Complexity Theory NP-Complete Problems

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Computational Logic

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Review

Further NP-complete Problems

Towards More NP-Complete Problems

Starting with SAT, one can readily show more problems \mathcal{P} to be NP-complete, each time performing two steps:

- (1) Show that $\mathcal{P} \in NP$
- (2) Find a known NP-complete problem \mathcal{P}' and reduce $\mathcal{P}' \leq_p \mathcal{P}$

Thousands of problem have now been shown to be NP-complete. (See Garey and Johnson for an early survey)

In this course:

 $\leq_{p} \text{Clique} \leq_{p} \text{Independent Set}$ Sat $\leq_{p} 3$ -Sat $\leq_{p} \text{Dir. Hamiltonian Path}$ $\leq_{p} \text{Subset Sum} \leq_{p} \text{Knapsack}$

DIRECTED HAMILTONIAN PATH

Input: A directed graph G.

Problem: Is there a directed path in *G* containing every vertex exactly once?

Theorem 9.1

DIRECTED HAMILTONIAN PATH *is* NP*-complete*.

DIRECTED	HAMILTONIAN	Ратн
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DIRECTED HAMILTONIAN PATH *is* NP*-complete*.

Proof.

• DIRECTED HAMILTONIAN PATH $\in NP$: Take the path to be the certificate.

Digression: How to design reductions

Task: Show that problem \mathcal{P} (DIR. HAMILTONIAN PATH) is NP-hard.

Arguably, the most important part is to decide where to start from.

That is, which problem to reduce to DIRECTED HAMILTONIAN PATH?

- Considerations:
 - ► Is there an NP-complete problem similar to *P*? (for example, CLIQUE and INDEPENDENT SET)
 - It is not always beneficial to choose a problem of the same type

(for example, reducing a graph problem to a graph problem)

- For instance, CLIQUE, INDEPENDENT SET are "local" problems (is there a set of vertices inducing some structure)
- Hamiltonian Path is a global problem (find a structure – the Hamiltonian path – containing all vertices)

Digression: How to design reductions

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- Considerations:
 - ► Is there an NP-complete problem similar to *P*? (for example, CLIQUE and INDEPENDENT SET)
 - It is not always beneficial to choose a problem of the same type

(for example, reducing a graph problem to a graph problem)

- ► For instance, CLIQUE, INDEPENDENT SET are "local" problems (is there a set of vertices inducing some structure)
- Hamiltonian Path is a global problem (find a structure – the Hamiltonian path – containing all vertices)
- How to design the reduction:
 - Does your problem come from an optimisation problem? If so: a maximisation problem? a minimisation problem?
 - Learn from examples, have good ideas.

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Proof.

- DIRECTED HAMILTONIAN PATH $\in NP$: Take the path to be the certificate.
- ► DIRECTED HAMILTONIAN PATH IS NP-hard: 3-Sat ≤_D Directed Hamiltonian Path

Proof idea: (see blackboard for details) Let $\varphi := \bigwedge_{i=1}^{k} C_i$ and $C_i := (L_{i,1} \lor L_{i,2} \lor L_{i,3})$

- For each variable X occurring in φ, we construct a directed graph ("gadget") that allows only two Hamiltonian paths: "true" and "false"
- Gadgets for each variable are "chained" in a directed fashion, so that all variables must be assigned one value
- Clauses are represented by vertices that are connected to the gadgets in such a way that they can only be visited on a Hamiltonian path that corresponds to an assignment where they are true

Details are also given in [Sipser, Theorem 7.46].

Example 9.2 (see blackboard)

 $\varphi := C_1 \land C_2$ where $C_1 := (X \lor \neg Y \lor Z)$ and $C_2 := (\neg X \lor Y \lor \neg Z)$

Towards More NP-Complete Problems

Starting with SAT, one can readily show more problems \mathcal{P} to be NP-complete, each time performing two steps:

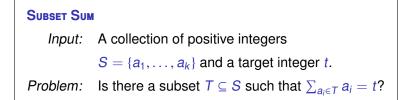
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NP-Completeness of SUBSET SUM



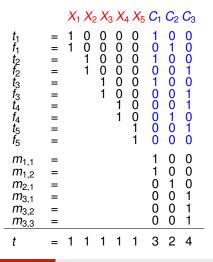
Theorem 9.3

SUBSET SUM *is* NP*-complete*.

Proof.

- SUBSET SUM \in NP: Take T to be the certificate.
- SUBSET SUM is NP-hard: SAT \leq_p SUBSET SUM

$(X_1 \lor X_2 \lor X_3) \land (\neg X_1 \lor \neg X_4) \land (X_4 \lor X_5 \lor \neg X_2 \lor \neg X_3)$



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Sat \leq_p Subset Sum

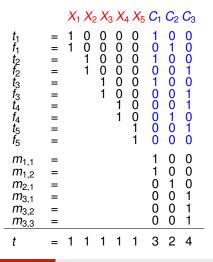
Given: $\varphi := C_1 \land \cdots \land C_k$ in conjunctive normal form.

(w.l.o.g. at most 9 literals per clause)

Let X_1, \ldots, X_n be the variables in φ . For each X_i let

$$t_i := a_1 \dots a_n c_1 \dots c_k \text{ where } a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \text{ and } c_j := \begin{cases} 1 & X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$$
$$f_i := a_1 \dots a_n c_1 \dots c_k \text{ where } a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \text{ and } c_j := \begin{cases} 1 & \neg X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$$

$(X_1 \lor X_2 \lor X_3) \land (\neg X_1 \lor \neg X_4) \land (X_4 \lor X_5 \lor \neg X_2 \lor \neg X_3)$



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Sat \leq_p Subset Sum

Further, for each clause C_i take $r := |C_i| - 1$ integers $m_{i,1}, \ldots, m_{i,r}$

where
$$m_{i,j} := c_i \dots c_k$$
 with $c_\ell := \begin{cases} 1 & \ell = i \\ 0 & \ell \neq i \end{cases}$
Definition of *S*: Let

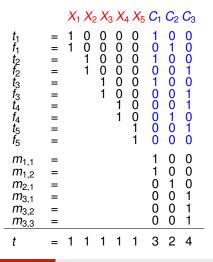
 $S := \{t_i, f_i \mid 1 \le i \le n\} \cup \{m_{i,j} \mid 1 \le i \le k, \quad 1 \le j \le |C_i| - 1\}$

Target: Finally, choose as target

 $t := a_1 \dots a_n c_1 \dots c_k$ where $a_i := 1$ and $c_i := |C_i|$

Claim: There is $T \subseteq S$ with $\sum_{a_i \in T} a_i = t$ iff φ is satisfiable.

$(X_1 \lor X_2 \lor X_3) \land (\neg X_1 \lor \neg X_4) \land (X_4 \lor X_5 \lor \neg X_2 \lor \neg X_3)$



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NP-Completeness of SUBSETSUM

Let $\varphi := \bigwedge C_i$ C_i : clauses

Show: If φ is satisfiable, then there is $T \subseteq S$ with $\sum_{s \in T} s = t$.

Let β be a satisfying assignment for φ

Set
$$T_1 := \{t_i \mid \beta(X_i) = 1 \quad 1 \le i \le m\} \cup \{f_i \mid \beta(X_i) = 0 \quad 1 \le i \le m\}$$

Further, for each clause C_i let r_i be the number of satisfied literals in C_i (with resp. to β).

Set
$$T_2 := \{m_{i,j} \mid 1 \le i \le k, 1 \le j \le |C_i| - r_i\}$$

and define $T := T_1 \cup T_2$.

It follows: $\sum_{s \in T} s = t$

NP-Completeness of SUBSET SUM

Show: If there is $T \subseteq S$ with $\sum_{s \in T} s = t$, then φ is satisfiable.

Let $T \subseteq S$ such that $\sum_{s \in T} s = t$

Define
$$\beta(X_i) = \begin{cases} 1 & \text{if } t_i \in T \\ 0 & \text{if } f_i \in T \end{cases}$$

This is well defined as for all *i*: $t_i \in T$ or $f_i \in T$ but not both.

Further, for each clause, there must be one literal set to 1 as for all *i*, the $m_{i,j} \in S$ do not sum up to the number of literals in the clause.

Knapsack and Strong NP-Completeness

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NP-completeness of KNAPSACK

Knapsack	
Input:	A set $I := \{1, \ldots, n\}$ of items
	each of value v_i and weight w_i for $1 \le i \le n$,
	target value t and weight limit ℓ
Problem:	Is there $T \subseteq I$ such that
	$\sum_{i \in T} v_i \ge t$ and $\sum_{i \in T} w_i \le \ell$?

Theorem 9.4

Кларsаск is NP-complete.

NP-completeness of KNAPSACK

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Input:	A set $I := \{1,, n\}$ of items
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Theorem 9.4

Кларsаск is NP-complete.

Proof.

- KNAPSACK \in NP: Take T to be the certificate.
- ► KNAPSACK IS NP-hard: Subset Sum ≤_p Knapsack

Subset Sum \leq_p Knapsack

Subset Sum:Given: $S := \{a_1, \dots, a_n\}$ collection of positive integersttarget integer

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Subset Sum \leq_p Knapsack

Subset Sum:Given: $S := \{a_1, \dots, a_n\}$ collection of positive integersttarget integer

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Reduction: From this input to SUBSET SUM construct

- set of items *l* := {1,..., *n*}
- weights and values $v_i = w_i = a_i$ for all $1 \le i \le n$
- target value t' := t and weight limit $\ell := t$

Subset Sum \leq_p Knapsack

Subset Sum:

Given: $S := \{a_1, \dots, a_n\}$ collection of positive integersttarget integer

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Reduction: From this input to SUBSET SUM construct

- set of items *I* := {1,..., *n*}
- weights and values $v_i = w_i = a_i$ for all $1 \le i \le n$
- target value t' := t and weight limit $\ell := t$

Clearly: For every $T \subseteq S$

$$\sum_{a_i \in T} a_i = t \quad \text{iff} \quad \sum_{a_i \in T} v_i \ge t' = t$$
$$\sum_{a_i \in T} w_i \le \ell = t$$

Hence: The reduction is correct and in polynomial time.

Complexity Theory

A Polynomial Time Algorithm for KNAPSACK

Кларваск can be solved in time $O(n\ell)$ using dynamic programming

Initialisation:

- Create an $(\ell + 1) \times (n + 1)$ matrix M
- Set M(w, 0) := 0 for all $1 \le w \le \ell$ and M(0, i) := 0 for all $1 \le i \le n$

Input $I = \{1, 2, 3, 4\}$ with Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

Weight: $w_1 = 1$ $w_2 = 1$ $w_3 = 3$ $w_4 = 2$

Weight limit: $\ell = 5$ Target value: t = 7

weight	max. total value from first <i>i</i> items				
limit w	<i>i</i> = 0	<i>i</i> = 1	<i>i</i> = 2	<i>i</i> = 3	<i>i</i> = 4
0					
1					
2					
3					
4					
5					

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0	0	0	0	0	0	
1	0					
2	0					
3	0					
4	0					
5	0					

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Computation: Assign further M(w, i) to be the largest total value obtainable by selecting from the first *i* items with weight limit *w*: For i = 0, 1, ..., n - 1 set M(w, i + 1) as

 $M(w, i+1) := \max \{ M(w, i), M(w - w_{i+1}, i) + v_{i+1} \}$

Here, if $w - w_{i+1} < 0$ we always take M(w, i).

Acceptance: If *M* contains an entry $\geq t$, accept. Otherwise reject.

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0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				
5	0				

Input $I = \{1, 2, 3, 4\}$ with Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$ Weight: $w_1 = 1$ $w_2 = 1$ $w_3 = 3$ $w_4 = 2$

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1	0	1			
2	0	1			
3	0	1			
4	0	1			
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0	0	0	0	0	0
1	0	1	3		
2	0	1			
3	0	1			
4	0	1			
5	0	1			

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1	0	1	3		
2	0	1	4		
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1	0	1	3		
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1	0	1	3		
2	0	1	4		
3	0	1	4		
4	0	1	4		
5	0	1	4		

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0	0	0	0	0	0
1	0	1	3	3	3
2	0	1	4	4	4
3	0	1	4	4	5
4	0	1	4	7	7
5	0	1	4	8	8

Did we prove P = NP?

Summary:

- ► Theorem 9.4: Кларѕаск is NP-complete
- KNAPSACK can be solved in time $O(n\ell)$ using dynamic programming

What went wrong?

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Pseudo-Polynomial Time

The previous algorithm is not sufficient to show that KNAPSACK is in ${\rm P}$

- The algorithm fills a $(\ell + 1) \times (n + 1)$ matrix M
- The size of the input to KNAPSACK is $O(n \log \ell)$

 \rightarrow the size of *M* is not bounded by a polynomial in the length of the input!

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The previous algorithm is not sufficient to show that $\mathsf{K}\mathtt{N}\mathtt{A}\mathtt{P}\mathtt{S}\mathtt{A}\mathtt{C}\mathtt{K}$ is in P

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- The size of the input to KNAPSACK is $O(n \log \ell)$

 \rightarrow the size of *M* is not bounded by a polynomial in the length of the input!

Definition 9.5 (Pseudo-Polynomial Time)

Problems decidable in time polynomial in the sum of the input length and the value of numbers occurring in the input.

Equivalently: Problems decidable in polynomial time when using unary encoding for all numbers in the input.

- If KNAPSACK is restricted to instances with ℓ ≤ p(n) for a polynomial p, then we obtain a problem in P.
- KNAPSACK is in polynomial time for unary encoding of numbers.

Strong NP-completeness

Pseudo Polynomial time: Algorithms polynomial in the maximum of the input length and the value of numbers occurring in the input.

Examples:

- KNAPSACK
- SUBSET SUM

Strong NP-completeness: Problems which remain NP-complete even if all numbers are bounded by a polynomial in the input length (equivalently: even for unary coding of numbers).

Examples:

- CLIQUE
- SAT
- HAMILTONIAN CYCLE

▶ ...

Note: Showing SAT \leq_{p} SUBSET SUM required exponentially large numbers.

CONP

The Class CONP

Recall that CONP is the complement class of NP.

Definition 9.6

- For a language $\mathcal{L} \subseteq \Sigma^*$ let $\overline{\mathcal{L}} := \Sigma^* \setminus \mathcal{L}$ be its complement
- For a complexity class C, we define $\operatorname{coC} := \{\mathcal{L} \mid \overline{\mathcal{L}} \in C\}$
- In particular $\operatorname{coNP} = \{\mathcal{L} \mid \overline{\mathcal{L}} \in \operatorname{NP}\}$

A problem belongs to coNP , if no-instances have short certificates.

Examples:

- No HAMILTONIAN PATH: Does the graph G not have a Hamiltonian path?
- TAUTOLOGY: Is the propositional logic formula φ a tautology (true under all assignments)?

CONP

coNP-completeness

Definition 9.7

A language $C \in \text{coNP}$ is coNP-complete, if $\mathcal{L} \leq_p C$ for all $\mathcal{L} \in \text{coNP}$.

Theorem 9.8

- P = COP
- Hence, $P \subseteq NP \cap CONP$

Open questions:

 \triangleright NP = coNP?

Most people do not think so.

 $P = NP \cap CONP?$

Again, most people do not think so.