TECHNISCHE UNIVERSITÄT DRESDEN

PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 6 ASP II *slides adapted from Torsten Schaub [Gebser et al.(2012)]

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Dresden

Agenda

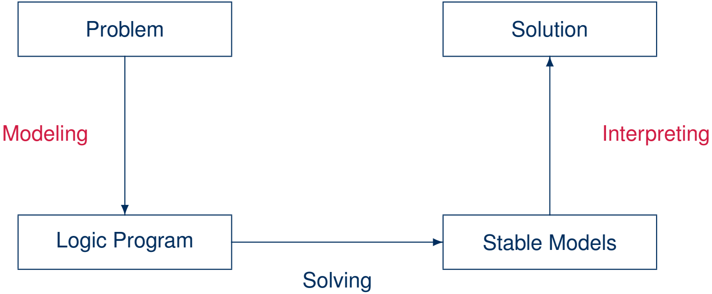
Overview ASP II

- Modeling
  1. Basic Modeling
  2. Methodology
- Language
  3. Motivation
  4. Core language

Modeling: Overview

Outline

Modeling and Interpreting



Modeling

- For solving a problem class **C** for a problem instance **I**, encode
  1. the problem instance **I** as a set $P_{\mathbf{I}}$ of facts and
  2. the problem class **C** as a set $P_{\mathbf{C}}$ of rules

  such that the solutions to **C** for **I** can be (polynomially) extracted from the stable models of $P_{\mathbf{I}} \cup P_{\mathbf{C}}$

- $P_{\mathbf{I}}$ is (still) called problem instance
- $P_{\mathbf{C}}$ is often called the problem encoding

- An encoding $P_{\mathbf{C}}$ is uniform, if it can be used to solve all its problem instances
  That is, $P_{\mathbf{C}}$ encodes the solutions to **C** for any set $P_{\mathbf{I}}$ of facts

Outline

Basic methodology

Methodology

Generate and Test (or: Guess and Check)

Generator: Generate potential stable model candidates (typically through non-deterministic constructs)

Tester: Eliminate invalid candidates (typically through integrity constraints)

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Methodology

Generate and Test (or: Guess and Check)

Generator	Generate potential stable model candidates (typically through non-deterministic constructs)
Tester	Eliminate invalid candidates (typically through integrity constraints)

Nutshell

Logic program = Data + Generator + Tester (+ Optimizer)

Outline

Satisfiability testing

- Problem Instance: A propositional formula ϕ in CNF
- Problem Class: Is there an assignment of propositional variables to true and false such that a given formula ϕ is true

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$$(a \vee \neg b) \wedge (\neg a \vee b)$$

- Logic Program:

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- Logic Program:

Generator	Tester	Stable models
$\{a, b\} \leftarrow$	$\leftarrow \quad \mathit{not}\ a, b$	$X_1 = \{a, b\}$
	$\leftarrow \quad a, \mathit{not}\ b$	$X_2 = \{\}$

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- Example: Consider formula

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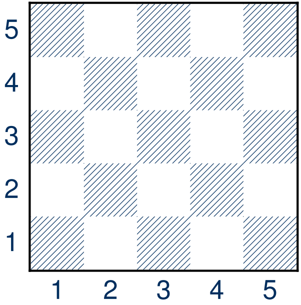
- Logic Program:

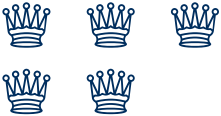
Generator		**Tester**		**Stable models**		
$\{a, b\}$	$\leftarrow$	$\leftarrow$	$not\ a, b$	X_1	=	$\{a, b\}$
		$\leftarrow$	$a, not\ b$	X_2	=	$\{\}$

Outline

The n-Queens Problem



- Place n queens on an $n \times n$ chess board
- Queens must not attack one another



Defining the Field

queens.lp

```
row(1..n).
col(1..n).
```

- Create file `queens.lp`
- Define the field
  - n rows
  - n columns

Defining the Field

Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
SATISFIABLE

Models       : 1
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Placing n Queens

queens.lp

```
row(1..n).
col(1..n).
n { queen(I,J) : row(I), col(J) }n.
```

- Guess a solution candidate
  by placing n queens on the board

Placing n Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
```

- Alternative formulation using integrity constraints.

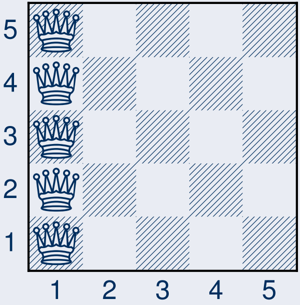
Placing n Queens

Running . . .

```
$ gringo queens.lp --const n=5 | clasp 2
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,1) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(1,2) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
...
```

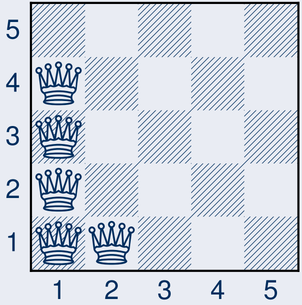
Placing n Queens: Answer 1

Answer 1



Placing n Queens: Answer 2

Answer 2



Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
n{ queen(I,J) : row(I), col(J) }n.
:- queen(I,J), queen(I,J'), J != J'.
```

- Forbid horizontal attacks

Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
n{ queen(I,J) : row(I), col(J) }n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
```

- Forbid horizontal attacks
- Forbid vertical attacks

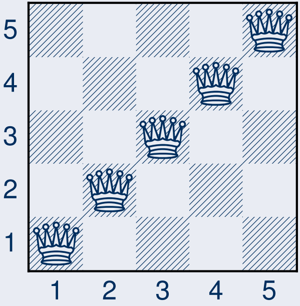
Horizontal and Vertical Attack

Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,5) queen(4,4) queen(3,3) \
queen(2,2) queen(1,1)
...
```

Horizontal and Vertical Attack: Answer 1

Answer 1



Diagonal Attack

queens.lp

```
row(1..n).
col(1..n).
n { queen(I,J) : row(I), col(J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I-J == I'-J'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I+J == I'+J'.
```

- Forbid diagonal attacks

Diagonal Attack

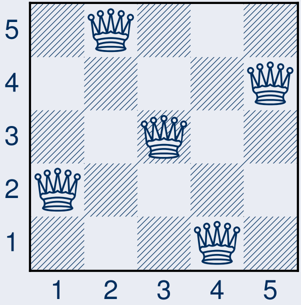
Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(4,5) queen(1,4) queen(3,3) queen(5,2) queen(2,1)
SATISFIABLE

Models      : 1+
Time        : 0.000
  Prepare   : 0.000
  Prepro.   : 0.000
  Solving   : 0.000
```

Diagonal Attack: Answer 1

Answer 1



Optimizing

queens-opt.lp

```
1 { queen(I,1..n) } 1 :- I = 1..n.
1 { queen(1..n,J) } 1 :- J = 1..n.
 :- 2 { queen(D-J,J) }, D =   2..2*n.
 :- 2 { queen(D+J,J) }, D = 1-n..n-1.
```

- Encoding can be optimized
- Much faster to solve

And sometimes it rocks

```
$ clingo -c n=5000 queens-opt-diag.lp -config=jumpy -q -stats=3
clingo version 4.1.0
Solving...
SATISFIABLE

Models       : 1+
Time         : 3758.143s (Solving: 1905.22s 1st Model: 1896.20s Unsat: 0.00s)
CPU Time     : 3758.320s

Choices      : 288594554
Conflicts    : 3442   (Analyzed: 3442)
Restarts     : 17     (Average: 202.47 Last: 3442)
Model-Level  : 7594728.0
Problems     : 1      (Average Length: 0.00 Splits: 0)
Lemmas       : 3442   (Deleted: 0)
  Binary     : 0      (Ratio:   0.00%)
  Ternary    : 0      (Ratio:   0.00%)
  Conflict   : 3442   (Average Length: 229056.5 Ratio: 100.00%)
  Loop       : 0      (Average Length:    0.0 Ratio:   0.00%)
  Other      : 0      (Average Length:    0.0 Ratio:   0.00%)

Atoms        : 75084857 (Original: 75069989 Auxiliary: 14868)
Rules        : 100129956 (1: 50059992/100090100 2: 39990/29856 3: 10000/10000)
Bodies       : 25090103
Equivalences : 125029999 (Atom=Atom: 50009999 Body=Body: 0 Other: 75020000)
Tight        : Yes
Variables    : 25024868 (Eliminated: 11781 Frozen: 25000000)
Constraints  : 66664  (Binary: 35.6% Ternary:  0.0% Other: 64.4%)

Backjumps    : 3442   (Average: 681.19 Max: 169512 Sum: 2344658)
  Executed   : 3442   (Average: 681.19 Max: 169512 Sum: 2344658 Ratio: 100.00%)
  Bounded    : 0      (Average:  0.00 Max:    0 Sum:       0 Ratio:   0.00%)
```

Outline

Traveling Salesperson

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).
```

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).

cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
cost(3,1,3).  cost(3,4,2).  cost(3,5,2).
cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).
```

Traveling Salesperson

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cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
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cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).

edge(X,Y) :- cost(X,Y,_).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).

#minimize { C,X,Y : cycle(X,Y), cost(X,Y,C) }.
```

Language: Overview

Outline

Basic language constructs

- A language can be enhanced by introducing new constructs
- To this end, we must address the following issues:
  - What is the syntax of the new language construct?
  - What is the semantics of the new language construct?
  - How to implement the new language construct?

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- A way of providing semantics is to furnish a translation removing the new constructs, eg. classical negation
- This translation might also be used for implementing the language constructs.

Outline

Outline

3 Motivation

4 Core language
- Integrity constraint
- Choice rule
- Cardinality rule
- Weight rule

Integrity constraint

- Idea Eliminate unwanted solution candidates
- Syntax An integrity constraint is of the form

$$\leftarrow a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$
- Example `:- edge(3,7), color(3,red), color(7,red).`

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- Example `:- edge(3,7), color(3,red), color(7,red).`
- Embedding The above integrity constraint can be turned into the normal rule

$$x \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n, not\ x$$

  where x is a new symbol, that is, $x \notin \mathcal{A}$.

Outline

Choice rule

- Idea Choices over subsets
- Syntax A choice rule is of the form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o$$

where $0 \leq m \leq n \leq o$ and each a_i is an atom for $1 \leq i \leq o$

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- Example

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{ buy(pizza); buy(wine); buy(corn) } :- at(grocery).
```

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```
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```
- Another Example $P = \{\{a\} \leftarrow b,\ b \leftarrow\}$ has two stable models: $\{b\}$ and $\{a, b\}$

Embedding in normal rules

- A choice rule of form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o$$

can be translated into $2m + 1$ normal rules

$$\begin{array}{lllllll} b & \leftarrow & a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o & & & & \\ a_1 & \leftarrow & b, not\ a'_1 & \ldots & a_m & \leftarrow & b, not\ a'_m \\ a'_1 & \leftarrow & not\ a_1 & \ldots & a'_m & \leftarrow & not\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \ldots, a'_m$.

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Outline

Cardinality rule

- Idea Control (lower) cardinality of subsets
- Syntax A cardinality rule is the form

$$a_0 \leftarrow l \{ a_1, \ldots, a_m, \textit{not } a_{m+1}, \ldots, \textit{not } a_n \}$$

where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
l is a non-negative integer.

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- Note l acts as a lower bound on the body

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```
pass(c42) :- 2 { pass(a1); pass(a2); pass(a3) }.
```

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- Example
  `pass(c42) :- 2 { pass(a1); pass(a2); pass(a3) }.`
- Another Example $P = \{a \leftarrow 1\{b, c\},\ b \leftarrow\}$ has stable model $\{a, b\}$

Embedding in normal rules

- Replace each cardinality rule

  $$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

  by $a_0 \leftarrow ctr(1, l)$

  where atom $ctr(i, j)$ represents the fact that at least j of the literals having an equal or greater index than i, are in a stable model

Embedding in normal rules

- Replace each cardinality rule

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where atom $ctr(i, j)$ represents the fact that at least j of the literals having an equal or greater index than i, are in a stable model

- The definition of $ctr/2$ is given for $0 \leq k \leq l$ by the rules

$$\begin{array}{rcll} ctr(i, k{+}1) & \leftarrow & ctr(i+1, k), a_i & \\ ctr(i, k) & \leftarrow & ctr(i+1, k) & \text{for } 1 \leq i \leq m \\ ctr(j, k{+}1) & \leftarrow & ctr(j+1, k), not\ a_j & \\ ctr(j, k) & \leftarrow & ctr(j+1, k) & \text{for } m+1 \leq j \leq n \\ ctr(n+1, 0) & \leftarrow & & \end{array}$$

Embedding in normal rules

- Replace each cardinality rule

$$a_0 \leftarrow l \, \{ \, a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n \, \}$$

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Embedding in normal rules

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Embedding in normal rules

- Replace each cardinality rule

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

  by $a_0 \leftarrow ctr(1, l)$

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An example

- Program $\{a \leftarrow,\ c \leftarrow 1\ \{a, b\}\}$ has the stable model $\{a, c\}$

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$$a \leftarrow$$

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\end{array}
\qquad
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. . . and vice versa

- A normal rule
$$a_0 \leftarrow a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n$$
can be represented by the cardinality rule
$$a_0 \leftarrow n\ \{a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n\}$$

Cardinality rules with upper bounds

- A rule of the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \; u \qquad (1)$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
  l and u are non-negative integers

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stands for

$$\begin{array}{rcl} a_0 & \leftarrow & b, not \; c \\ b & \leftarrow & l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \} \\ c & \leftarrow & u{+}1 \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \} \end{array}$$

where b and c are new symbols

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- Note The single constraint in the body of the cardinality rule (1) is referred to as a cardinality constraint

Cardinality constraints

- Syntax A cardinality constraint is of the form

  $$l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \; u$$

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- In other words, if

$$l \leq | \, (\{a_1, \ldots, a_m\} \cap X) \cup (\{a_{m+1}, \ldots, a_n\} \setminus X) \, | \leq u$$

Cardinality constraints as heads

- A rule of the form

  $$l\ \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, not\ a_{o+1}, \ldots, not\ a_p$$

  where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $1 \leq i \leq p$;
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- Example `1{ color(v42,red); color(v42,green); color(v42,blue) }1.`

Outline

Weight rule

- Syntax A weight rule is the form

$$a_0 \leftarrow l \, \{ \, w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : \mathit{not}\ a_{m+1}, \ldots, w_n : \mathit{not}\ a_n \, \}$$

  where $0 \leq m \leq n$ and each a_i is an atom;
  l and w_i are integers for $1 \leq i \leq n$
- A weighted literal $w_i : \ell_i$ associates each literal ℓ_i with a weight w_i

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- A weighted literal $w_i : \ell_i$ associates each literal ℓ_i with a weight w_i
- Note A cardinality rule is a weight rule where $w_i = 1$ for $0 \leq i \leq n$

Weight constraints

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$$l \; \{ \; w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : \mathit{not} \; a_{m+1}, \ldots, w_n : \mathit{not} \; a_n \; \} \; u$$

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$$l \leq \left(\sum_{1 \leq i \leq m, a_i \in X} w_i + \sum_{m < i \leq n, a_i \notin X} w_i \right) \leq u$$

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- Example

```
10 { 4:course(db); 6:course(ai); 8:course(project); 3:course(xml) } 20
```

References

Martin Gebser, Benjamin Kaufmann Roland Kaminski, and Torsten Schaub.
Answer Set Solving in Practice.
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doi=10.2200/S00457ED1V01Y201211AIM019.

- See also: `http://potassco.sourceforge.net`