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Complexity Theory

Exercise 7: Diagonalization

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Exercise 7.1. Show the following.

1. $\textsc{Time}(2^n) = \textsc{Time}(2^{n+1})$
2. $\textsc{Time}(2^n) \subset \textsc{Time}(2^{2n})$
3. $\text{DT}\textsc{ime}(n) \subset \text{PS}\textsc{pace}$

Exercise 7.2. Find the fault in the following proof of $\text{P} \neq \text{NP}$:

Assume that $\text{P} = \text{NP}$. Then **SAT** $\in \text{P}$ and thus there exists a $k \in \mathbb{N}$ such that **SAT** $\in \text{DT}\textsc{ime}(n^k)$. Because every language in NP is polynomial-time reducible to **SAT** we have $\text{NP} \subseteq \text{DT}\textsc{ime}(n^k)$. It follows that $\text{P} \subseteq \text{DT}\textsc{ime}(n^k)$. But by the Time Hierarchy Theorem there exist languages in $\text{DT}\textsc{ime}(n^{k+1})$ that are not in $\text{DT}\textsc{ime}(n^k)$, contradicting $\text{P} \subseteq \text{DT}\textsc{ime}(n^k)$. Therefore, $\text{P} \neq \text{NP}$.

Exercise 7.3. Consider the function $\mathsf{pad}\colon \Sigma^* \times \mathbb{N} \to \Sigma^*\#^*$ defined as $\mathsf{pad}(s, \ell) = s\#^j$, where $j = \max(0, \ell - |s|)$. In other words, $\mathsf{pad}(s, \ell)$ adds enough copies of $\#$ to the end of s so that the length is at least ℓ.

For some language $\mathbf{A} \subseteq \Sigma^*$ and $f\colon \mathbb{N} \to \mathbb{N}$ define $\mathsf{pad}(\mathbf{A}, f) = \{\, \mathsf{pad}(s, f(|s|)) \mid s \in \mathbf{A} \,\}$.

1. Show that if $\mathbf{A} \in \text{DT}\textsc{ime}(n^6)$, then $\mathsf{pad}(\mathbf{A}, n^2) \in \text{DT}\textsc{ime}(n^3)$.
2. Show that if $\text{NE}\textsc{xp}\text{T}\textsc{ime} \neq \text{E}\textsc{xp}\text{T}\textsc{ime}$, then $\text{P} \neq \text{NP}$.
3. Show for every $\mathbf{A} \subseteq \Sigma^*$ and every $k \in \mathbb{N}$ that $\mathbf{A} \in \text{P}$ if and only if $\mathsf{pad}(\mathbf{A}, n^k) \in \text{P}$.
4. Show that $\text{P} \neq \text{DS}\textsc{pace}(n)$.
5. Show that $\text{NP} \neq \text{DS}\textsc{pace}(n)$.

Exercise 7.4. Show that there exists a function that is not time-constructible.