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The Complexity of Fuzzy Description Logics over Finite Lattices with Nominals

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Abstract

The complexity of reasoning in fuzzy description logics (DLs) over finite lattices usually does not exceed that of the underlying classical DLs. This has recently been shown for the logics between $L\text{-}\mathfrak{I}\mathcal{ALC}$ and $L\text{-}\mathfrak{I}\mathcal{SCHI}$ using a combination of automata- and tableau-based techniques. In this report, this approach is modified to deal with nominals and constants in $L\text{-}\mathfrak{I}\mathcal{SCHOI}$. Reasoning w.r.t. general TBoxes is ExpTime-complete, and PSpace-completeness is shown under the restriction to acyclic terminologies in two sublogics. The latter implies two previously unknown complexity results for the classical DLs $\mathcal{ALCHO}$ and $\mathcal{SO}$.

1 Introduction

Fuzzy extensions of DLs have first been studied in [27, 31, 33] to model concepts that do not have a precise meaning. Such concepts occur in many application domains. For example, a physician may base a diagnosis on the patient having a *high fever*, which is not clearly characterized even by the precise body temperature. The main idea behind fuzzy DLs is that concepts are not interpreted as sets, but rather as fuzzy sets, which assign a *membership degree* from $[0,1]$ to each domain element. As a fuzzy concept, HighFever could assign degree 0.7 to a patient with a body temperature of 38 °C, and 0.9 when the body temperature is 39 °C.

The first fuzzy DLs were based on the so-called *Zadeh* semantics that is derived from fuzzy set theory [34]. Later, it was proposed [22] to view fuzzy DLs from the point of view of Mathematical Fuzzy Logic [21] and *t-norm-based* semantics were introduced. A t-norm is a binary operator on $[0,1]$ that determines how the conjunction of two fuzzy statements is evaluated. Unfortunately, it was shown that many t-norm-based fuzzy DLs allowing general TBoxes have undecidable consistency problems [3, 11, 14]. This can be avoided by either choosing a t-norm that allows the consistency problem to be trivially reduced to classical reasoning [9], restricting to acyclic TBoxes [5], or taking the truth values from a finite structure, usually a total order [7, 8, 28] or a lattice [10, 12, 23, 29]. Recently, it was shown that the complexity of reasoning in fuzzy DLs over finite lattices with (generalized) t-norms often matches that of the underlying classical DLs [12, 13].

In this report, we analyze the complexity of fuzzy extensions of $\mathcal{SHOI}$ using a finite lattice L. In the classical case, deciding consistency of ontologies with general TBoxes is ExpTime-complete in all logics between $\mathcal{ALC}$ and $\mathcal{SHOI}$ [17, 25], and we show that this also holds for $L\text{-}\mathfrak{I}\mathcal{SCHOI}$. The additional letters $\mathfrak{I}$ and $\mathcal{C}$ in the name of the logic denote the presence of the constructors for implication and involutive negation, respectively. This nomenclature was introduced to make the subtle differences between different fuzzy DLs more explicit [11, 15]. As all fuzzy DLs considered in this report have both $\mathfrak{I}$ and $\mathcal{C}$, it is safe to ignore these letters here and simply read L-$\mathcal{SHOI}$ instead of L-$\mathfrak{I}\mathcal{SCHOI}$.

Consistency remains ExpTime-complete in the classical DLs $\mathcal{ALCOI}$ and $\mathcal{SH}$ even w.r.t. the *empty* TBox [19, 30]. However, when restricting to acyclic (or empty) TBoxes in $\mathcal{SI}$, it is only PSpace-complete [1, 20]. Similar results have been shown before under finite lattice semantics in L-$\mathfrak{I}\mathcal{ALCHI}$ and L-$\mathfrak{I}\mathcal{SCI}_\mathsf{c}$ [12]. The latter restricts all roles to be *crisp*, i.e. they are allowed to take only the two classical truth values. Here, we extend these results to L-$\mathfrak{I}\mathcal{ALCHO}$ and L-$\mathfrak{I}\mathcal{SCO}_\mathsf{c}$, which also shows previously unknown complexity results for the classical DLs $\mathcal{ALCHO}$ and $\mathcal{SO}$ with acyclic TBoxes.

2 Preliminaries

We first introduce looping automata on infinite trees and several helpful notions from [1], which will be used later for our reasoning procedures. Afterwards, we briefly recall relevant definitions from lattice theory [16].

2.1 Looping Automata

We consider the infinite tree of fixed arity $k \in \mathbb{N}$, represented by the set K^* of its *nodes*, where K abbreviates $\{1, \dots k\}$. Here, ε represents the root node, and ui, $i \in K$, is the i-th successor of the node $u \in K^*$. An *ancestor* of $u \in K^*$ is a node $u' \in K^*$ for which there is a $u'' \in K^*$ with $u = u'u''$. A *path* in this tree is a sequence $u_1, \dots, u_m$ of nodes such that $u_1 = \varepsilon$ and, for every i, $1 \leq i \leq m-1$, u_{i+1} is a successor of u_i.

Definition 2.1 (looping automaton)**.** *A* looping (tree) automaton *is a tuple* $\mathsf{A} = (Q, I, \Delta)$ *where Q is a finite set of* states*, $I \subseteq Q$ is a set of* initial states*, and* $\Delta \subseteq Q^{k+1}$ *is the* transition relation*. A* run *of* A *is a mapping* $r\colon K^* \to Q$ *such that* $r(\varepsilon) \in I$ *and* $(r(u), r(u1), \dots, r(uk)) \in \Delta$ *for every* $u \in K^*$*. The* emptiness problem *is to decide whether a given looping automaton has a run.*

The emptiness problem for such automata is decidable in polynomial time [32]. However, the automata we construct in Section 4 are exponential in the size of the input. In order to obtain PSpace decision procedures, we need to identify the length of the longest possible path in a run that does not repeat any states.

Definition 2.2 (invariant, blocking)**.** *Let* $\mathsf{A} = (Q, I, \Delta)$ *be a looping automaton and* $\hookleftarrow$ *a binary relation over Q, called the* blocking relation*.* A *is* $\hookleftarrow$-invariant *if* $(q_0, q_1, \dots, q_i, \dots, q_k) \in \Delta$ *and* $q_i \hookleftarrow q_i'$ *always imply* $(q_0, q_1, \dots, q_i', \dots, q_k) \in \Delta$*.*

If this is the case, then A *is* m-blocking *for* $m \in \mathbb{N}$ *if in every path* $u_1, \ldots, u_m$ *of length* m *in a run* r *of* A *there are two indices* $1 \leq i < j \leq m$ *with* $r(u_j) \looparrowleft r(u_i)$.

The notion of blocking is similar to that used in tableau algorithms for DLs [4, 20]. If q is blocked by its ancestor q' ($q \looparrowleft q'$), then we do not need to consider the subtree below q since every transition involving q can be replaced by one using q' instead. Of course, every looping automaton is $=$-invariant and $|Q|$-blocking. However, as mentioned above the size of Q may already be exponential in some external parameter. To obtain m-blocking automata with m bounded polynomially in the size of the input, we can use a faithful family of functions to prune the transition relation.

Definition 2.3 (faithful). *Let* $\mathsf{A} = (Q, I, \Delta)$ *be a looping automaton. A family* $\mathfrak{f} = (f_q)_{q \in Q}$ *of functions* $f_q \colon Q \to Q$ *is called* faithful *(w.r.t.* A*) if*

- *for all* $(q, q_1, \ldots, q_k) \in \Delta$*, we have* $(q, f_q(q_1), \ldots, f_q(q_k)) \in \Delta$*, and*
- *for all* $(q_0, q_1, \ldots, q_k) \in \Delta$*, we have* $(f_q(q_0), f_q(q_1), \ldots, f_q(q_k)) \in \Delta$.

The subautomaton $\mathsf{A}^{\mathfrak{f}} := (Q, I, \Delta^{\mathfrak{f}})$ *induced by* $\mathfrak{f}$ *is defined by*

$$\Delta^{\mathfrak{f}} := \{(q, f_q(q_1), \ldots, f_q(q_k)) \mid (q, q_1, \ldots, q_k) \in \Delta\}.$$

The name *faithful* reflects the fact that the resulting subautomaton simulates all runs of A. The following connection between the two automata was shown in [1].

Proposition 2.4. *Let* A *be a looping automaton and* $\mathfrak{f}$ *be a faithful family of functions for* A*. Then* A *has a run iff* $\mathsf{A}^{\mathfrak{f}}$ *has a run.*

Together with some other assumptions, polynomial blocking allows us to test emptiness in polynomial space.

Definition 2.5 (PSpace on-the-fly construction). *Let* I *be a set of* inputs*. A construction that yields, for each* $\mathsf{i} \in \mathsf{I}$*, an* m_{i}*-blocking looping automaton* A_{i} *over* k_{i}*-ary trees is called a* PSpace on-the-fly construction *if there is a polynomial* P *such that, for every input* i *of size* n*,*

(i) $m_{\mathsf{i}} \leq P(n)$ *and* $k_{\mathsf{i}} \leq P(n)$*,*

(ii) the size of every state of A_{i} *is bounded by* $P(n)$*, and*

(iii) one can guess in time bounded by $P(n)$ *an initial state, and, given a state* q*, a transition* $(q, q_1, \ldots, q_k)$ *of* A_{i}.

The following result is again taken from [1].

Proposition 2.6. *If the looping automata* A_{i} *are obtained by a* PSpace *on-the-fly construction, then emptiness of* A_{i} *can be decided in* PSpace *in the size of* i.

2.2 Residuated Lattices

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ with the two commutative, associative, and idempotent binary operators *supremum* ($\vee$) and *infimum* ($\wedge$) that satisfy $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ for all $x, y \in L$. The natural partial order on L is given by $x \leq y$ iff $x \wedge y = x$ for all $x, y \in L$. An *antichain* is a set $S \subseteq L$ of incomparable elements. The *width* of the lattice L is the maximum cardinality of all its antichains. This lattice is *complete* if suprema and infima of arbitrary subsets $S \subseteq L$ exist; these are denoted by $\bigvee_{x \in S} x$ and $\bigwedge_{x \in S} x$, respectively. It is *distributive* if $\wedge$ and $\vee$ distribute over each other, *finite* if L is finite, and *bounded* if it has a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$. Every finite lattice is complete, and every complete lattice is bounded by $\mathbf{0} := \bigwedge_{x \in L} x$ and $\mathbf{1} := \bigvee_{x \in L} x$.

A *De Morgan lattice* is a distributive lattice L with a unary involutive operator $\sim$ on L satisfying ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$ and ${\sim}(x \wedge y) = {\sim}x \vee {\sim}y$ for all $x, y \in L$. A *t-norm* over a bounded lattice L is a commutative, associative, monotone binary operator $\otimes$ on L that has $\mathbf{1}$ as its unit. A *residuated lattice* is a bounded lattice L with a t-norm $\otimes$ and a *residuum* $\Rightarrow : L \times L \to L$ satisfying $x \otimes y \leq z$ iff $y \leq x \Rightarrow z$ for all $x, y, z \in L$. We always assume that $\otimes$ is *join-preserving*, that is, $x \otimes \bigvee_{y \in S} y = \bigvee_{y \in S} x \otimes y$ holds for all $x \in L$ and $S \subseteq L$. This is a natural assumption that corresponds to the left-continuity assumption for t-norms over the standard fuzzy interval $[0, 1]$ [21].

3 L-$\mathcal{ISCHOI}$

Since fuzzy DLs over infinite lattices easily become undecidable when dealing with GCIs [3, 11, 13, 14], we now fix a *finite* residuated De Morgan lattice L. For the complexity analysis, we assume that L is given as a list of its elements and that all lattice operations are computable in polynomial time.[1]

The syntax of the fuzzy description logic L-$\mathcal{ISCHOI}$ is similar to that of classical $\mathcal{SHOI}$: complex roles and concepts are constructed from disjoint sets $\mathsf{N_C}$ of *concept names*, $\mathsf{N_R}$ of *role names*, and $\mathsf{N_I}$ of *individual names*.

Definition 3.1 (syntax)**.** *The set* $\mathsf{N_R^-}$ *of* (complex) roles *is* $\{r, r^- \mid r \in \mathsf{N_R}\}$. *The set of* (complex) concepts *is constructed as follows:*

- *every concept name is a concept, and*
- *for concepts* C, D, $r \in \mathsf{N_R^-}$, $a \in \mathsf{N_I}$, *and* $p \in L$, *the following are also concepts:* $\overline{p}$ *(constant),* $\{a\}$ *(nominal),* $\neg C$ *(negation),* $C \sqcap D$ *(conjunc-*

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[1] If instead the size of the input encoding of L is logarithmic in the cardinality of L, then all complexity results except Theorem 5.8 remain valid.

tion), $C \to D$ (implication), $\exists r.C$ (existential restriction), and $\forall r.C$ (value restriction).

For a complex role s, the *inverse of* s (written $\overline{s}$) is s^- if $s \in \mathsf{N_R}$ and r if $s = r^-$.

Definition 3.2 (semantics). *A* (fuzzy) interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty* domain $\Delta^\mathcal{I}$ *and an* interpretation function $\cdot^\mathcal{I}$ *that assigns to every* $A \in \mathsf{N_C}$ *a fuzzy set* $A^\mathcal{I}\colon \Delta^\mathcal{I} \to L$*, to every* $r \in \mathsf{N_R}$ *a fuzzy binary relation* $r^\mathcal{I}\colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to L$*, and to every* $a \in \mathsf{N_I}$ *a domain element* $a^\mathcal{I} \in \Delta^\mathcal{I}$*. This function is extended to complex roles and concepts as follows for all* $x, y \in \Delta^\mathcal{I}$*:*

- $(r^-)^\mathcal{I}(x, y) := r^\mathcal{I}(y, x)$*;*
- $\overline{p}^\mathcal{I}(x) := p$*;*
- $\{a\}^\mathcal{I}(x) := \mathbf{1}$ *if* $x = a^\mathcal{I}$*, and* $\{a\}^\mathcal{I}(x) := \mathbf{0}$ *otherwise;*
- $(\neg C)^\mathcal{I}(x) := {\sim}C^\mathcal{I}(x)$*;*
- $(C \sqcap D)^\mathcal{I}(x) := C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$*;*
- $(C \to D)^\mathcal{I}(x) := C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)$*;*
- $(\exists r.C)^\mathcal{I}(x) := \bigvee_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$*; and*
- $(\forall r.C)^\mathcal{I}(x) := \bigwedge_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) \Rightarrow C^\mathcal{I}(y)$*.*

One can express *fuzzy nominals* [6] of the form $\{p_1/a_1, \dots, p_n/a_n\}$ with $p_i \in L$ and $a_i \in \mathsf{N_I}$, $1 \leq i \leq n$, by $(\{a_1\} \sqcap \overline{p_1}) \sqcup \cdots \sqcup (\{a_n\} \sqcap \overline{p_n})$, where $C \sqcup D$ abbreviates $\neg(\neg C \sqcap \neg D)$. Unlike in classical DLs, existential and value restrictions need not be dual to each other, i.e. in general we have $(\neg\exists r.C)^\mathcal{I} \neq (\forall r.\neg C)^\mathcal{I}$.

Definition 3.3 (ontology). *An* axiom *is a* concept assertion $\langle a{:}C \bowtie p\rangle$*, a* concept definition $\langle A \doteq C \geq p\rangle$*, a* general concept inclusion (GCI) $\langle C \sqsubseteq D \geq p\rangle$*, a* role inclusion $\langle r \sqsubseteq s\rangle$*, or a* transitivity axiom $\mathsf{trans}(r)$*, where* C, D *are concepts,* $r, s \in \mathsf{N_R^-}$*,* $a \in \mathsf{N_I}$*,* $A \in \mathsf{N_C}$*,* $p \in L$*, and* $\bowtie \in \{<, \leq, =, \geq, >\}$*.*

An acyclic TBox *is a finite set* $\mathcal{T}$ *of concept definitions where every* $A \in \mathsf{N_C}$ *has at most one definition* $\langle A \doteq C \geq p\rangle$ *in* $\mathcal{T}$ *and the relation* $>_\mathcal{T}$ *on* $\mathsf{N_C}$ *is acyclic, where* $A >_\mathcal{T} B$ *iff* B *occurs in the definition of* A*. A* general TBox *is a finite set of GCIs, an* ABox *a finite set of concept assertions, and an* RBox *a finite set of role inclusions and transitivity axioms. An* ontology *is a triple* $(\mathcal{A}, \mathcal{T}, \mathcal{R})$ *consisting of an ABox* $\mathcal{A}$*, an (acyclic or general) TBox* $\mathcal{T}$*, and an RBox* $\mathcal{R}$*.*

An interpretation $\mathcal{I}$ satisfies *(or is a* model *of)*

- *an assertion* $\langle a{:}C \bowtie p\rangle$ *if* $C^\mathcal{I}(a^\mathcal{I}) \bowtie p$*.*

- *a concept definition* $\langle A \doteq C \geq p\rangle$ *if for every element* $x \in \Delta^{\mathcal{I}}$ *it holds that* $(A^{\mathcal{I}}(x) \Rightarrow C^{\mathcal{I}}(x)) \otimes (C^{\mathcal{I}}(x) \Rightarrow A^{\mathcal{I}}(x)) \geq p$*.*
- *a GCI* $\langle C \sqsubseteq D \geq p\rangle$ *if for every* $x \in \Delta^{\mathcal{I}}$ *we have* $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq p$*.*
- *a role inclusion* $\langle r \sqsubseteq s\rangle$ *if* $r^{\mathcal{I}}(x,y) \leq s^{\mathcal{I}}(x,y)$ *holds for all* $x, y \in \Delta^{\mathcal{I}}$*.*
- *a transitivity axiom* $\mathsf{trans}(r)$ *if* $r^{\mathcal{I}}(x,y) \otimes r^{\mathcal{I}}(y,z) \leq r^{\mathcal{I}}(x,z)$ *holds for all* $x, y, z \in \Delta^{\mathcal{I}}$*.*
- *an ABox, TBox, RBox, or ontology if it satisfies all axioms in it.*

We denote by $\mathsf{N_I}(\mathcal{O})$ and $\mathsf{N_R}(\mathcal{O})$ the sets of individual names and role names, respectively, occurring in an ontology $\mathcal{O}$, and set $\mathsf{N_R^-}(\mathcal{O}) := \{r, r^- \mid r \in \mathsf{N_R}(\mathcal{O})\}$. As usual, for an ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ we define the *role hierarchy* $\sqsubseteq_{\mathcal{R}}$ as the reflexive transitive closure of $\{(r,s) \in \mathsf{N_R^-}(\mathcal{O}) \mid r \sqsubseteq_{\mathcal{R}} s \in \mathcal{R} \text{ or } \overline{r} \sqsubseteq_{\mathcal{R}} \overline{s} \in \mathcal{R}\}$, and we call a role r *transitive* if either $\mathsf{trans}(r) \in \mathcal{R}$ or $\mathsf{trans}(\overline{r}) \in \mathcal{R}$.

For an acyclic TBox $\mathcal{T}$, all concept names that occur on the left-hand side of a definition in $\mathcal{T}$ are called *defined.* All other concept names occurring in $\mathcal{T}$ are *primitive.* In a general TBox, all concept names are *primitive.*

We do not consider *role assertions* of the form $\langle (a,b){:}r \bowtie p\rangle$ since in the presence of nominals they can be simulated by concept assertions, e.g. $\langle a{:}\exists r.\{b\} \bowtie p\rangle$.

Definition 3.4 (reasoning)**.** *Let* C, D *be concepts,* $\mathcal{O}$ *an ontology, and* $p \in L$*.*

- $\mathcal{O}$ *is* consistent *if it has a model.*
- C *is* p-satisfiable *w.r.t.* $\mathcal{O}$ *if there is a model* $\mathcal{I}$ *of* $\mathcal{O}$ *and an element* $x \in \Delta^{\mathcal{I}}$ *such that* $C^{\mathcal{I}}(x) \geq p$*.*
- C *is* p-subsumed *by* D *w.r.t.* $\mathcal{O}$ *if every model of* $\mathcal{O}$ *is also a model of* $\langle C \sqsubseteq D \geq p\rangle$*.*
- *The* best satisfiability degree *for* C *w.r.t.* $\mathcal{O}$ *is the supremum of all* $p' \in L$ *such that* C *is* p'*-satisfiable w.r.t.* $\mathcal{O}$*.*
- *The* best subsumption degree *of* C *and* D *w.r.t.* $\mathcal{O}$ *is the supremum of all* $p' \in L$ *such that* C *is* p'*-subsumed by* D *w.r.t.* $\mathcal{O}$*.*

Observe that C is p-satisfiable w.r.t. $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ iff $(\mathcal{A} \cup \{\langle a{:}C \geq p\rangle\}, \mathcal{T}, \mathcal{R})$ is consistent, where a is a fresh individual name. Similarly, C is p-subsumed by D w.r.t. $\mathcal{O}$ iff $(\mathcal{A} \cup \{\langle a{:}C \to D < p\rangle\}, \mathcal{T}, \mathcal{R})$ is inconsistent. To compute the *best* degrees to which these inferences hold, one has to solve polynomially many consistency problems (cf. [13]). Thus, in the following we focus on deciding consistency.

4 Deciding Consistency

Consistency in L-$\mathfrak{I}\mathcal{SCHOI}$ with general TBoxes is ExpTime-complete, matching the complexity of classical $\mathcal{SHOI}$ [17]. To show this, we adapt the automata-based procedures from [1, 12] to this more expressive logic. The conditions for the role hierarchy, inverse roles, and transitive roles are similar to the tableaux rules used in [20]. To deal with nominals, we employ *pre-completions* inspired by the approaches in [2, 13, 18]. In Section 5, we derive additional complexity results for consistency in the sublogics L-$\mathfrak{I}\mathcal{ALCHO}$ (without transitivity and inverse roles) and L-$\mathfrak{I}\mathcal{SCO}_\mathsf{c}$ (without role inclusions, inverse roles, and fuzzy roles) with acyclic TBoxes.

It was shown in [12] that over a finite lattice L every interpretation $\mathcal{I}$ is n-*witnessed*, where n is the width of the lattice. This means that for every concept C, $r \in \mathsf{N}_\mathsf{R}^-$, and $x \in \Delta^\mathcal{I}$ there are n *witnesses* $y_1, \dots, y_n \in \Delta^\mathcal{I}$ such that $(\exists r.C)^\mathcal{I}(x) = \bigvee_{i=1}^n r^\mathcal{I}(x, y_i) \otimes C^\mathcal{I}(y_i)$, and similarly for the value restrictions. For the sake of simplicity, we present the following reasoning procedure only for the case of $n = 1$, i.e. we assume that all interpretations are *(1-)witnessed.* It can be generalized to handle arbitrary n by easy adaptations of the following definitions, in particular the introduction of more than one witness in Definition 4.3.

We now consider an ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ that we want to test for consistency. The main idea of the algorithm is to find an abstract representation of a tree-shaped model of $\mathcal{O}$, a so-called *Hintikka tree.* Every node of this tree consists of a *Hintikka function* that describes the values of all relevant concepts for one domain element of the model. Additionally, each Hintikka function stores the values of all role connections from the parent node. We define the set $\mathsf{sub}(\mathcal{O})$ to contain all subconcepts of concepts occurring in $\mathcal{O}$, together with all $\exists s.C$ (and $\forall s.C$) for which $\exists r.C$ ($\forall r.C$) occurs in $\mathcal{O}$, $s \sqsubseteq_\mathcal{R} r$, and s is transitive.

Definition 4.1 (Hintikka function)**.** *A* Hintikka function *for* $\mathcal{O}$ *is a partial function* $H \colon \mathsf{sub}(\mathcal{O}) \cup \mathsf{N}_\mathsf{R}^-(\mathcal{O}) \to L$ *satisfying the following conditions:*

- *$H(s)$ is defined for all $s \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$;*
- *if $H(\overline{p})$ is defined, then $H(\overline{p}) = p$;*
- *if $H(\{a\})$ is defined, then $H(\{a\}) \in \{\mathbf{0}, \mathbf{1}\}$;*
- *if $H(C \sqcap D)$ is defined, then $H(C)$ and $H(D)$ are also defined and it holds that $H(C \sqcap D) = H(C) \otimes H(D)$; and similarly for $\neg C$ and $C \to D$.*

This function is compatible *with*

- *an assertion $\langle a{:}C \bowtie \ell \rangle$ if, whenever $H(\{a\}) = \mathbf{1}$, then $H(C)$ is defined and $H(C) \bowtie \ell$.*

- *a concept definition* $\langle A \doteq C \geq \ell \rangle$ *if, whenever* $H(A)$ *is defined, then* $H(C)$ *is defined and* $(H(A) \Rightarrow H(C)) \otimes (H(C) \Rightarrow H(A)) \geq \ell$*.*
- *a GCI* $\langle C \sqsubseteq D \geq \ell \rangle$ *if* $H(C)$ *and* $H(D)$ *are defined and* $H(C) \Rightarrow H(D) \geq \ell$*.*
- *a role inclusion* $r \sqsubseteq s$ *if* $H(r) \leq H(s)$*.*
- *an ABox/TBox/RBox/ontology if it is compatible with all axioms in it.*

The support *of* H *is the set* $\mathsf{supp}(H)$ *of all* $C \in \mathsf{sub}(\mathcal{O})$ *for which* H *is defined, and* $\mathsf{Ind}(H)$ *is the set of all* $a \in \mathsf{N_I}(\mathcal{O})$ *for which* $H(\{a\}) = \mathbf{1}$*.*

To deal with nominals, our algorithm maintains a polynomial amount of global information about the named domain elements, called a *pre-completion.* Since one domain element can have several names, we first consider a partition of $\mathsf{N_I}(\mathcal{O})$ that specifies which names are interpreted by the same elements. The pre-completion further contains one Hintikka function for each named individual, and the values of all role connections between them.

Definition 4.2 (pre-completion)**.** *A* pre-completion *for the ontology* $\mathcal{O}$ *is a triple* $\mathfrak{P} = (\mathcal{P}, \mathcal{H}_\mathcal{P}, \mathcal{R}_\mathcal{P})$*, where* $\mathcal{P}$ *is a partition of* $\mathsf{N_I}(\mathcal{O})$*,* $\mathcal{H}_\mathcal{P} = (H_X)_{X \in \mathcal{P}}$ *is a family of Hintikka functions for* $\mathcal{O}$*, and* $\mathcal{R}_\mathcal{P} = (r_\mathcal{P})_{r \in \mathsf{N_R}(\mathcal{O})}$ *is a family of fuzzy binary relations* $r_\mathcal{P} \colon \mathcal{P} \times \mathcal{P} \to L$*, such that, for all* $X \in \mathcal{P}$*,*

- $\mathsf{Ind}(H_X) = X$ *and*
- H_X *is compatible with* $\mathcal{O}$*.*

A Hintikka function H *for* $\mathcal{O}$ *is* compatible *with* $\mathfrak{P}$ *if for all* $a \in \mathsf{Ind}(H)$*, we have* $H|_{\mathsf{sub}(\mathcal{O})} = H_{[a]_\mathcal{P}}|_{\mathsf{sub}(\mathcal{O})}$*.*

We further set $r^-_\mathcal{P}(X, Y) := r_\mathcal{P}(Y, X)$ for all $X, Y \in \mathcal{P}$ and $r \in \mathsf{N_R}(\mathcal{O})$.

The arity k of our Hintikka trees is the number of existential and value restrictions in $\mathsf{sub}(\mathcal{O})$. Each successor in the tree describes the witness for one restriction. For the following definition, we consider $K := \{1, \ldots, k\}$ as before and fix a bijection $\varphi \colon \{C \mid C \in \mathsf{sub}(\mathcal{O}) \text{ is of the form } \exists r.D \text{ or } \forall r.D\} \to K$.

Definition 4.3 (Hintikka condition)**.** *The tuple* $(H_0, H_1, \ldots, H_k)$ *of Hintikka functions for* $\mathcal{O}$ *satisfies the* Hintikka condition *if the following hold:*

a) For every existential restriction $\exists r.C \in \mathsf{sub}(\mathcal{O})$*:*

  - *If* $\exists r.C \in \mathsf{supp}(H_0)$ *and* $i = \varphi(\exists r.C)$*, then we have* $C \in \mathsf{supp}(H_i)$ *and* $H_0(\exists r.C) = H_i(r) \otimes H_i(C)$*.*

- *If* $\exists r.C \in \mathsf{supp}(H_0)$*, then for all* $i \in K$*, we have* $C \in \mathsf{supp}(H_i)$ *and* $H_0(\exists r.C) \geq H_i(r) \otimes H_i(C)$*; moreover, for all transitive roles* $s \sqsubseteq_{\mathcal{R}} r$*, we have* $\exists s.C \in \mathsf{supp}(H_i)$ *and* $H_0(\exists r.C) \geq H_i(s) \otimes H_i(\exists s.C)$*.*
- *For all* $i \in K$ *with* $\exists r.C \in \mathsf{supp}(H_i)$*, we have* $C \in \mathsf{supp}(H_0)$ *and* $H_i(\exists r.C) \geq H_i(\overline{r}) \otimes H_0(C)$*; moreover, for all transitive roles* $s \sqsubseteq_{\mathcal{R}} \overline{r}$*, we have* $\exists s.C \in \mathsf{supp}(H_0)$ *and* $H_i(\exists r.C) \geq H_i(s) \otimes H_0(\exists s.C)$*.*

b) For every value restriction $\forall r.C \in \mathsf{sub}(\mathcal{O})$*:*

- *If* $\forall r.C \in \mathsf{supp}(H_0)$ *and* $i = \varphi(\forall r.C)$*, then we have* $C \in \mathsf{supp}(H_i)$ *and* $H_0(\forall r.C) = H_i(r) \Rightarrow H_i(C)$*.*
- *If* $\forall r.C \in \mathsf{supp}(H_0)$*, then for all* $i \in K$*, we have* $C \in \mathsf{supp}(H_i)$ *and* $H_0(\forall r.C) \leq H_i(r) \Rightarrow H_i(C)$*; moreover, for all transitive roles* $s \sqsubseteq_{\mathcal{R}} r$*, we have* $\forall s.C \in \mathsf{supp}(H_i)$ *and* $H_0(\forall r.C) \leq H_i(s) \Rightarrow H_i(\forall s.C)$*.*
- *For all* $i \in K$ *with* $\forall r.C \in \mathsf{supp}(H_i)$*, we have* $C \in \mathsf{supp}(H_0)$ *and* $H_i(\forall r.C) \leq H_i(\overline{r}) \Rightarrow H_0(C)$*; moreover, for all transitive roles* $s \sqsubseteq_{\mathcal{R}} \overline{r}$*, we have* $\forall s.C \in \mathsf{supp}(H_0)$ *and* $H_i(\forall r.C) \leq H_i(s) \Rightarrow H_0(\forall s.C)$*.*

c) For all $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$ *and* $i, j \in K$ *such that* $a \in \mathsf{Ind}(H_i)$*,* $b \in \mathsf{Ind}(H_j)$*, and* $[a]_\mathcal{P} = [b]_\mathcal{P}$*, we have* $H_i(r) = H_j(r)$*.*

d) For all $a \in \mathsf{Ind}(H_0)$*,* $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$*,* $i \in K$*, and* $b \in \mathsf{Ind}(H_i)$*, it holds that* $H_i(r) = r_\mathcal{P}([a]_\mathcal{P}, [b]_\mathcal{P})$*.*

Intuitively, Condition a) ensures that the designated successor satisfies the witnessing condition for $\exists r.C$, and that the other successors do not interfere; this includes the parent node, which is a $\overline{r}$-predecessor. Additionally, existential restrictions are transferred along transitive roles, similar to the $\forall_+$-rule in [20]. Conditions c) and d) are concerned with the behavior of named successors; in particular, the values for the role connections between named individuals specified by the pre-completion should be respected.

Given a pre-completion $\mathfrak{P} = (\mathcal{P}, \mathcal{H}_\mathcal{P}, \mathcal{R}_\mathcal{P})$, a *Hintikka tree for* $\mathcal{O}$ *starting with* H_X, $X \in \mathcal{P}$, is a mapping $\mathbf{T}$ that assigns to each $u \in K^*$ a Hintikka function $\mathbf{T}(u)$ for $\mathcal{O}$ that is compatible with $\mathcal{T}$, $\mathcal{R}$, and $\mathfrak{P}$ such that $\mathbf{T}(\varepsilon) = H_X$ and every tuple $(\mathbf{T}(u), \mathbf{T}(u1), \dots, \mathbf{T}(uk))$ satisfies the Hintikka condition.

Lemma 4.4. *$\mathcal{O}$ is consistent iff there exist a pre-completion* $\mathfrak{P} = (\mathcal{P}, \mathcal{H}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ *for* $\mathcal{O}$ *and, for each* $X \in \mathcal{P}$*, a Hintikka tree for* $\mathcal{O}$ *starting with* H_X*.*

Proof. Assume that such a pre-completion and Hintikka trees $\mathbf{T}_X$ for $\mathcal{O}$ starting with H_X exist. We first remove irrelevant nodes in these Hintikka trees. A node $u \in K^*$ is *relevant* in $\mathbf{T}_X$ if $\mathsf{Ind}(\mathbf{T}_X(u')) = \emptyset$ for all (non-empty) ancestors $u' \in K^+$ of u. The idea is that if $a \in \mathsf{Ind}(\mathbf{T}_X(u'))$, then by the compatibility with $\mathfrak{P}$ the Hintikka function $\mathbf{T}_X(u')$ agrees with $H_{[a]_\mathcal{P}} = \mathbf{T}_{[a]_\mathcal{P}}(\varepsilon)$ on the values of all concepts in $\mathsf{sub}(\mathcal{O})$, and thus $\mathbf{T}_X(u')$ can be replaced with $\mathbf{T}_{[a]_\mathcal{P}}(\varepsilon)$. The root nodes are always relevant since they are needed to represent the named individuals. We now define the interpretation $\mathcal{I}$ with domain

$$\Delta^\mathcal{I} := \{(X,u) \in \mathcal{P} \times K^* \mid u \text{ is relevant in } \mathbf{T}_X\}.$$

We set $a^\mathcal{I} := ([a]_\mathcal{P}, \varepsilon)$ for all $a \in \mathsf{N_I}(\mathcal{O})$. For $r \in \mathsf{N_R}$, we first define the fuzzy binary relation $r^\mathbf{T}$ on $\Delta^\mathcal{I}$ as follows for all $(X,u),(Y,v) \in \Delta^\mathcal{I}$:

- $r^\mathbf{T}((X,u),(Y,v)) := \mathbf{T}_X(ui)(r)$ if $r \in \mathsf{N}^-_\mathsf{R}(\mathcal{O})$ and for $i \in K$ it holds that (i) $(Y,v) = (X,ui)$ or (ii) $v = \varepsilon$ and $\mathsf{Ind}(\mathbf{T}_X(ui)) \cap Y \neq \emptyset$;
- $r^\mathbf{T}((X,u),(Y,v)) := \mathbf{T}_Y(vi)(r^-)$ if $r^- \in \mathsf{N}^-_\mathsf{R}(\mathcal{O})$ and for $i \in K$ it holds that (i) $(X,u) = (Y,vi)$ or (ii) $u = \varepsilon$ and $\mathsf{Ind}(\mathbf{T}_Y(vi)) \cap X \neq \emptyset$; and
- $r^\mathbf{T}((X,u),(Y,v)) := \mathbf{0}$ otherwise.

To see that this is well-defined, consider the following three cases.

- If $r \in \mathsf{N}^-_\mathsf{R}(\mathcal{O})$ and there are $i,j \in K$ such that $v = \varepsilon$, $\mathsf{Ind}(\mathbf{T}_X(ui)) \cap Y \neq \emptyset$, and $\mathsf{Ind}(\mathbf{T}_X(uj)) \cap Y \neq \emptyset$, then from Condition c) of Definition 4.3 we get $\mathbf{T}_X(ui)(r) = \mathbf{T}_X(uj)(r)$.
- If $r^- \in \mathsf{N}^-_\mathsf{R}(\mathcal{O})$, $i,j \in K$, $u = \varepsilon$, and $\mathsf{Ind}(\mathbf{T}_Y(vi)) \cap X$ and $\mathsf{Ind}(\mathbf{T}_Y(vj)) \cap X$ are non-empty, we have $\mathbf{T}_Y(vi)(r^-) = \mathbf{T}_Y(vj)(r^-)$ by the same condition.
- If $r, r^- \in \mathsf{N}^-_\mathsf{R}(\mathcal{O})$, $u = v = \varepsilon$ and there are $i,j \in K$ with $a \in \mathsf{Ind}(\mathbf{T}_X(i)) \cap Y$ and $b \in \mathsf{Ind}(\mathbf{T}_Y(j)) \cap X$, then $Y = [a]_\mathcal{P}$ and $X = [b]_\mathcal{P}$. By Condition d) of Definition 4.3, we obtain $\mathbf{T}_X(i)(r) = r_\mathcal{P}(X,Y) = r^-_\mathcal{P}(Y,X) = \mathbf{T}_Y(j)(r^-)$.

We also set $(r^-)^\mathbf{T}((X,u),(Y,v)) := r^\mathbf{T}((Y,v),(X,u))$ for all $(X,u),(Y,v) \in \Delta^\mathcal{I}$. Before we proceed to define $\mathcal{I}$, we show that this definition satisfies the following property, which mainly follows from the Hintikka condition:

Claim 1. *For all* $\exists r.C \in \mathsf{sub}(\mathcal{O})$ *and* $(X,u),(Y,v) \in \Delta^\mathcal{I}$ *such that* $\mathbf{T}_X(u)(\exists r.C)$ *is defined, we have* $\mathbf{T}_X(u)(\exists r.C) \geq r^\mathbf{T}((X,u),(Y,v)) \otimes \mathbf{T}_Y(v)(C)$*, and, for all transitive roles* $s \sqsubseteq_\mathcal{R} r$*,* $\mathbf{T}_X(u)(\exists r.C) \geq s^\mathbf{T}((X,u),(Y,v)) \otimes \mathbf{T}_Y(v)(\exists s.C)$*.*

The first part is trivial if $r^\mathbf{T}((X,u),(Y,v)) = \mathbf{0}$; otherwise, there must be an index $i \in K$ such that (A) $r^\mathbf{T}((X,u),(Y,v)) = \mathbf{T}_X(ui)(r)$ and (A.i) $(Y,v) = (X,ui)$ or (A.ii) $v = \varepsilon$ and $\mathsf{Ind}(\mathbf{T}_X(ui)) \cap Y \neq \emptyset$; or (B) $r^\mathbf{T}((X,u),(Y,v)) = \mathbf{T}_Y(vi)(\overline{r})$ and (B.i) $(X,u) = (Y,vi)$ or (B.ii) $u = \varepsilon$ and $\mathsf{Ind}(\mathbf{T}_Y(vi)) \cap X \neq \emptyset$.

In Case (A), the Hintikka condition implies that $\mathbf{T}_X(ui)(C)$ is defined and we have $\mathbf{T}_X(u)(\exists r.C) \geq \mathbf{T}_X(ui)(r) \otimes \mathbf{T}_X(ui)(C)$. It thus suffices to show that $\mathbf{T}_Y(v)(C) = \mathbf{T}_X(ui)(C)$. In Case (A.i), this is immediate; in Case (A.ii), we have $\mathbf{T}_Y(v)(C) = H_Y(C) = \mathbf{T}_X(ui)(C)$ by the compatibility with $\mathfrak{P}$.

In Case (B.i), we get $\mathbf{T}_X(u)(\exists r.C) = \mathbf{T}_Y(vi)(\exists r.C)$; in Case (B.ii), we also have $\mathbf{T}_X(u)(\exists r.C) = H_X(\exists r.C) = \mathbf{T}_Y(vi)(\exists r.C)$ by the compatibility with $\mathfrak{P}$. In both cases, we have $\mathbf{T}_X(u)(\exists r.C) = \mathbf{T}_Y(vi)(\exists r.C) \geq \mathbf{T}_Y(vi)(\overline{r}) \otimes \mathbf{T}_Y(v)(C)$ by the Hintikka condition.

The remaining part of Claim 1 can be shown by similar arguments, using the parts of Definition 4.3 about transitive roles.

To properly interpret transitive roles, we now set, for all $x_1, \dots, x_n \in \Delta^{\mathcal{I}}$ with $n \geq 3$, $r^{\mathbf{T}}(x_1, \dots, x_n) := r^{\mathbf{T}}(x_1, x_2) \otimes \dots \otimes r^{\mathbf{T}}(x_{n-1}, x_n)$ and

$$r^{\mathcal{I}}(x, y) := r^{\mathbf{T}}(x, y) \vee \bigvee_{\substack{s \sqsubseteq_{\mathcal{R}} r \\ s \text{ transitive}}} \bigvee_{n \geq 1} \bigvee_{z_1, \dots, z_n \in \Delta^{\mathcal{I}}} s^{\mathbf{T}}(x, z_1, \dots, z_n, y)$$

for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$. By the above definitions, the same expression is valid for inverse roles. Furthermore, if r is transitive, then $r^{\mathcal{I}}$ is the transitive closure of $r^{\mathbf{T}}$, and thus a transitive fuzzy binary relation. For every $r \sqsubseteq s \in \mathcal{R}$ and $x, y \in \Delta^{\mathcal{I}}$, we have $r^{\mathbf{T}}(x, y) \leq s^{\mathbf{T}}(x, y)$ by the compatibility with $\mathcal{R}$. Since $r' \sqsubseteq_{\mathcal{R}} r$ then implies that $r' \sqsubseteq_{\mathcal{R}} s$, we have $r^{\mathcal{I}}(x, y) \leq s^{\mathcal{I}}(x, y)$, and thus $\mathcal{I}$ satisfies $\mathcal{R}$.

We now define the interpretation of concept names under $\mathcal{I}$. For every primitive concept name A, we simply set $A^{\mathcal{I}}(X, u) := \mathbf{T}_X(u)(A)$ for all $(X, u) \in \Delta^{\mathcal{I}}$. $\mathcal{I}$ is extended to the defined concept names while showing the following claim:

Claim 2. *For all $(X, u) \in \Delta^{\mathcal{I}}$ and all $C \in \mathsf{sub}(\mathcal{O})$ for which $\mathbf{T}_X(u)(C)$ is defined, we have $C^{\mathcal{I}}(X, u) = \mathbf{T}_X(u)(C)$.*

We prove this by induction on the *weight* $o(C)$:

- $o(A) := o(\overline{p}) := o(\{a\}) := 0$ for every primitive concept name A, $p \in L$, and $a \in \mathsf{N_I}$;
- $o(A) := o(C) + 1$ for every definition $\langle A \doteq C \geq \ell \rangle \in \mathcal{T}$;
- $o(\neg C) := o(C) + 1$;
- $o(C \sqcap D) := o(C \to D) := \max\{o(C), o(D)\} + 1$; and
- $o(\exists r.C) := o(\forall r.C) := o(C) + 1$.

This weight is well-defined for general and acyclic TBoxes.

For every constant concept, Claim 2 follows immediately from Definition 4.1. For a primitive concept name A, it holds by the definition of $A^{\mathcal{I}}$ above.

If $\mathbf{T}_X(u)(\{a\})$ is defined for some $a \in \mathsf{N_I}(\mathcal{O})$, then by Definition 4.1 this value is either $\mathbf{0}$ or $\mathbf{1}$. If it is $\mathbf{0}$, then we cannot have $\mathbf{T}_X(u) = H_{[a]_{\mathcal{P}}}$ by Definition 4.2.

Thus, $a^{\mathcal{I}} = ([a]_{\mathcal{P}}, \varepsilon) \neq (X, u)$, and hence $\{a\}^{\mathcal{I}}(X, u) = \mathbf{0} = \mathbf{T}_X(u)(\{a\})$. Otherwise, we have $\mathbf{T}_X(u)(\{a\}) = \mathbf{1}$, i.e. $a \in \mathsf{Ind}(\mathbf{T}_X(u))$. Since u is relevant in $\mathbf{T}_X$, we infer that $u = \varepsilon$. By Definition 4.2, we get $a \in \mathsf{Ind}(\mathbf{T}_X(u)) = \mathsf{Ind}(H_X) = X$, and thus $a^{\mathcal{I}} = ([a]_{\mathcal{P}}, \varepsilon) = (X, u)$. We conclude $\{a\}^{\mathcal{I}}(X, u) = \mathbf{1} = \mathbf{T}_X(u)(\{a\})$.

Consider now a defined concept name A with the definition $\langle A \doteq C \geq \ell\rangle \in \mathcal{T}$. If $\mathbf{T}_X(u)(A)$ is defined, then by the compatibility with $\mathcal{T}$ the value $\mathbf{T}_X(u)(C)$ is also defined and $\big(\mathbf{T}_X(u)(A) \Rightarrow \mathbf{T}_X(u)(C)\big) \otimes \big(\mathbf{T}_X(u)(C) \Rightarrow \mathbf{T}_X(u)(A)\big) \geq \ell$. Since $o(C) < o(A)$, we get $C^{\mathcal{I}}(X, u) = \mathbf{T}_X(u)(C)$ by induction. Thus, we can define $A^{\mathcal{I}}(X, u) := \mathbf{T}_X(u)(A)$ to ensure that $\mathcal{I}$ satisfies $\langle A \doteq C \geq \ell\rangle$ at (X, u). Whenever $\mathbf{T}_X(u)(A)$ is undefined, we can set $A^{\mathcal{I}}(X, u) := C^{\mathcal{I}}(X, u)$ to satisfy this concept definition without violating the claim.

If $\mathbf{T}_X(u)(\neg C)$ is defined, then $\mathbf{T}_X(u)(C)$ is also defined. By induction, we obtain $(\neg C)^{\mathcal{I}}(X, u) = {\sim}C^{\mathcal{I}}(X, u) = {\sim}\mathbf{T}_X(u)(C) = \mathbf{T}_X(u)(\neg C)$. Similar arguments show Claim 2 for conjunctions and implications.

Assume now that $\ell := \mathbf{T}_X(u)(\exists r.C)$ is defined for $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and consider $i := \varphi(\exists r.C)$. We first prove the existence of an element $(Y, v) \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}((X, u), (Y, v)) \otimes C^{\mathcal{I}}(Y, v) \geq \ell$. By the Hintikka condition, we know that $\mathbf{T}_X(ui)(C)$ is defined and $\ell = \mathbf{T}_X(ui)(r) \otimes \mathbf{T}_X(ui)(C)$. Since u is relevant in $\mathbf{T}_X$, ui can only be irrelevant in $\mathbf{T}_X$ if $\mathsf{Ind}(\mathbf{T}_X(ui)) \neq \emptyset$. We make a case distinction on whether ui is relevant or not.

- If there exists $a \in \mathsf{Ind}(\mathbf{T}_X(ui))$, then by compatibility of $\mathbf{T}_X(ui)$ with $\mathfrak{P}$ the value $\mathbf{T}_{[a]_{\mathcal{P}}}(\varepsilon)(C) = H_{[a]_{\mathcal{P}}}(C) = \mathbf{T}_X(ui)(C)$ is defined. Since the root ε is relevant in $\mathbf{T}_{[a]_{\mathcal{P}}}$, by induction we get $C^{\mathcal{I}}([a]_{\mathcal{P}}, \varepsilon) = \mathbf{T}_{[a]_{\mathcal{P}}}(\varepsilon)(C)$. Since also $r^{\mathcal{I}}((X, u), ([a]_{\mathcal{P}}, \varepsilon)) \geq r^{\mathbf{T}}((X, u), ([a]_{\mathcal{P}}, \varepsilon)) = \mathbf{T}_X(ui)(r)$ and $\otimes$ is monotone, we can choose $(Y, v) := ([a]_{\mathcal{P}}, \varepsilon)$.
- Otherwise, we have $\mathsf{Ind}(\mathbf{T}_X(ui)) = \emptyset$ and $(X, ui) \in \Delta^{\mathcal{I}}$. By induction, this implies that $C^{\mathcal{I}}(X, ui) = \mathbf{T}_X(ui)(C)$, and from the definition of $r^{\mathcal{I}}$ we obtain $r^{\mathcal{I}}((X, u), (X, ui)) \geq r^{\mathbf{T}}((X, u), (X, ui)) = \mathbf{T}_X(ui)(r)$, which allows us to choose $(Y, v) := (X, ui)$.

If we can show that $r^{\mathcal{I}}((X, u), (Z, w)) \otimes C^{\mathcal{I}}(Z, w) \leq \ell$ holds for all $(Z, w) \in \Delta^{\mathcal{I}}$, then we obtain $(\exists r.C)^{\mathcal{I}}(X, u) = \ell$, as desired. By the definition of $r^{\mathcal{I}}$ and since $\otimes$ is join-preserving, it suffices to show that (a) $r^{\mathbf{T}}((X, u), (Z, w)) \otimes C^{\mathcal{I}}(Z, w) \leq \ell$ and (b) $s^{\mathbf{T}}((X, u), (Y_1, v_1), \dots, (Y_n, v_n), (Z, w)) \otimes C^{\mathcal{I}}(Z, w) \leq \ell$ for all transitive roles $s \sqsubseteq_{\mathcal{R}} r$ and $(Y_i, v_i) \in \Delta^{\mathcal{I}}$, $1 \leq i \leq n$, with $n \geq 1$.

(a) We have $\ell = \mathbf{T}_X(u)(\exists r.C) \geq r^{\mathbf{T}}((X, u), (Z, w)) \otimes C^{\mathcal{I}}(Z, w)$ by Claim 1 and induction.

(b) Again, by Claim 1 we have $\ell \geq s^{\mathbf{T}}((X, u), (Y_1, v_1)) \otimes \mathbf{T}_{Y_1}(v_1)(\exists s.C)$, and moreover $\mathbf{T}_{Y_j}(v_j)(\exists s.C) \geq s^{\mathbf{T}}((Y_j, v_j), (Y_{j+1}, v_{j+1})) \otimes \mathbf{T}_{Y_{j+1}}(v_{j+1})(\exists s.C)$ for all j, $1 \leq j \leq n-1$. Also, $\mathbf{T}_{Y_n}(v_n)(\exists s.C) \geq s^{\mathbf{T}}((Y_n,v_n),(Z,w)) \otimes \mathbf{T}_Z(w)(C)$, and thus $\ell \geq s^{\mathbf{T}}((X,u),(Y_1,v_1),\dots,(Y_n,v_n),(Z,w)) \otimes C^{\mathcal{I}}(Z,w)$ by monotonicity of $\otimes$ and induction.

The remaining case of Claim 2 for value restrictions can be shown using similar arguments and a variant of Claim 1.

We have thus defined an interpretation $\mathcal{I}$ that satisfies all concept definitions in $\mathcal{T}$. In the case that $\mathcal{T}$ is a general TBox, consider any GCI $\langle C \sqsubseteq D \geq \ell\rangle \in \mathcal{T}$ and $(X,u) \in \Delta^{\mathcal{I}}$. By the compatibility of $\mathbf{T}_X(u)$ with $\mathcal{T}$, we know that $\mathbf{T}_X(u)(C)$ and $\mathbf{T}_X(u)(D)$ are defined and $\mathbf{T}_X(u)(C) \Rightarrow \mathbf{T}_X(u)(D) \geq \ell$. By Claim 2, we thus have $C^{\mathcal{I}}(X,u) \Rightarrow D^{\mathcal{I}}(X,u) \geq \ell$, which shows that $\mathcal{I}$ satisfies the GCI. Finally, consider an assertion $\langle a{:}C \bowtie \ell\rangle \in \mathcal{A}$. By the compatibility of $H_{[a]_\mathcal{P}}$ with $\mathcal{A}$ (see Definition 4.2), we know that $H_{[a]_\mathcal{P}}(C)$ is defined and $H_{[a]_\mathcal{P}}(C) \bowtie \ell$. By Claim 2, we conclude $C^{\mathcal{I}}(a^{\mathcal{I}}) = C^{\mathcal{I}}([a]_\mathcal{P},\varepsilon) = \mathbf{T}_{[a]_\mathcal{P}}(\varepsilon)(C) = H_{[a]_\mathcal{P}}(C) \bowtie \ell$; that is, $\mathcal{I}$ satisfies the assertion.

Conversely, let $\mathcal{I}$ be a model of $\mathcal{O}$. We define a pre-completion $\mathfrak{P} := (\mathcal{P},\mathcal{H}_\mathcal{P},\mathcal{R}_\mathcal{P})$ for $\mathcal{O}$ based on the partition $\mathcal{P} := \{\{b \in \mathsf{N_I}(\mathcal{O}) \mid a^{\mathcal{I}} = b^{\mathcal{I}}\} \mid a \in \mathsf{N_I}(\mathcal{O})\}$. For all $r \in \mathsf{N_R}(\mathcal{O})$ and $X,Y \in \mathcal{P}$, we set $r_\mathcal{P}(X,Y) := r^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}})$, where (a,b) is an arbitrary element of $X \times Y$. Similarly, we set $H_X(r) := \mathbf{0}$ for every $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$ and $H_X(C) := C^{\mathcal{I}}(a^{\mathcal{I}})$ for every $C \in \mathsf{sub}(\mathcal{O})$ to define the family $\mathcal{H}_\mathcal{P} = (H_X)_{X\in\mathcal{P}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this obviously defines Hintikka functions that are compatible with $\mathcal{T}$ and $\mathcal{R}$, and we also have $\mathsf{Ind}(H_X) = X$ for every $X \in \mathcal{P}$. Furthermore, for every $\langle a{:}C \bowtie \ell\rangle \in \mathcal{A}$, we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \ell$, and thus $H_{[a]_\mathcal{P}}(C) \bowtie \ell$, which shows that $\mathfrak{P}$ is indeed a pre-completion for $\mathcal{O}$.

For a given $X \in \mathcal{P}$, we now define the Hintikka tree $\mathbf{T}_X$ starting with H_X by inductively constructing a mapping $g_X\colon K^* \to \Delta^{\mathcal{I}}$ that specifies which elements of $\Delta^{\mathcal{I}}$ represent the nodes of $\mathbf{T}_X$ and satisfies the following property:

Claim 3. *For all $u \in K^*$, $C \in \mathsf{sub}(\mathcal{O})$, $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$, and $i \in K$, we have $\mathbf{T}_X(u)(C) = C^{\mathcal{I}}(g_X(u))$ and $\mathbf{T}_X(ui)(r) = r^{\mathcal{I}}(g_X(u),g_X(ui))$.*

This in particular ensures that all constructed Hintikka functions are compatible with $\mathcal{T}$, $\mathcal{R}$, and $\mathfrak{P}$.

We start the construction by setting $\mathbf{T}_X(\varepsilon) := H_X$ and $g_X(\varepsilon) := a^{\mathcal{I}}$ for an arbitrary $a \in X$. Thus, $\mathbf{T}_X$ starts with H_X and Claim 3 is satisfied at ε by the definition of H_X above. Let now $u \in K^*$ be a node for which $\mathbf{T}_X$ and g_X have already been defined while satisfying Claim 3, and consider any $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and $i := \varphi(\exists r.C)$. Since $\mathcal{I}$ is witnessed, there must be a $y \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(g_X(u)) = r^{\mathcal{I}}(g_X(u),y) \otimes C^{\mathcal{I}}(y)$. We now set $g_X(ui) := y$, $\mathbf{T}_X(ui)(s) := s^{\mathcal{I}}(g_X(u),y)$ for all $s \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$, and $\mathbf{T}_X(ui)(C) := C^{\mathcal{I}}(y)$ for all $C \in \mathsf{sub}(\mathcal{O})$ to satisfy Claim 3 at ui. Likewise, for any $\forall r.C \in \mathsf{sub}(\mathcal{O})$ there must be a $y \in \Delta^{\mathcal{I}}$ with $(\forall r.C)^{\mathcal{I}}(g_X(u)) = r^{\mathcal{I}}(g_X(u),y) \Rightarrow C^{\mathcal{I}}(y)$, and we proceed as above to define $\mathbf{T}_X$ and g_X at ui for $i := \varphi(\forall r.C)$.

We now show that every tuple $(\mathbf{T}_X(u), \mathbf{T}_X(u1), \dots, \mathbf{T}_X(uk))$, $u \in K^*$, satisfies the Hintikka condition. The first point of Condition a) from Definition 4.3 is obviously satisfied by the above construction. Consider now any $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and $i \in K$. By Claim 3 and the semantics of existential restrictions, we obtain

$$\begin{aligned}\mathbf{T}_X(u)(\exists r.C) &= (\exists r.C)^{\mathcal{I}}(g_X(u)) \\ &\geq r^{\mathcal{I}}(g_X(u), g_X(ui)) \otimes C^{\mathcal{I}}(g_X(ui)) \\ &= \mathbf{T}_X(ui)(r) \otimes \mathbf{T}_X(ui)(C),\end{aligned}$$

and, for all transitive roles $s \sqsubseteq_{\mathcal{R}} r$,

$$\begin{aligned}\mathbf{T}_X(u)(\exists r.C) &= (\exists r.C)^{\mathcal{I}}(g_X(u)) \\ &= \bigvee_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(g_X(u), y) \otimes C^{\mathcal{I}}(y) \\ &\geq \bigvee_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(g_X(u), y) \otimes C^{\mathcal{I}}(y) \\ &\geq \bigvee_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(g_X(u), g_X(ui)) \otimes s^{\mathcal{I}}(g_X(ui), y) \otimes C^{\mathcal{I}}(y) \\ &= s^{\mathcal{I}}(g_X(u), g_X(ui)) \otimes (\exists s.C)^{\mathcal{I}}(g_X(ui)) \\ &= \mathbf{T}_X(ui)(s) \otimes \mathbf{T}_X(ui)(\exists s.C).\end{aligned}$$

The remaining part of a) and b) can be shown by similar arguments. For c), consider $u \in K^*$, $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$, $i, j \in K$, $a \in \mathsf{Ind}(\mathbf{T}_X(ui))$, and $b \in \mathsf{Ind}(\mathbf{T}_X(uj))$ with $[a]_\mathcal{P} = [b]_\mathcal{P}$. Then Claim 3 yields $g_X(ui) = a^{\mathcal{I}} = b^{\mathcal{I}} = g_X(uj)$, and thus $\mathbf{T}_X(ui)(r) = r^{\mathcal{I}}(g_X(u), a^{\mathcal{I}}) = \mathbf{T}_X(uj)(r)$. For d), let $u \in K^*$, $a \in \mathsf{Ind}(\mathbf{T}_X(u))$, $r \in \mathsf{N}_\mathsf{R}^-(\mathcal{O})$, $i \in K$, and $b \in \mathsf{Ind}(\mathbf{T}_X(ui))$. By Claim 3, $g_X(u) = a^{\mathcal{I}}$, $g_X(ui) = b^{\mathcal{I}}$, and $\mathbf{T}_X(ui)(r) = r^{\mathcal{I}}(g_X(u), g_X(ui)) = r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = r_\mathcal{P}([a]_\mathcal{P}, [b]_\mathcal{P})$. □

Given a pre-completion $\mathfrak{P} = (\mathcal{P}, \mathcal{H}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ for $\mathcal{O}$ and $X \in \mathcal{P}$, the *Hintikka automaton* for $\mathcal{O}$ and H_X is the looping automaton $\mathsf{A}_{\mathcal{O},H_X} := (Q_\mathcal{O}, I_{H_X}, \Delta_\mathcal{O})$, where $Q_\mathcal{O}$ consists of all pairs (H, i) of Hintikka functions H for $\mathcal{O}$ that are compatible with $\mathcal{T}$, $\mathcal{R}$, and $\mathfrak{P}$ and indices $i \in K$, $I_{H_X} := \{(H_X, 1)\}$, and $\Delta_\mathcal{O}$ is the set of all tuples $((H_0, i_0), (H_1, 1), \dots, (H_k, k))$ such that $(H_0, \dots, H_k)$ satisfies the Hintikka condition. It is easy to see that the first components of the runs of $\mathsf{A}_{\mathcal{O},H_X}$ are exactly the Hintikka trees for $\mathcal{O}$ starting with H_X, and the second components simply store the index of the existential or value restriction for which the state acts as a witness. By Lemma 4.4, consistency of $\mathcal{O}$ is thus equivalent to the existence of a pre-completion and the non-emptiness of the Hintikka automata $\mathsf{A}_{\mathcal{O},H_X}$ for each equivalence class X.

Since the number of pre-completions is bounded exponentially in the size of the input ($\mathcal{O}$ and L) and each pre-completion is of size polynomial in the size of the input, we can enumerate all pre-completions in exponential time and for each of them check emptiness of the polynomially many automata $\mathsf{A}_{\mathcal{O},H_X}$. Since the size of these automata is exponential in the size of the input, by [32] we obtain the following complexity result. ExpTime-hardness holds already in $\mathcal{ALC}$ [25].

Theorem 4.5. *In L-$\mathfrak{I}\mathcal{SCHOI}$ over a finite residuated De Morgan lattice L, consistency w.r.t. general TBoxes is* ExpTime*-complete.*

5 Acyclic TBoxes

We now extend the previous complexity results for lattice-based fuzzy DLs with acyclic TBoxes [12, 13] by showing that consistency in L-$\mathfrak{I}\mathcal{ALCHO}$ and L-$\mathfrak{I}\mathcal{SCO}_\mathsf{c}$ is PSpace-complete in this setting. Recall that in L-$\mathfrak{I}\mathcal{SCO}_\mathsf{c}$, roles must always be interpreted as *crisp* functions that only take the values $\mathbf{0}$ and $\mathbf{1}$. Due to the absence of inverse roles, in the following we can restrict all definitions to use $\mathsf{N}_\mathsf{R}(\mathcal{O})$ instead of $\mathsf{N}_\mathsf{R}^-(\mathcal{O})$, and we can remove Condition d) and the last items of Conditions a) and b) from Definition 4.3.

Let now $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be such that $\mathcal{T}$ is acyclic. We can guess a triple $\mathfrak{P} = (\mathcal{P}, \mathcal{H}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ and verify the conditions of Definition 4.2 in (nondeterministic) polynomial space. Thus, if emptiness of the polynomially many Hintikka automata $\mathsf{A}_{\mathcal{O},H_X}$ could be decided in polynomial space, we would obtain a PSpace upper bound for consistency [24]. The idea is to modify the construction of $\mathsf{A}_{\mathcal{O},H_X}$ using a faithful family of functions to obtain a PSpace on-the-fly construction. As in [12], these automata already satisfy most of Definition 2.5, except the polynomial bound on the maximal length a path before (equality) blocking occurs. The faithful families of functions we use are very similar to those employed in [12] for L-$\mathfrak{I}\mathcal{ALCHI}$ and L-$\mathfrak{I}\mathcal{SCI}_\mathsf{c}$.

For the subsequent constructions to work, we need to change the notion of compatibility of a Hintikka function H with $\mathfrak{P}$ to a weaker variant: we only require that for every $a \in \mathsf{Ind}(H)$ and every $C \in \mathsf{sub}(\mathcal{O})$ for which $H(C)$ is defined, $H_{[a]_\mathcal{P}}(C)$ is also defined and $H(C) = H_{[a]_\mathcal{P}}(C)$. This new definition does not work in the presence of inverse roles. However, in L-$\mathfrak{I}\mathcal{ALCHO}$ and L-$\mathfrak{I}\mathcal{SCO}_\mathsf{c}$, all previous results remain valid. The only changes necessary are in two places of the proof of Lemma 4.4, belonging to the proofs of Claims 1 and 2 for existential (and value) restrictions. In both cases, it suffices to infer from $a \in \mathsf{Ind}(H)$ and $C \in \mathsf{supp}(H)$ that also $C \in \mathsf{supp}(H_{[a]_\mathcal{P}})$ and $H(C) = H_{[a]_\mathcal{P}}$, which is precisely the new definition given above.

5.1 L-$\mathfrak{I}\mathcal{ALCHO}$

We now present a faithful family of functions for the case that $\mathcal{O}$ is formulated in L-$\mathfrak{I}\mathcal{ALCHO}$. For this, we denote by $\mathsf{rd}_\mathcal{T}(C)$ the role depth of the unfolding of a concept C w.r.t. the acyclic TBox $\mathcal{T}$, by $\mathsf{rd}_{\mathcal{T}}(H)$ for a Hintikka function H the maximal $\mathsf{rd}_{\mathcal{T}}(C)$ of a concept $C \in \mathsf{supp}(H)$, and by $\mathsf{sub}^{\leq n}(\mathcal{O})$ the restriction of $\mathsf{sub}(\mathcal{O})$ to concepts C with $\mathsf{rd}_{\mathcal{T}}(C) \leq n$.

Definition 5.1 (family $\mathfrak{f}$)**.** *We define* $\mathfrak{f} = (f_q)_{q \in Q_{\mathcal{O}}}$ *for all* $q = (H, i) \in Q_{\mathcal{O}}$ *with* $n := \mathsf{rd}_{\mathcal{T}}(H)$ *and all* $q' = (H', i') \in Q_{\mathcal{O}}$ *by* $f_q(q') := (H'', i')$*, where, for every* $C \in \mathsf{sub}(\mathcal{O})$ *and* $r \in \mathsf{N_R}(\mathcal{O})$*,*

$$H''(C) := \begin{cases} H'(C) & \textit{if } C \in \mathsf{sub}^{\leq n-1}(\mathcal{O}), \\ \textit{undefined} & \textit{otherwise;} \end{cases}$$

$$H''(r) := \begin{cases} H'(r) & \textit{if } \mathsf{supp}(H) \neq \emptyset, \\ \mathbf{0} & \textit{otherwise.} \end{cases}$$

For all $q, q' \in Q_{\mathcal{O}}$, we have that $f_q(q')$ is again a state of $\mathsf{A}_{\mathcal{O},H_X}$ (according to the new definition of compatibility with $\mathfrak{P}$). The idea of this definition is to reduce the maximal role depth of the Hintikka function in every transition of the automaton.

Lemma 5.2. *In L-*$\mathfrak{I}\mathcal{ALCHO}$*, the family* $\mathfrak{f}$ *is faithful w.r.t.* $\mathsf{A}_{\mathcal{O},H_X}$*.*

Proof. Consider states $q = (H, i)$, $q_0 = (H_0, i_0)$, and $q_j = (H_j, j)$, $1 \leq j \leq k$, and define $n := \mathsf{rd}_{\mathcal{T}}(H)$, $q'_0 := (H'_0, i_0) := f_q(q_0)$, and $q'_j := (H'_j, j) := f_q(q_j)$ for each j, $1 \leq j \leq k$. Assuming that $(H, H_1, \ldots, H_k)$ satisfies the Hintikka condition, we have to verify it for $(H, H'_1, \ldots, H'_k)$. Note that we consider neither inverse nor transitive roles, and thus half of this condition is vacuous.

For a), consider any $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and $j \in K$. If $\exists r.C \in \mathsf{supp}(H)$, then $\mathsf{rd}_{\mathcal{T}}(C) < \mathsf{rd}_{\mathcal{T}}(\exists r.C) \leq \mathsf{rd}_{\mathcal{T}}(H)$. Since $H_j(C)$ is defined, we have $H'_j(C) = H_j(C)$. Furthermore, $\mathsf{supp}(H) \neq \emptyset$, and thus $H'_j(r) = H_j(r)$, which shows that the required (in)equalities remain satisfied after applying f_q. Similar arguments can be used for b). For c), let $r \in \mathsf{N_R}(\mathcal{O})$ and $j_1, j_2 \in K$. If there are $a \in \mathsf{Ind}(H'_{j_1})$ and $b \in \mathsf{Ind}(H'_{j_2})$ with $[a]_{\mathcal{P}} = [b]_{\mathcal{P}}$, this must already have been true for H_{j_1} and H_{j_2}. Since $\mathsf{supp}(H)$ cannot be empty, we have $H'_{j_1}(r) = H_{j_1}(r) = H_{j_2}(r) = H'_{j_2}(r)$.

For the second condition of Definition 2.3, we show that $(H'_0, H'_1, \ldots, H'_k)$ satisfies the Hintikka condition whenever $(H_0, H_1, \ldots, H_k)$ does. For all $\exists r.C \in \mathsf{supp}(H'_0)$ and $j \in K$, we have $H_0(\exists r.C) = H'_0(\exists r.C)$, $\mathsf{rd}_{\mathcal{T}}(C) < \mathsf{rd}_{\mathcal{T}}(\exists r.C) < \mathsf{rd}_{\mathcal{T}}(H)$, and $\mathsf{supp}(H)$. Thus, we get $H'_j(C) = H_j(C)$ and $H'_j(r) = H_j(r)$ as before. The remaining conditions follow from the same arguments as above. □

It remains to show that emptiness of the induced subautomaton $\mathsf{A}^{\mathfrak{f}}_{\mathcal{O},H_X}$ can be decided in PSpace. For the following result, we use the equality on $Q_{\mathcal{O}}$ as the blocking relation.

Lemma 5.3. *In L-*$\mathfrak{I}\mathcal{ALCHO}$*, the construction of* $\mathsf{A}^{\mathfrak{f}}_{\mathcal{O},H_X}$ *from* L*,* $\mathcal{O}$*, and* H_X *is a* PSpace *on-the-fly construction.*

Proof. We show that $\mathsf{A}^{\mathfrak{f}}_{\mathcal{O},H_X}$ is polynomially blocking (with equality as blocking relation). Consider any path in a run of this automaton. Since the maximal role depth of the Hintikka functions is decreased in each transition, after at most $m := \max\{\mathsf{rd}_{\mathcal{T}}(C) \mid C \in \mathsf{sub}(\mathcal{O})\} + 1$ transitions, we must reach a state (H, i) with $\mathsf{supp}(H) = \emptyset$. From the next transition on, the first component of each state additionally assigns $\mathbf{0}$ to all role names. Thus, after $m + k + 2$ transitions, we have seen at least one state twice. This number is linear in the size of $\mathcal{O}$. □

Propositions 2.4 and 2.6 yield the desired complexity result. PSpace-hardness holds already in classical $\mathcal{ALC}$ w.r.t. the empty TBox [26].

Theorem 5.4. *In L-$\mathfrak{I}\mathcal{ALCHO}$ over a finite residuated De Morgan lattice L, consistency w.r.t. acyclic TBoxes is* PSpace*-complete.*

5.2 L-$\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}$

For L-$\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}$, the construction is a little more involved. Since now the interpretations of roles are restricted to $\mathbf{0}$ and $\mathbf{1}$, all Hintikka functions H for $\mathcal{O}$ need to satisfy the additional condition that $H(r) \in \{\mathbf{0}, \mathbf{1}\}$ for all $r \in \mathsf{N_R}(\mathcal{O})$. We further denote by $\varphi_r(\mathcal{O})$ for $r \in \mathsf{N_R}(\mathcal{O})$ the set of all indices $i \in K$ such that $i = \varphi(C)$ for a concept C of the form $\exists r.D$ or $\forall r.D$. We then replace K in Definition 4.3 by $\varphi_r(\mathcal{O})$. The idea is that in the absence of role inclusions it suffices to consider one role for each successor. The resulting definition is closer to the Hintikka condition from [12].

Lemma 4.4 remains valid under these modifications. Again, it is only necessary to change the proof of the "if" direction. In particular, in the definition of $r^{\mathbf{T}}$ we have to replace the first occurrence of K by $\varphi_r(\mathcal{O})$, and the second one by $\varphi_{r^-}(\mathcal{O})$. Moreover, all following references to K have to be changed to $\varphi_r(\mathcal{O})$ or $\varphi_{r^-}(\mathcal{O})$ as appropriate.

Given a Hintikka function H for $\mathcal{O}$ and a role name r, we define the sets

$$\begin{aligned} H|_r &:= \{C \in \mathsf{supp}(H) \mid C = \exists r.D \text{ or } C = \forall r.D\}, \\ H^{-r} &:= \{C \in \mathsf{supp}(H) \mid \exists r.C \text{ or } \forall r.C \in \mathsf{sub}(\mathcal{O})\}. \end{aligned}$$

Definition 5.5 (family $\mathfrak{g}$). *We define $\mathfrak{g} = (g_q)_{q \in Q_{\mathcal{O}}}$ for all $q = (H, i) \in Q_{\mathcal{O}}$ with $n := \mathsf{rd}_{\mathcal{T}}(H)$ and all $q' = (H', i') \in Q_{\mathcal{O}}$ and $r' \in \mathsf{N_R}(\mathcal{O})$ such that $i' \in \varphi_{r'}(\mathcal{O})$ by*

$g_q(q') := (H'', i')$, where, for all $C \in \mathsf{sub}(\mathcal{O})$ and $r \in \mathsf{N_R}(\mathcal{O})$:

$$P := \begin{cases} \mathsf{sub}^{\leq n}(\mathcal{O}) \cap H'|_{r'} & \text{if } r' \text{ is transitive,} \\ \emptyset & \text{otherwise;} \end{cases}$$

$$H''(C) := \begin{cases} H'(C) & \text{if } C \in \mathsf{sub}^{\leq n-1}(\mathcal{O}) \cup P, \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$H''(r) := \begin{cases} H'(r) & \text{if } \mathsf{supp}(H) \neq \emptyset \text{ and } r = r', \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Again, the resulting pair (H'', i') is an element of $Q_{\mathcal{O}}$. In contrast to the previous section, we cannot always reduce the role depth of the Hintikka functions, but have to keep some restrictions over transitive roles.

Lemma 5.6. *In $L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}$, the family $\mathfrak{g}$ is faithful w.r.t. $\mathsf{A}_{\mathcal{O},H_X}$.*

Proof. Let $q = (H, i)$, $q_0 = (H_0, i_0)$, $q_j = (H_j, j)$, $q_0' := (H_0', i_0) := g_q(q_0)$, and $q_j' := (H_j', j) := g_q(q_j)$, $1 \leq j \leq k$, be states of $\mathsf{A}_{\mathcal{O},H_X}$. We let $n := \mathsf{rd}_{\mathcal{T}}(H)$ and r_j be the unique role name with $j \in \varphi_{r_j}(\mathcal{O})$, $1 \leq j \leq k$. We assume that $(q, q_1, \dots, q_k) \in \Delta_{\mathcal{O}}$, and verify that then also $(q, q_1', \dots, q_k') \in \Delta_{\mathcal{O}}$.

For Condition a) of Definition 4.3, consider any $\exists r.C \in \mathsf{supp}(H)$. This implies that $\mathsf{rd}_{\mathcal{T}}(\exists r.C) \leq \mathsf{rd}_{\mathcal{T}}(H) = n$ and $\mathsf{supp}(H) \neq \emptyset$. For every $j \in \varphi_r(\mathcal{O})$, we thus have $H_j'(C) = H_j(C)$ and $H_j'(r) = H_j(r)$. This shows that the equality and the first inequality are still satisfied. Consider now a transitive role $s \sqsubseteq_{\mathcal{R}} r$, which must be equal to r since $\mathcal{O}$ does not contain any role inclusions. By the Hintikka condition, we have $\exists r.C \in \mathsf{supp}(H_j)$, and thus $\exists r.C \in H_j|_r$ and $H_j'(\exists r.C) = H_j(\exists r.C)$, which proves the final inequality. Condition b) can be shown by similar arguments. For c), let $r \in \mathsf{N_R}(\mathcal{O})$, $j_1, j_2 \in \varphi_r(\mathcal{O})$, $a \in \mathsf{Ind}(H_{j_1})$, and $b \in \mathsf{Ind}(H_{j_2})$ with $[a]_{\mathcal{P}} = [b]_{\mathcal{P}}$. Since $\mathsf{supp}(H) \neq \emptyset$, we have $H_{j_1}'(r) = H_{j_1}(r) = H_{j_2}(r) = H_{j_2}'(r)$ by the Hintikka condition.

The proof of the second condition of Definition 2.3 is analogous. □

To prove the counterpart of Lemma 5.3 for $L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}$, we use the blocking relation $\hookleftarrow_{L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}}$ on $Q_{\mathcal{O}}$ defined by $(H, i) \hookleftarrow_{L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}} (H', i')$ iff

A. $i = i' = \varphi(E)$ for $E \in \mathsf{sub}(\mathcal{O})$ of the form $\exists r.F$ or $\forall r.F$;

B. $\mathsf{Ind}(H) = \mathsf{Ind}(H') = \emptyset$ or there is some $X \in \mathcal{P}$ such that $\mathsf{Ind}(H) \cap X \neq \emptyset$ and $\mathsf{Ind}(H') \cap X \neq \emptyset$; and

C. one of the following alternatives holds:

   i. $H = H'$;

ii. $H(r) = H'(r) = \mathbf{0}$ and $H|_r \cup H^{-r} = H'|_r \cup H'^{-r}$; or

iii. 1. r is transitive, $H(r) = H'(r) = \mathbf{1}$, $H(F) = H'(F)$,
    2. $H(C) = H'(C)$ for every concept C in $\mathcal{Q}(H, H', r) := H|_r \cup H'|_r$, and
    3. $H'(C) \leq H'(\exists r.C)$ for every $\exists r.C \in H'|_r$ and $H'(C) \geq H'(\forall r.C)$ for every $\forall r.C \in H'|_r$.

This is an extended version of the blocking relation used for $L\text{-}\mathfrak{I}\mathcal{SCI}_{\mathsf{c}}$ in [12].

We now verify that $\mathsf{A}^{\mathfrak{g}}_{\mathcal{O},H_X}$ is $\hookleftarrow_{L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}}$-invariant. Condition B ensures that Condition c) of Definition 4.3 remains satisfied, and thus we only need to consider the influence of C.i–C.iii on a) (for b) the arguments are similar):

i. The equality relation is always invariant.

ii. The (in)equalities of the Hintikka condition remain satisfied when replacing one successor H with $H(r) = \mathbf{0}$ by an H' that also satisfies $H'(r) = \mathbf{0}$. Thus, H' only needs to be defined for all relevant concepts, which is expressed by the second part of this condition.

iii. Condition 1 ensures that the first equality is still satisfied. Condition 2 restricts all existential restrictions that are transferred by r to be evaluated by identical values, and thus the second inequality remains satisfied. Finally, Condition 3 yields the first inequality: Since $H_0(\exists r.C) \geq H'(r) \otimes H'(\exists r.C)$ and $H'(\exists r.C) \geq H'(C)$, it follows that also $H_0(\exists r.C) \geq H'(r) \otimes H'(C)$.

It remains to show that these definitions ensure polynomial blocking.

Lemma 5.7. *In $L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}$, the construction of $\mathsf{A}^{\mathfrak{g}}_{\mathcal{O},H_X}$ from L, $\mathcal{O}$, and H_X is a* PSPACE *on-the-fly construction.*

Proof. We show that the automata are polynomially blocking w.r.t. $\hookleftarrow_{L\text{-}\mathfrak{I}\mathcal{SCO}_{\mathsf{c}}}$. Consider three consecutive states (H_0, i_0), (H_1, i_1), (H_2, i_2) of a path in a run of $\mathsf{A}^{\mathfrak{g}}_{\mathcal{O},H_X}$, and let r_j be such that $i_j \in \varphi_{r_j}(\mathcal{O})$, $0 \leq j \leq 2$. By the definition of $g_{(H,i)}$, we have $\mathsf{rd}_{\mathcal{T}}(H_0) \geq \mathsf{rd}_{\mathcal{T}}(H_1) \geq \mathsf{rd}_{\mathcal{T}}(H_2)$. If r_1 is not transitive, then $\mathsf{rd}_{\mathcal{T}}(H_0) > \mathsf{rd}_{\mathcal{T}}(H_1)$. Furthermore, if $r_1 \neq r_2$, then $\mathsf{rd}_{\mathcal{T}}(H_0) > \mathsf{rd}_{\mathcal{T}}(H_2)$, whether r_1 and r_2 are transitive or not. Thus, after $\max\{\mathsf{rd}_{\mathcal{T}}(C) \mid C \in \mathsf{sub}(\mathcal{O})\} + 1$ transitions using non-transitive roles or different consecutive roles we must reach a state (H, i) where $\mathsf{supp}(H)$ is empty.

However, if $r_1 = r_2$ is transitive, then the role depth need not decrease. By the Hintikka condition, we know that $H_1|_{r_1} \subseteq H_2|_{r_1}$ and $H_1^{-r_1} \subseteq H_2^{-r_1}$. Thus, there can be at most $2 \cdot |\mathsf{sub}(\mathcal{O})|$ many transitions using the same transitive role r_1 with $H(r_1) = \mathbf{0}$ without triggering Condition C.ii of the blocking relation.

Finally, if $H_1(r_1) > \mathbf{0}$, then we must have $H_1(r_1) = \mathbf{1}$. Thus, by the Hintikka condition we have $H_0(\exists r_1.C) \geq H_1(r_1) \otimes H_1(\exists r_1.C) = H_1(\exists r_1.C)$ for all $\exists r_1.C \in \mathsf{supp}(H_0)$, and dually for all value restrictions over r_1. Hence, after at most $|L||\mathsf{sub}(\mathcal{O})|$ transitions with the transitive role r_1 to degree $\mathbf{1}$, the values of all concepts in $H|_{r_1}$ remain fixed (cf. Condition C.iii.2). For the next transition, we have $H_1(\exists r_1.C) = H_0(\exists r_1.C) \geq H_1(r_1) \otimes H_1(C) = H_1(C)$, and thus Condition C.iii.3 is also satisfied.

An additional number of $k|L|(|\mathcal{P}| + 1)$ transitions ensure that the we find two states (H, i), (H', i') that also satisfy the remaining Conditions A and B of $\hookleftarrow_{L\text{-}\mathfrak{I}\mathcal{SCO}_\mathsf{c}}$.

In total, every path longer than $2(|L|+1)(|\mathsf{sub}(\mathcal{O})|+1)^4$ must contain two nodes that are in the blocking relation. This number is polynomial in the size of the input. $\square$

Propositions 2.4 and 2.6 and [26] now entail the following result.

Theorem 5.8. *In $L\text{-}\mathfrak{I}\mathcal{SCO}_\mathsf{c}$ over a finite residuated De Morgan lattice L, consistency w.r.t. acyclic TBoxes is* PSPACE*-complete.*

As a side effect, we obtain new, albeit not surprising, complexity results for the underlying classical description logics.

Corollary 5.9. *In classical $\mathcal{ALCHO}$ and $\mathcal{SO}$, consistency w.r.t. acyclic TBoxes is* PSPACE*-complete.*

6 Conclusions

We have extended previous complexity results about fuzzy DLs with finite lattice semantics to cover nominals. This required extensive adaptations of the automata-based algorithm used for $L\text{-}\mathfrak{I}\mathcal{SCHI}$ and its sublogics in [12]. We employed pre-completions similar to those in [2, 13, 18] to show complexity results for ontology consistency. Due to the expressivity of our ABoxes, these easily transfer to other standard reasoning problems. In particular, we have shown that consistency in $L\text{-}\mathfrak{I}\mathcal{SCHOI}$ w.r.t. general TBoxes can be decided in EXPTIME. This drops to PSPACE when restricting to acyclic TBoxes in the sublogics $L\text{-}\mathfrak{I}\mathcal{ALCHO}$ and $L\text{-}\mathfrak{I}\mathcal{SCO}_\mathsf{c}$. To the best of our knowledge, only the sublogics $\mathcal{SI}$ [1, 20] and $\mathcal{ALCHI}$ [12, 13] of classical $\mathcal{SHOI}$ were known to have PSPACE-complete reasoning problems w.r.t. acyclic TBoxes. On the other hand, in $\mathcal{ALCOI}$ and $\mathcal{SH}$ reasoning is already EXPTIME-hard without any TBox [19, 30]. The present results for $\mathcal{ALCHO}$ and $\mathcal{SO}$ thus complete the picture about reasoning w.r.t. acyclic TBoxes in the logics between $\mathcal{ALC}$ and $\mathcal{SHOI}$ (see Figure 1).

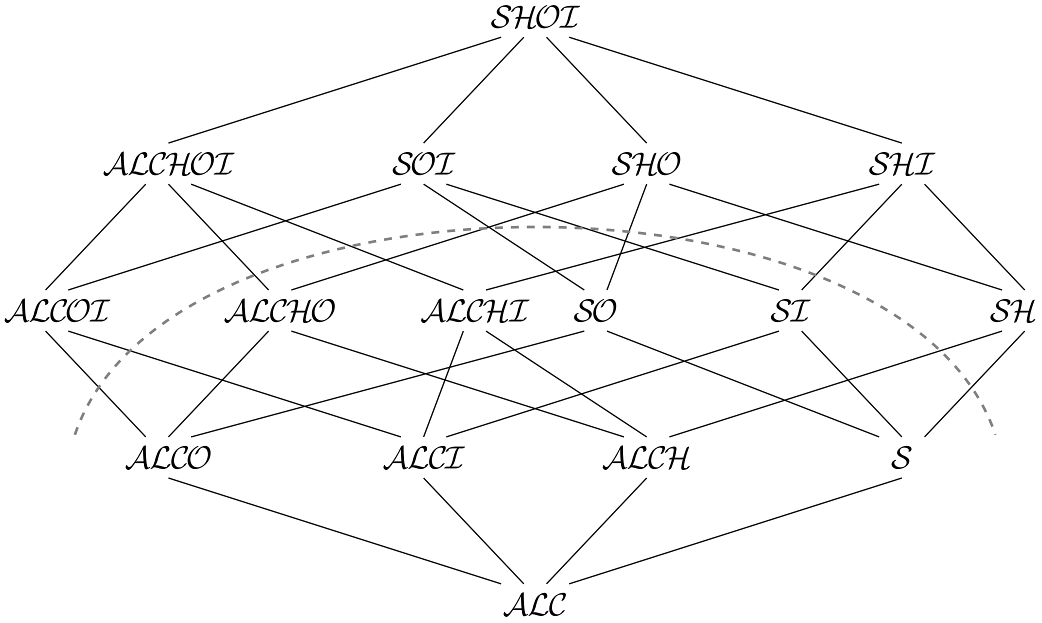

Figure 1: The PSpace/ExpTime boundary in classical DLs with acyclic TBoxes

It would be interesting to extend the presented results to deal with fuzzy role inclusions ($\langle r \sqsubseteq s \geq p \rangle$) or cardinality restrictions ($\geq n\, r.C$), although it is not clear how to define the semantics of the latter in a setting where already a simple existential restriction may entail the existence of $n > 1$ witnessing role successors. We also plan to extend the automata-based algorithm for the fuzzy DL $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ based on the so-called Gödel t-norm over the truth degrees from $[0,1]$ to more expressive logics using the ideas presented here and in [12, 13].

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