution θ of $\mathcal{I}$ with $\theta(X_A) \neq \emptyset$.*

The set of all solutions of $\mathcal{I}$ is closed under point-wise union, i.e., if θ_1 and θ_2 are solutions of $\mathcal{I}$, then $\theta_1 \cup \theta_2$ is also one, where $(\theta_1 \cup \theta_2)(X) := \theta_1(X) \cup \theta_2(X)$ for each indeterminate X of $\mathcal{I}$. Thus, $\mathcal{I}$ has a finite, admissible solution iff for each $X \in N_v$ there is a constant $A \in N_c$ such that $\mathcal{A}(X, A)$ accepts a non-empty language. Since the union of local solutions is again local, it is possible to construct a finite, local, admissible solution of $\mathcal{I}$ in exponential time in the size of Γ if there exists a finite, admissible solution of $\mathcal{I}$.

To summarize, assume that Γ is unifiable. Then we enumerate all elements $\mathcal{I}$ of $\mathfrak{F}_\Gamma$ and check whether they have a finite, admissible solution. By Lemma 1, at least one of them must have such a solution. Lemmata 3 and 4 show that one can construct a finite, local, admissible solution θ of $\mathcal{I}$ that is of size at most exponential in the size of Γ. Using Lemma 2, we can then construct a local $\mathcal{EL}^{-\top}$-unifier of Γ that is of size at most exponential in the size of Γ.

It is shown in [3] that this exponential bound is optimal, i.e., there is a sequence Γ_n of solvable $\mathcal{EL}^{-\top}$-unification problems of size polynomial in n such that any local $\mathcal{EL}^{-\top}$-unifier of Γ_n has size at least exponential in n.

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