PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 6 Answer-Set Programming Motivation and Introduction

*slides adapted from Torsten Schaub [Gebser et al.(2012)]

Sarah Gaggl

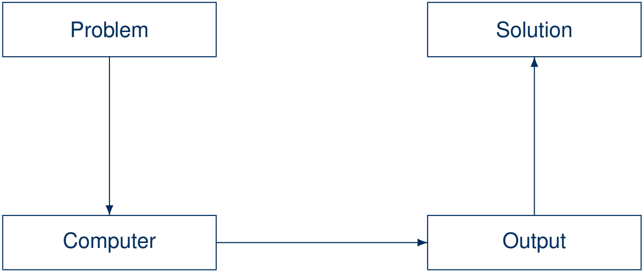
Dresden, 13th May 2019

Agenda

1. Introduction
2. Constraint Satisfaction (CSP)
3. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
4. Local Search, Stochastic Hill Climbing, Simulated Annealing
5. Tabu Search
6. Answer-set Programming (ASP)
7. Evolutionary Algorithms/ Genetic Algorithms
8. Structural Decomposition Techniques (Tree/Hypertree Decompositions)

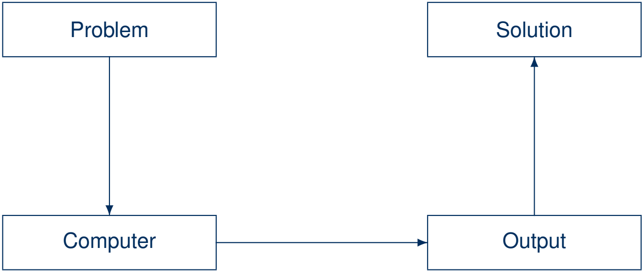
Outline

Informatics



Informatics

"What is the problem?" versus "How to solve the problem?"



Traditional programming

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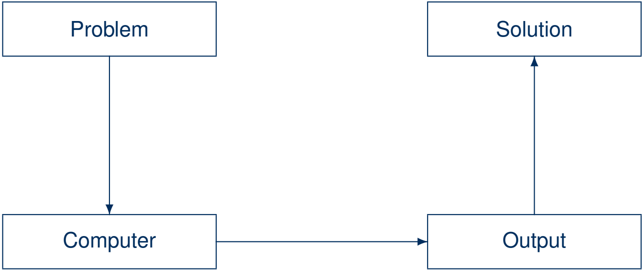

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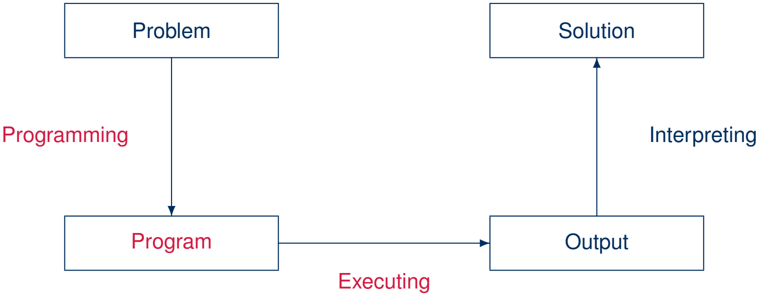

Declarative problem solving

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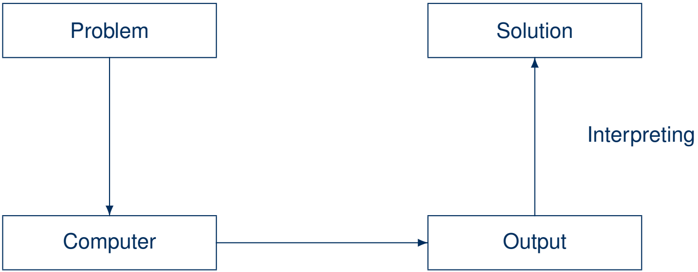

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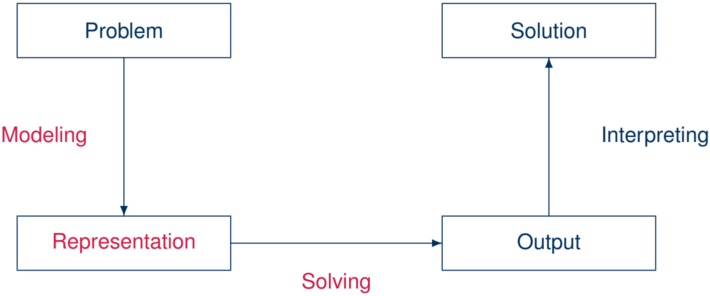

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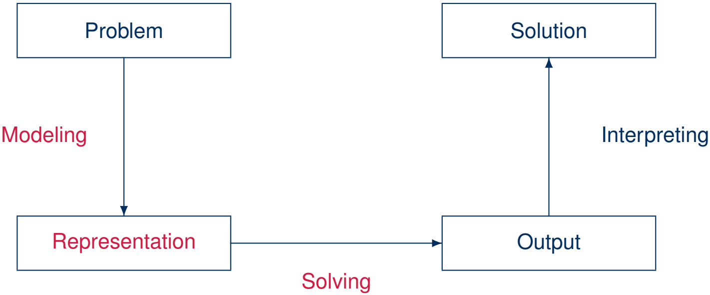

Answer Set Programming

in a Nutshell

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  - a rich yet simple modeling language
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  - (logic-based) knowledge representation and (nonmonotonic) reasoning
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- ASP embraces many emerging application areas

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ASP = DB+LP+KR+SAT

KR's shift of paradigm

Theorem Proving based approach (eg. Prolog)

1. Provide a representation of the problem
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Model Generation based Problem Solving

Representation	Solution
constraint satisfaction problem	assignment
propositional horn theories	smallest model
propositional theories	models
propositional theories	minimal models
propositional theories	stable models
propositional programs	minimal models
propositional programs	supported models
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first-order theories	models
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first-order theories	Herbrand models
auto-epistemic theories	expansions
default theories	extensions
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LP-style playing with blocks

Prolog program

```
on(a,b).
on(b,c).

above(X,Y) :- on(X,Y).
above(X,Y) :- on(X,Z), above(Z,Y).
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Shuffled Prolog program

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Prolog queries (answered via fixed execution)

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?- above(a,c).

Fatal Error: local stack overflow.
```

SAT-style playing with blocks

Formula

$$\begin{array}{ll} & on(a,b) \\ \wedge & on(b,c) \\ \wedge & (on(X,Y) \rightarrow above(X,Y)) \\ \wedge & (on(X,Z) \wedge above(Z,Y) \rightarrow above(X,Y)) \end{array}$$

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Herbrand model

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➥ Answer Set Programming (ASP)

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Answer Set Programming commonly

Representation	Solution
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Answer Set Programming in practice

Representation	Solution
propositional programs	stable models
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Answer Set Programming in practice

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Stable Herbrand model

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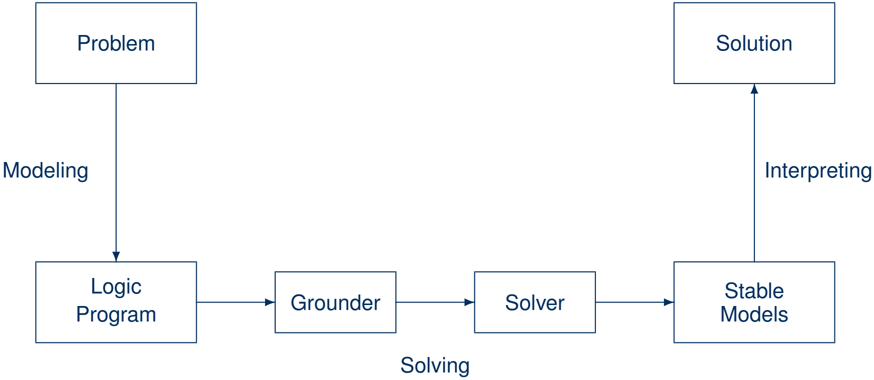
ASP versus LP

ASP	Prolog
Model generation	Query orientation
Bottom-up	Top-down
Modeling language	Programming language
Rule-based format	
Instantiation Flat terms	Unification Nested terms
(Turing +) $NP^{(NP)}$	Turing

ASP versus SAT

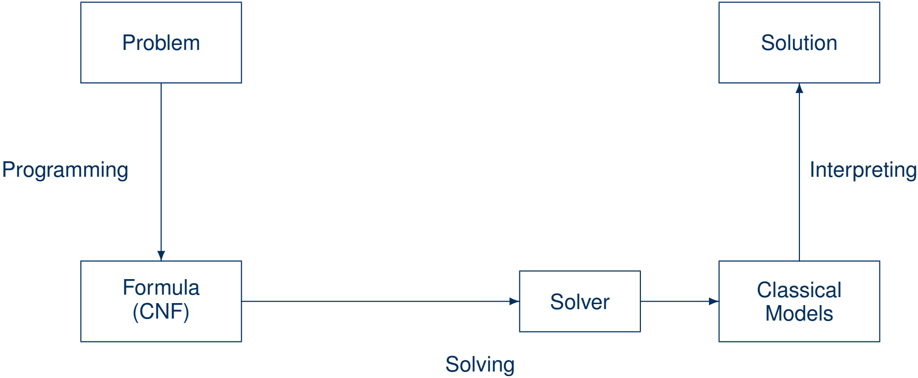
ASP	SAT
Model generation	
Bottom-up	
Constructive Logic	Classical Logic
Closed (and open) world reasoning	Open world reasoning
Modeling language	—
Complex reasoning modes	Satisfiability testing
Satisfiability	Satisfiability
Enumeration/Projection	—
Optimization	—
Intersection/Union	—
(Turing +) $NP^{(NP)}$	NP

ASP solving



Problem
Solution
Modeling
Interpreting
Logic Program
Grounder
Solver
Stable Models
Solving

SAT solving


Problem
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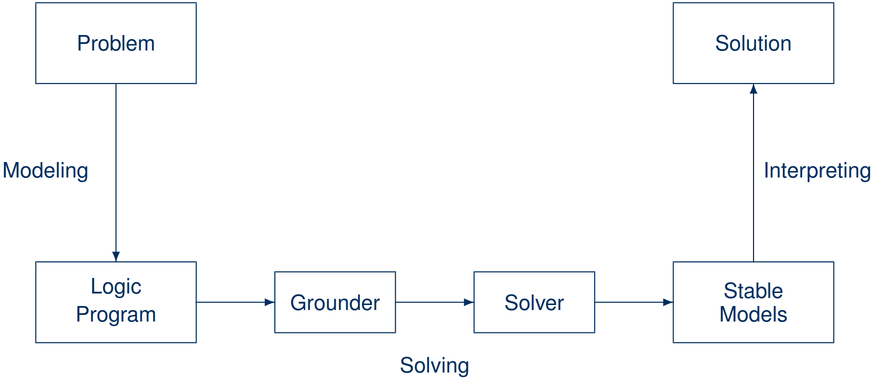
Rooting ASP solving



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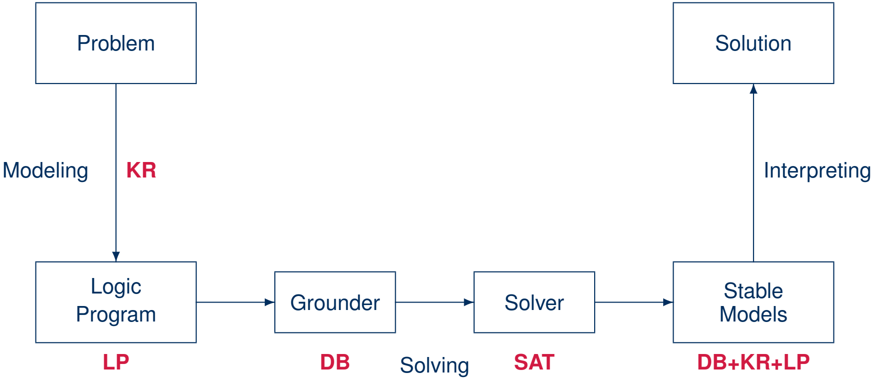

Two sides of a coin

- ASP as High-level Language
  - Express problem instance(s) as sets of facts
  - Encode problem (class) as a set of rules
  - Read off solutions from stable models of facts and rules

- ASP as Low-level Language
  - Compile a problem into a logic program
  - Solve the original problem by solving its compilation

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  - Automated Planning
  - Code Optimization
  - Composition of Renaissance Music
  - Database Integration
  - Decision Support for NASA shuttle controllers
  - Model Checking
  - Product Configuration
  - Robotics
  - System Biology
  - System Synthesis
  - (industrial) Team-building
  - and many many more

What does ASP offer?

- Integration of DB, KR, and SAT techniques
- Succinct, elaboration-tolerant problem representations
  - Rapid application development tool
- Easy handling of dynamic, knowledge intensive applications
  - including: data, frame axioms, exceptions, defaults, closures, etc

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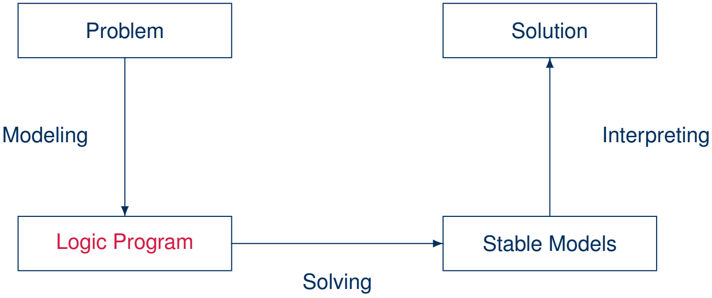
Problem solving in ASP: Syntax



Normal logic programs

- A (normal) logic program over a set $\mathcal{A}$ of atoms is a finite set of rules
- A (normal) rule, r, is of the form

$$a_0 \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n$$

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- Notation

$$\begin{array}{rcl} head(r) & = & a_0 \\ body(r) & = & \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\} \\ body(r)^+ & = & \{a_1, \ldots, a_m\} \\ body(r)^- & = & \{a_{m+1}, \ldots, a_n\} \end{array}$$

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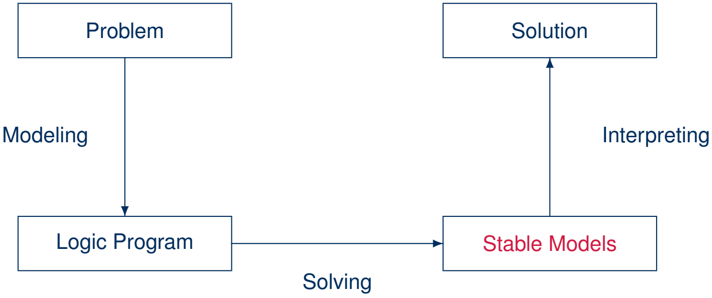
- A program is called positive if $body(r)^- = \emptyset$ for all its rules

Rough notational convention

We sometimes use the following notation interchangeably
in order to stress the respective view:

	true, false	if	and	or	iff	default negation	classical negation
source code		`:-`	`,`	`|`		`not`	`-`
logic program		$\leftarrow$	$,$	$;$		*not*	$\neg$
formula	$\bot, \top$	$\rightarrow$	$\wedge$	$\vee$	$\leftrightarrow$	$\sim$	$\neg$

Problem solving in ASP: Semantics



Formal Definition

Stable models of positive programs

- A set of atoms X is closed under a positive program P iff for any $r \in P$, $head(r) \in X$ whenever $body(r)^+ \subseteq X$
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- The smallest set of atoms which is closed under a positive program P is denoted by $Cn(P)$
  - $Cn(P)$ corresponds to the $\subseteq$-smallest model of P (ditto)
- The set $Cn(P)$ of atoms is the stable model of a positive program P

Some “logical” remarks

- Positive rules are also referred to as definite clauses
  - Definite clauses are disjunctions with exactly one positive atom:

$$a_0 \vee \neg a_1 \vee \cdots \vee \neg a_m$$

  - A set of definite clauses has a (unique) smallest model

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- Horn clauses are clauses with at most one positive atom
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  - Non-definite Horn clauses can be regarded as integrity constraints
  - A set of Horn clauses has a smallest model or none

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- This smallest model is the intended semantics of such sets of clauses
  - Given a positive program P, $Cn(P)$ corresponds to the smallest model of the set of definite clauses corresponding to P

Basic idea

Consider the logical formula Φ and its three (classical) models:

$$\{p,q\}, \{q,r\}, \text{ and } \{p,q,r\}$$

Φ $\boxed{q \;\wedge\; (q \wedge \neg r \rightarrow p)}$

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$$\boxed{\begin{array}{ccc} p & \mapsto & 1 \\ q & \mapsto & 1 \\ r & \mapsto & 0 \end{array}}$$

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Φ $\boxed{q \;\wedge\; (q \wedge \neg r \rightarrow p)}$

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Informally, a set X of atoms is a stable model of a logic program P

- if X is a (classical) model of P and
- if all atoms in X are justified by some rule in P

(rooted in intuitionistic logics HT (Heyting, 1930) and G3 (Gödel, 1932))

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Formal Definition

Stable model of normal programs

- The Gelfond-Lifschitz Reduct[Gelfond and Lifschitz(1991)], P^X, of a program P relative to a set X of atoms is defined by

$$P^X = \{head(r) \leftarrow body(r)^+ \mid r \in P \text{ and } body(r)^- \cap X = \emptyset\}$$

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- Note: $Cn(P^X)$ is the $\subseteq$–smallest (classical) model of P^X
- Note: Every atom in X is justified by an “applying rule from P”

A closer look at P^X

- In other words, given a set X of atoms from P,

  P^X is obtained from P by deleting
  1. each rule having $not\ a$ in its body with $a \in X$
     and then
  2. all negative atoms of the form $not\ a$
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- Note: Only negative body literals are evaluated w.r.t. X

A first example

$$P = \{p \leftarrow p,\ q \leftarrow \mathit{not}\ p\}$$

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X		$Cn(P^X)$
$\emptyset$		
$\{p\}$		
$\{q\}$		
$\{p, q\}$		

A first example

$$P = \{p \leftarrow p,\ q \leftarrow not\ p\}$$

X	P^X	$Cn(P^X)$
$\emptyset$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$
$\{p\}$	$p \leftarrow p$	$\emptyset$
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Some properties

- A logic program may have zero, one, or multiple stable models!

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- A logic program may have zero, one, or multiple stable models!
- If X is an stable model of a logic program P,
  then X is a model of P (seen as a formula)
- If X and Y are stable models of a normal program P,
  then $X \not\subset Y$

Programs with Variables

Let P be a logic program

- Let $\mathcal{T}$ be a set of (variable-free) terms
- Let $\mathcal{A}$ be a set of (variable-free) atoms constructable from $\mathcal{T}$

Programs with Variables

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- Let $\mathcal{T}$ be a set of variable-free terms (also called Herbrand universe)
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Programs with Variables

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- Let $\mathcal{A}$ be a set of (variable-free) atoms constructable from $\mathcal{T}$

- Ground Instances of $r \in P$: Set of variable-free rules obtained by replacing all variables in r by elements from $\mathcal{T}$:

$$ground(r) = \{r\theta \mid \theta : var(r) \to \mathcal{T}, var(r\theta) = \emptyset\}$$

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  θ is a (ground) substitution

- Ground Instantiation of P: $ground(P) = \bigcup_{r \in P} ground(r)$

An example

$$P = \{\ r(a,b) \leftarrow,\ r(b,c) \leftarrow,\ t(X,Y) \leftarrow r(X,Y)\ \}$$

$$\mathcal{T} = \{a,b,c\}$$

$$\mathcal{A} = \left\{ \begin{array}{c} r(a,a), r(a,b), r(a,c), r(b,a), r(b,b), r(b,c), r(c,a), r(c,b), r(c,c), \\ t(a,a), t(a,b), t(a,c), t(b,a), t(b,b), t(b,c), t(c,a), t(c,b), t(c,c) \end{array} \right\}$$

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- Intelligent Grounding aims at reducing the ground instantiation

Stable models of programs with Variables

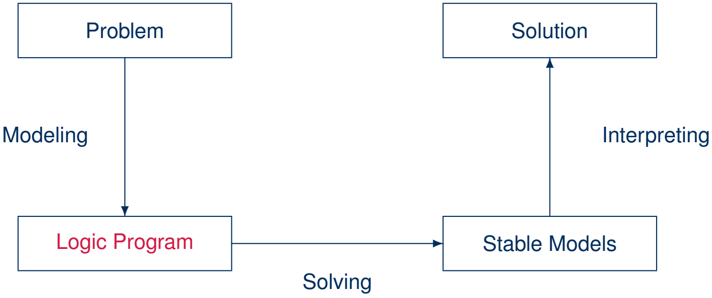
Let P be a normal logic program with variables

Stable models of programs with Variables

Let P be a normal logic program with variables

- A set X of (ground) atoms is a stable model of P,
  if $Cn(ground(P)^X) = X$

Problem solving in ASP: Extended Syntax



Language Constructs

Language Constructs

- Variables (over the Herbrand Universe)
  - `p(X) :- q(X)` over constants $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ stands for
    `p(a) :- q(a), p(b) :- q(b), p(c) :- q(c)`

Language Constructs

- Conditional Literals
  - `p :- q(X) : r(X)` given `r(a), r(b), r(c)` stands for
    `p :- q(a), q(b), q(c)`

Language Constructs

- Disjunction
  - `p(X) | q(X) :- r(X)`

Language Constructs

- Integrity Constraints
  - `:- q(X), p(X)`

Language Constructs

- Choice
  - `2 { p(X,Y) : q(X) } 7 :- r(Y)`

Language Constructs

- Aggregates
  - `s(Y) :- r(Y), 2 #count { p(X,Y) : q(X) } 7`
  - also: `#sum`, `#avg`, `#min`, `#max`, `#even`, `#odd`

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Modeling

- For solving a problem class **C** for a problem instance **I**, encode
  1. the problem instance **I** as a set $P_{\mathbf{I}}$ of facts and
  2. the problem class **C** as a set $P_{\mathbf{C}}$ of rules

  such that the solutions to **C** for **I** can be (polynomially) extracted from the stable models of $P_{\mathbf{I}} \cup P_{\mathbf{C}}$

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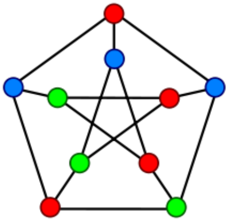
- An encoding $P_{\mathbf{C}}$ is uniform, if it can be used to solve all its problem instances
  That is, $P_{\mathbf{C}}$ encodes the solutions to **C** for any set $P_{\mathbf{I}}$ of facts

Example 3-Colorability



- Vertices are represented with predicates $\mathrm{node}(X)$;
- Edges are represented with predicates $\mathrm{edge}(X, Y)$.

Question: Is there a valid assignment of three colors for an input graph G such that no two adjacent vertices have the same color?

Graph coloring

```
node(1..6).
```

Graph coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).
```

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edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).
```

Graph coloring

```
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edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
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Problem instance

Graph coloring

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edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).


1 { color(X,C) : col(C) } 1 :- node(X).
```

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:- edge(X,Y), color(X,C), color(Y,C).
```

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Problem instance

```
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Problem encoding

color.lp

```
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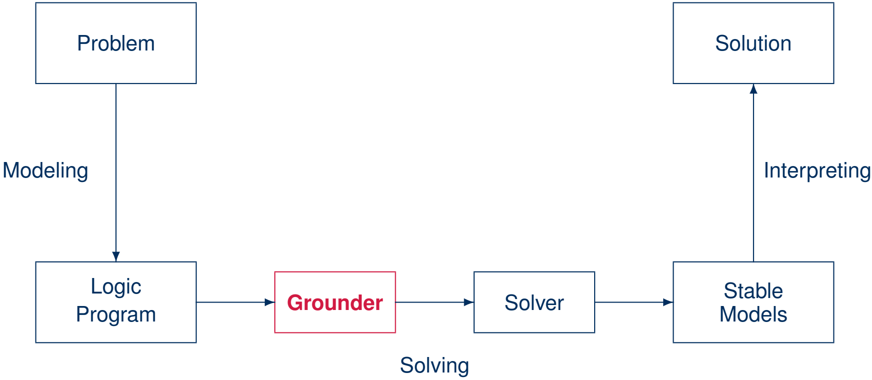
ASP solving process



Graph coloring: Grounding

```
$ gringo --text color.lp
```

Graph coloring: Grounding

```
$ gringo --text color.lp

node(1).  node(2).  node(3).  node(4).  node(5).  node(6).

edge(1,2).  edge(1,3).  edge(1,4).  edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).  edge(4,1).  edge(4,2).  edge(5,3).
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col(r).  col(b).  col(g).

1 {color(1,r), color(1,b), color(1,g)} 1.
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1 {color(4,r), color(4,b), color(4,g)} 1.
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1 {color(6,r), color(6,b), color(6,g)} 1.

 :- color(1,r), color(2,r).  :- color(2,g), color(5,g).  ...  :- color(6,r), color(2,r).
 :- color(1,b), color(2,b).  :- color(2,r), color(6,r).       :- color(6,b), color(2,b).
 :- color(1,g), color(2,g).  :- color(2,b), color(6,b).       :- color(6,g), color(2,g).
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```
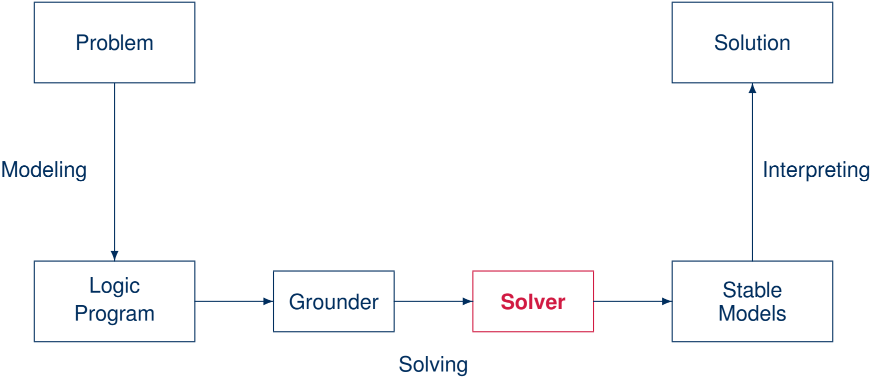

ASP solving process



Graph coloring: Solving

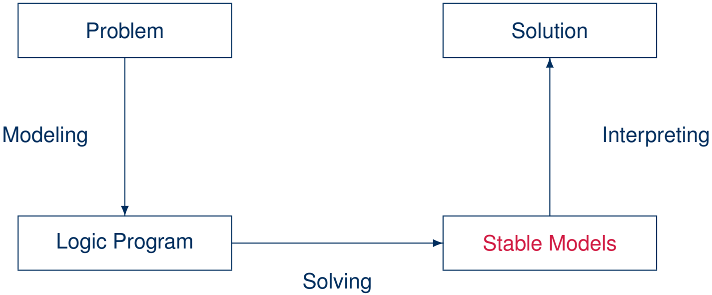
```
$ gringo color.lp | clasp 0
```

Graph coloring: Solving

```
$ gringo color.lp | clasp 0
```

```
clasp version 2.1.0
Reading from stdin
Solving...
Answer: 1
edge(1,2) ... col(r) ... node(1) ... color(6,b) color(5,g) color(4,b) color(3,r) color(2,r
Answer: 2
edge(1,2) ... col(r) ... node(1) ... color(6,r) color(5,g) color(4,r) color(3,b) color(2,b
Answer: 3
edge(1,2) ... col(r) ... node(1) ... color(6,g) color(5,b) color(4,g) color(3,r) color(2,r
Answer: 4
edge(1,2) ... col(r) ... node(1) ... color(6,r) color(5,b) color(4,r) color(3,g) color(2,g
Answer: 5
edge(1,2) ... col(r) ... node(1) ... color(6,g) color(5,r) color(4,g) color(3,b) color(2,b
Answer: 6
edge(1,2) ... col(r) ... node(1) ... color(6,b) color(5,r) color(4,b) color(3,g) color(2,g
SATISFIABLE

Models      : 6
Time        : 0.002s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
```

Problem solving in ASP: Reasoning Modes



Reasoning Modes

- Satisfiability
- Enumeration[†]
- Projection[†]
- Intersection[‡]
- Union[‡]
- Optimization

- and combinations of them

[†] without solution recording

[‡] without solution enumeration

References

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- See also: `http://potassco.sourceforge.net`