Refining Labelled Systems for Modal and Constructive Logics with Applications

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logics LOGICAL METHODS IN COMPUTER SCIENCE

FWF Der Wissenschaftsfonds.

PhD Thesis Defense

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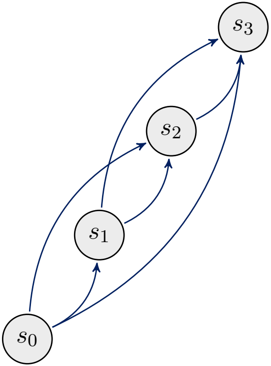
Modal and Constructive Logics?

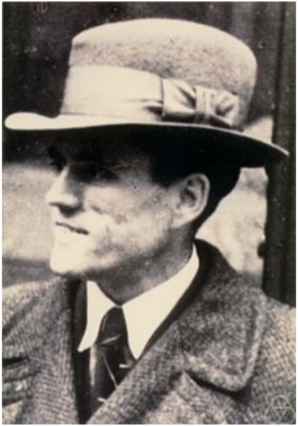
$\Diamond\phi$ $[a]\phi$ $K_a\phi$ $\mathcal{P}\phi$ $\mathsf{G}\phi$ $\blacklozenge\phi$ $\phi \mathsf{U} \psi$ $[\alpha\ \mathit{stit} : \phi]$ $\phi \supset \psi$ $\phi \prec \psi$

- Program Verification (e.g. LTL, PDL)
- Knowledge and Information Change (e.g. Dynamic Epistemic Logic)
- Constructive Reasoning (e.g. Intuitionistic Logic)
- Normative Reasoning (e.g. Deontic STIT Logic)
- Knowledge Representation (e.g. $\mathcal{ALC}$)



Proof Theory?

- Offers **constructive** and **syntactic** approach to studying (meta-)logical properties of logics; e.g.
  - Consistency
  - Decidability
  - Interpolation
- **Fruitful** approach to **automated reasoning**; e.g.
  - Complexity optimal decision algorithms with witnesses
  - Knowledge base querying



Gerhard Gentzen (1945)

A Prominent Desideratum: Analyticity

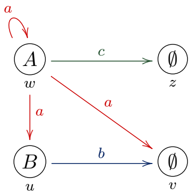
"A proof is **analytic** if it does not go beyond its **subject matter**."

Bernard Bolzano

Our Interpretation: A proof is **analytic** if it only contains **subformulae** of the **conclusion**.

Labelled Calculi (Semantic-Based Systems)

$R_a ww, R_a wu, R_b uv, R_a wv, R_c wz \Rightarrow w : A, u : B$

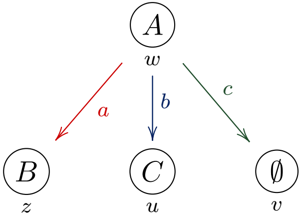


Pros
Semantic Clauses + Frame Properties ⇝ Inference Rules
Easy to Construct (Process can be automated)
General Theorems of Fundamental Properties

Cons
Redundant and Complex Syntax
Not analytic (in a strong sense)

Nested Sequent Systems

$A, (a)\{B\}, (b)\{C\}, (c)\{\emptyset\}$



Pros	Cons
Analytic	Lacking General Method of Construction
Minimized Bureaucracy/ Simpler Syntax	Lacking General Theorems Ensuring Properties

The Method of Structural Refinement

The Method:

1. Semantics to Labelled Systems
2. Labelled to Nested/Refined Labelled via
   - Structural Rule Elimination
   - Domain Atom Removal
3. Establish Inheritance

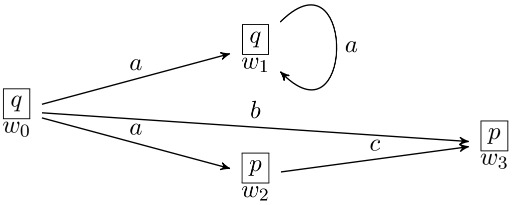
The Benefits:

- Best of both worlds
- Improved theory of nested sequents
- Compression in calculus and proof size
- Simplification of sequents
- Structural rules traded for logical (propagation) rules

Grammar Logics

- **The class of logics includes**:
  - Epistemic Logics (reasoning about knowledge)
  - Temporal Logics (reasoning about time)
  - Description Logics (knowledge representation)
- **Language**:

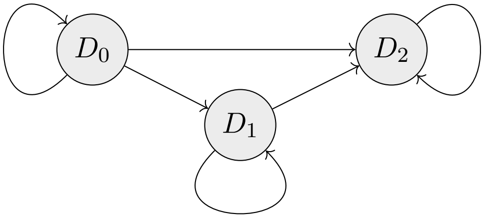
$$A ::= p \mid \neg p \mid A \vee A \mid A \wedge A \mid [\alpha]A \mid \langle \alpha \rangle A \ \textit{with} \ \alpha \in \{a, \overline{a}, b, \overline{b}, c, \overline{c}, \ldots\}$$



First-Order Intuitionistic Logics

- **Logics Considered**:
  - First-Order Intuitionistic Logic with Non-Constant Domains
  - First-Order Intuitionistic Logic with Constant Domains
- **Language**:

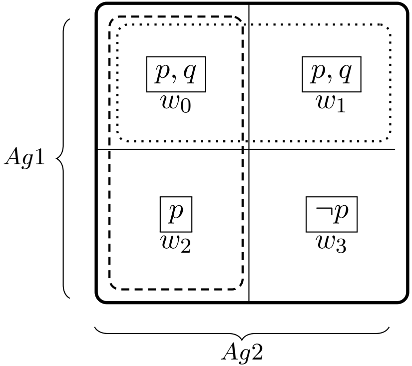
$$A ::= p(x_1, \ldots, x_n) \mid \bot \mid A \vee A \mid A \wedge A \mid A \supset A \mid \exists x A \mid \forall x A$$



Deontic STIT Logics

- **Used to model (deontic) agential choice making**
- **Language**:

$$A ::= p \mid \neg p \mid A \vee A \mid A \wedge A \mid [i]A \mid \langle i \rangle A \mid \Box A \mid \Diamond A \mid \otimes_i A \mid \ominus_i A$$

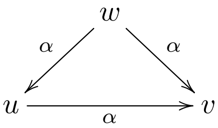


An Example: Km5

Classical Propositional Logic +

Axioms:	Inference Rules:
$[\alpha](A \to B) \to ([\alpha]A \to [\alpha]B)$ $A \to [\alpha]\langle\overline{\alpha}\rangle A$ $\langle\overline{\alpha}\rangle\langle\alpha\rangle A \to \langle\alpha\rangle A$	$\dfrac{A}{[\alpha]A}\,\text{nec}$

Euclideanity:



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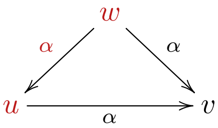

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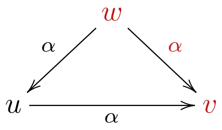

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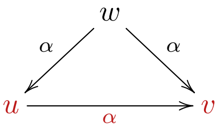

Labelled Sequents

Labelled Sequents: $\mathcal{R} \Rightarrow \Gamma$ and

$$\mathcal{R} ::= R_{\alpha}wu \mid L_1, L_1 \qquad \Gamma ::= w : A \mid L_2, L_2$$

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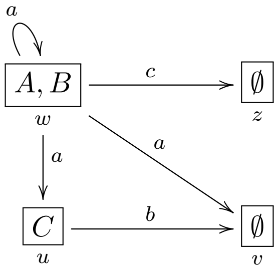

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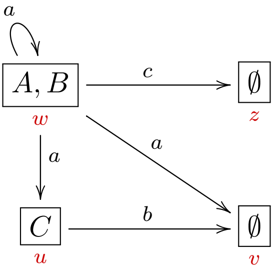

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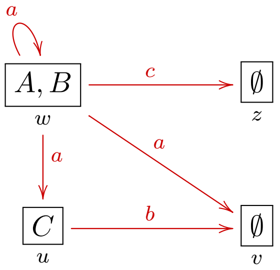

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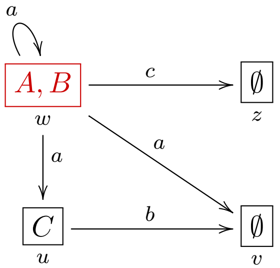

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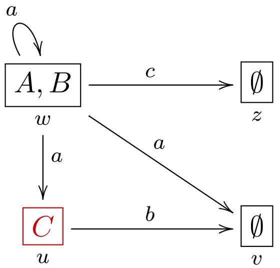

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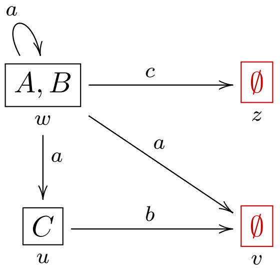


The Labelled Calculus G3Km5

$$\frac{}{\mathcal{R} \Rightarrow w : p, w : \neg p, \Gamma}\ (id)$$

$$\frac{\mathcal{R} \Rightarrow w : A, w : B, \Gamma}{\mathcal{R} \Rightarrow w : A \vee B, \Gamma}\ (\vee) \qquad \frac{\mathcal{R} \Rightarrow w : A, \Gamma \qquad \mathcal{R} \Rightarrow w : B, \Gamma}{\mathcal{R} \Rightarrow w : A \wedge B, \Gamma}\ (\wedge)$$

$$\frac{\mathcal{R}, R_{\alpha}wu \Rightarrow w : \langle \alpha \rangle A, u : A, \Gamma}{\mathcal{R}, R_{\alpha}wu \Rightarrow w : \langle \alpha \rangle A, \Gamma}\ (\langle \alpha \rangle) \qquad \frac{\mathcal{R}, R_{a}wu \Rightarrow u : A, \Gamma}{\mathcal{R} \Rightarrow w : [\alpha]A, \Gamma}\ ([\alpha])^{\dagger}$$

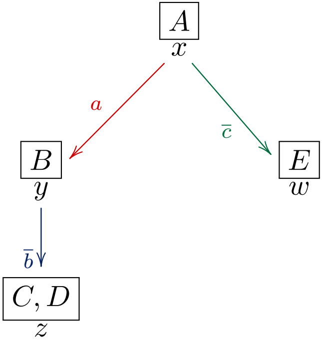
$$\frac{\mathcal{R}, R_{\overline{\alpha}}uw, R_{\alpha}wv, R_{\alpha}wv \Rightarrow \Gamma}{\mathcal{R}, R_{\overline{\alpha}}uw, R_{\alpha}wv \Rightarrow \Gamma}\ (euc) \qquad \frac{\mathcal{R}, R_{\alpha}uw, R_{\overline{\alpha}}wu \Rightarrow \Gamma}{\mathcal{R}, R_{\alpha}uw \Rightarrow \Gamma}\ (cv)$$

Nested Sequents

- **Nested Sequents** [Tiu et al. 2012]:

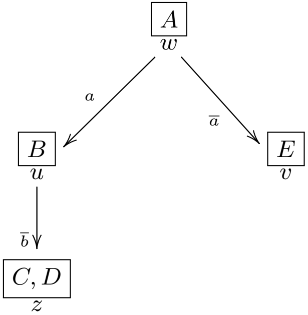
$$\Sigma ::= A \mid \Sigma, \Sigma \mid (\alpha)\{\Sigma\}$$

- **Example**: $A, (a)\{B, (\bar{b})\{C, D\}\}, (\bar{c})\{E\}$

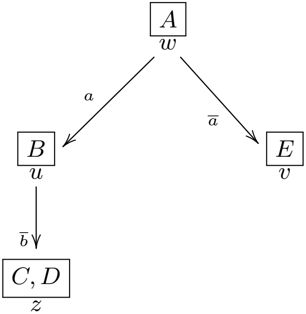


Translating Labelled and Nested

$R_a wu, R_{\overline{b}} uv, R_{\overline{a}} wv \Rightarrow w : A, u : B, z : C, z : D, v : E$



$A, (a)\{B, (\overline{b})\{C, D\}\}, (\overline{a})\{E\}$



- Treelike Labelled Sequent = Nested Sequent

The Nested Calculus DKm5 [Tiu et al. 2012]

$$\frac{}{\Sigma[p, \neg p]}\ (id) \qquad \frac{\Sigma[A, B]}{\Sigma[A \vee B]}\ (\vee) \qquad \frac{\Sigma[A] \qquad \Sigma[B]}{\Sigma[A \wedge B]}\ (\wedge)$$

$$\frac{\Sigma[(\alpha)\{A\}]}{\Sigma[[\alpha]A]}\ ([\alpha]) \qquad \frac{\Sigma[\langle\alpha\rangle A][A]}{\Sigma[\langle\alpha\rangle A][\emptyset]}\ (p)^{\dagger}$$

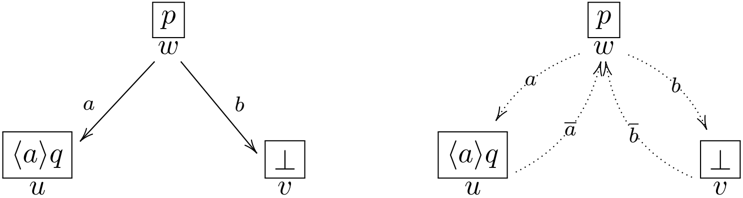
Propagation Rules and Graphs [Tiu et al. 2012]

- **Propagation rules**

  1. Nested sequent ⇝ Propagation graph
  2. Axioms ⇝ Formal grammars
  3. Accepted string ⇝ Rule application

- **Propagation graphs**

$$p, (a)\{\langle a \rangle q\}, (b)\{\bot\}$$



Propagation Rule Example

$$\langle\overline{a}\rangle\langle a\rangle A \rightarrow \langle a\rangle A \quad \leftrightsquigarrow \quad a \longrightarrow \overline{a} \cdot a$$

$$\frac{r, (a)\{p, q\}, (a)\{\langle a\rangle p\}}{r, (a)\{q\}, (a)\{\langle a\rangle p\}}\,(p)$$

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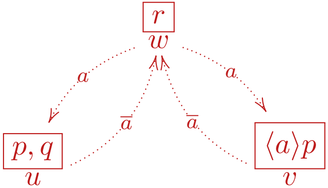

1. Premise $\rightsquigarrow$ propagation graph

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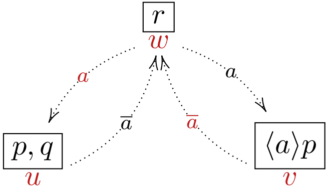

1. Premise $\rightsquigarrow$ propagation graph
2. a implies a certain path

Propagation Rule Example

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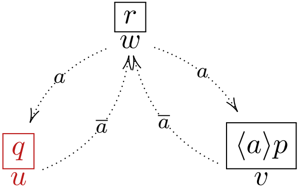
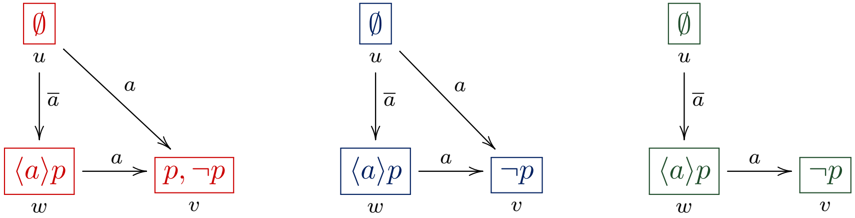


1. Premise $\rightsquigarrow$ propagation graph
2. a implies a certain path
3. Apply Rule

Elimination Example

$$
\cfrac{\cfrac{\cfrac{}{R_{\overline{a}}uw, R_a wv, R_a uv \Rightarrow v : p, v : \neg p, w : \langle a \rangle p}\ (id)}{R_{\overline{a}}uw, R_a wv, R_a uv \Rightarrow v : \neg p, w : \langle a \rangle p}\ (\langle a \rangle)}{R_{\overline{a}}uw, R_a wv \Rightarrow v : \neg p, w : \langle a \rangle p}\ (euc)
$$



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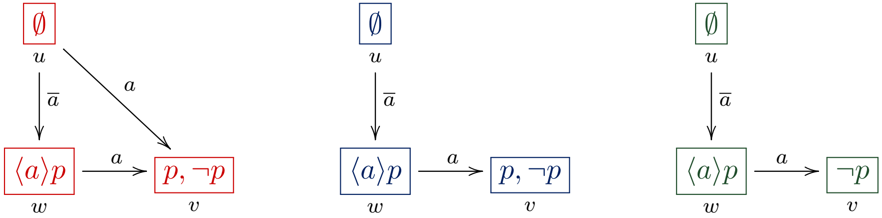

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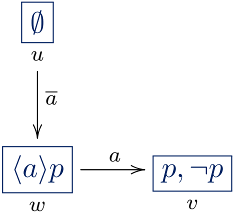
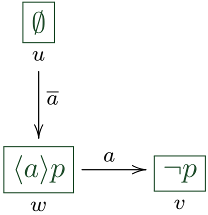
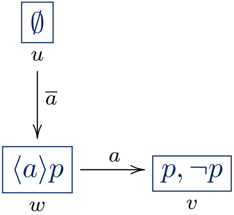
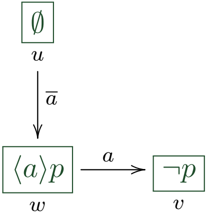




Elimination Example

$$\dfrac{\dfrac{}{(\overline{a})\{\langle a\rangle p, (a)\{p, \neg p\}\}}\ (id)}{(\overline{a})\{\langle a\rangle p, (a)\{\neg p\}\}}\ (p)$$





Rules and Consequences

- **Propagation rules** $\rightsquigarrow$ **Structural Rule Elimination**
- **Structural Rule Elimination** $\rightsquigarrow$ **Treelike Sequents**
- **Propagation rules**: Propagate formulae along paths
- **Reachability rules**: Also check if data exists along paths
- **Benefits** of parameterizing with grammars:
  - Modularity: Change Grammar $\rightsquigarrow$ Change Logic
  - Generality: Can define calculi for sizable classes of logics

Summary of Decidability and Interpolation Results

- Decidability of Deontic STIT Logics via Proof-Search
  - Returns proof of validity
  - Returns counter-model of invalidity
- Interpolation: If $A \to B \in \mathbf{L}$, then
  - $\exists I$ such that $A \to I, I \to B \in \mathbf{L}$;
  - I is in 'common language' of A and B.
- Interpolation for Context-Free Grammar Logics with Converse
  - Constructive: Builds Interpolant
  - Verifiable/Explainable: Returns proofs witnessing interpolation
  - General: Applicable to other classes of logics

Results Obtained via Refinement Work

- New Method: Semantics $\rightsquigarrow$ Labelled systems $\rightsquigarrow$ Nested systems
- Existing nested calculi can be derived via method:
  - Nested calculi for grammar logics [Tiu et al. 2012]
  - Nested calculi for FO Intuitionistic logics [Fitting 2014]
- New systems for FO Intuitionistic Logics
  - Utilizes reachability rules
  - Modularity: (Sub-)(Bi-)Intuitionistic Logics
- New systems for deontic STIT logics
- Suitable for “applications”: Decidability and Interpolation

List of Publications Used in Thesis

1. Ciabattoni, A., Lyon, T., & Ramanayake, R. (2018). From Display to Labelled Proofs for Tense Logics. LFCS 2018
2. Berkel, K., & Lyon, T. (2019). Cut-Free Calculi and Relational Semantics for Temporal STIT Logics. JELIA 2019
3. Lyon, T., & Berkel, K. (2019). Automating Agential Reasoning: Proof-Calculi and Syntactic Decidability for STIT Logics. PRIMA 2019
4. Lyon, T., Tiu, A., Goré, R., & Clouston, R. (2020). Syntactic Interpolation for Tense Logics and Bi-Intuitionistic Logic via Nested Sequents. CSL 2020
5. Lyon, T. (2020). On Deriving Nested Calculi for Intuitionistic Logics from Semantic Systems. LFCS 2020
6. Lyon, T. (2021). On the Correspondence between Nested Calculi and Semantic Systems for Intuitionistic Logics. Journal of Logic and Computation.
7. Berkel, K. & Lyon, T. (2021). The Varieties of Ought-Implies-Can and Deontic STIT Logic. DEON 2021
8. Ciabattoni, A., Lyon, T., Ramanayake, R., & Tiu, A. (2021). Display to Labelled Proofs and Back Again for Tense Logics. ACM Transactions on Computational Logic.

Future Work

- Find General Conditions for Extracting Nested from Labelled
- Refining calculi for other logics:
  - Free logics
  - Modal logics with nominals
  - Constructive Modal Logics
- Provide Nested/Analytic Calculi for Logics w/o One
- Decidability, Interpolation, etc.
- Extend Method: Indexed-Nested, Linear Nested, and Hypersequents