Exercise 4.1. Show that $\mathsf{P}$ is closed under star.

Exercise 4.2. Show that the word-problem of nondeterministic polynomial-time Turing machines is $\mathsf{NP}$-complete.

Exercise 4.3. Consider the problem $\text{CLIQUE}$:

Input: An undirected graph $G$ and some $k \in \mathbb{N}$

Question: Does there exist a clique in $G$ of size at least $k$?

Suppose $\text{CLIQUE}$ can be solved in time $T(n)$ for some $T : \mathbb{N} \rightarrow \mathbb{N}$ with $T(n) \geq n$ for all $n \in \mathbb{N}$. Show that then the optimization problem

Input: An undirected graph $G$

Compute: A clique in $G$ of maximal size

can be computed in time $O(n \cdot T(n))$.

(You can assume that $T$ is monotone.)

Exercise 4.4. Show that the following problem is $\mathsf{NP}$-complete:

Input: A propositional formula $\phi$ in CNF

Question: Does $\phi$ have at least 2 different satisfying assignments?

Exercise 4.5. Let $G = (V, E)$ be an undirected graph. A set $X \subseteq V$ is called a vertex cover if for each $\{u, v\} \in E$ we have $X \cap \{u, v\} \neq \emptyset$. The problem $\text{VERTEX COVER}$ is

Input: A graph $G$ and some $k \in \mathbb{N}$

Question: Does there exist a vertex cover in $G$ of size at most $k$?

Show that $\text{VERTEX COVER}$ is $\mathsf{NP}$-complete.

Hint: Try to find a reduction from $\text{INDEPENDENT SET}$.

Exercise 4.6. 1. Show that if any $\mathsf{coNP}$-complete problem is in $\mathsf{NP}$, then $\mathsf{NP} = \mathsf{coNP}$.

2. Show that if a language $L$ is $\mathsf{NP}$-complete, then $\overline{L}$ is $\mathsf{coNP}$-complete.

* Exercise 4.7. Let $A \subseteq 1^*$. Show that if $A$ is $\mathsf{NP}$-complete, then $\mathsf{P} = \mathsf{NP}$.

Proceed as follows: Consider a polynomial-time reduction $f$ from $\text{SAT}$ to $A$. For a formula $\phi$, let $\phi_{0100}$ be the reduced formula where variables $x_1, x_2, x_3, x_4$ in $\phi$ are set to the values 0, 1, 0, 0, respectively. (The particular choice of 4 variables as well as of 0100 is arbitrary here) What happens when one applies $f$ to all of these exponentially many reduced formulas?