Finite and Algorithmic Model Theory

Lecture 4 (Dresden 02.11.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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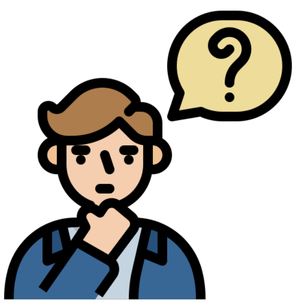

Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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Let τ be a *finite* signature, and let $m \in \mathbb{N}$. $\mathrm{FO}_m[\tau]$ is set of all FO formulae over τ with q.r. $\leq m$.

Notation: $\mathfrak{A} \equiv_m^\tau \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy precisely the same $\mathrm{FO}_m[\tau]$ sentences (τ often omitted).

Lemma (Finiteness of $\mathrm{FO}_m[\tau]$ with $\leq k$ free variables)

The set of all $\mathrm{FO}_m[\tau]$ formulae with at most k free variables is finite up to logical equivalence.

Measuring complexity of a formula: quantifier rank

The quantifier rank $\mathrm{qr}(\varphi)$ of φ is its depth of quantifier nesting.

- $\mathrm{qr}(\varphi) := 0$ for atomic φ
- $\mathrm{qr}(\neg\varphi) := \mathrm{qr}(\varphi)$
- $\mathrm{qr}(\varphi \oplus \varphi') := \max(\mathrm{qr}(\varphi), \mathrm{qr}(\varphi'))$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
- $\mathrm{qr}(\exists x\ \varphi) = \mathrm{qr}(\forall x\ \varphi) := \mathrm{qr}(\varphi) + 1$

Examples:

$\mathrm{qr}(\exists x \forall y \forall z\ \mathrm{R}(x, y, z)) = 3$

$\mathrm{qr}(\exists x\, [\mathrm{A}(x) \wedge (\forall y \mathrm{R}(y)) \vee (\exists z \top)]) = 2$

for φ in PNF $\mathrm{qr}(\varphi) = \#$quantifiers.

Quantifier rank can be exponentially smaller than the total number of quantifiers.

$\varphi_0(x, y) := \mathrm{E}(x, y), \quad \varphi_{n+1}(x, y) := \exists z\ (\varphi_n(x, z) \wedge \varphi_n(z, y)) \quad \rightsquigarrow\ \mathrm{qr}(\varphi_n) = n$ but φ_n has $2^n - 1$ quants.

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The set of all $\mathrm{FO}_m[\tau]$ formulae with at most k free variables is finite up to logical equivalence.

Proof

Idea: characterise $\mathrm{FO}_0[\tau]$ with a "truth table" of equality between constants/variables + induction!

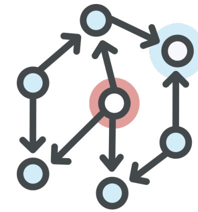
Ehrenfeucht-Fraïssé games

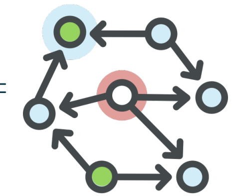
Ehrenfeucht-Fraïssé games

- Duration: *m rounds*.

Ehrenfeucht-Fraïssé games

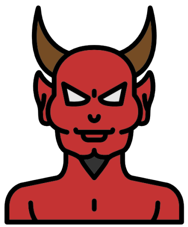
- Duration: m rounds.
- Playground: two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$.

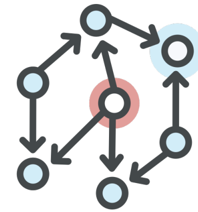
$\mathfrak{A} :=$ 

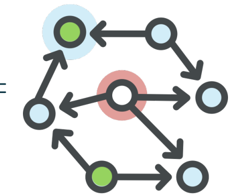
$\mathfrak{B} :=$ 

Ehrenfeucht-Fraïssé games

- Duration: *m* rounds.
- Playground: two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$.
- Two players: Spoil∃r



$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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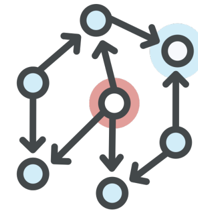
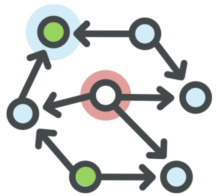
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

- Duration: m rounds.
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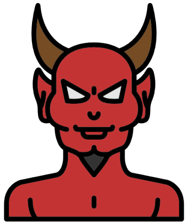

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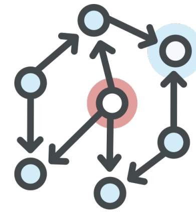
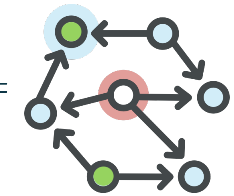
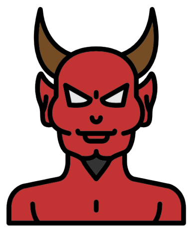


$\mathfrak{B} :=$





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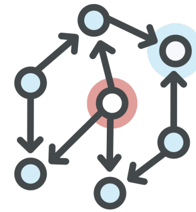
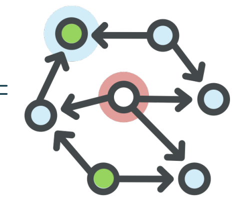
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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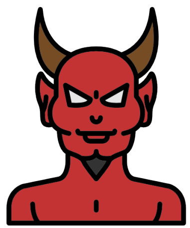

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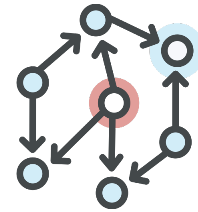
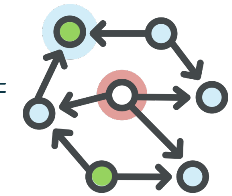
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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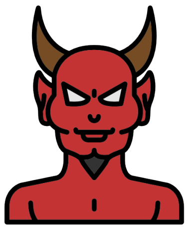
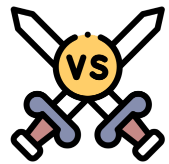
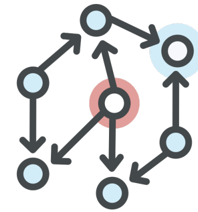
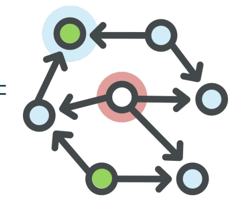




Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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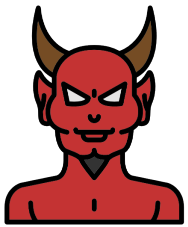
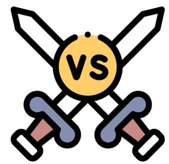
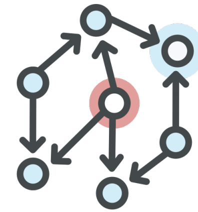
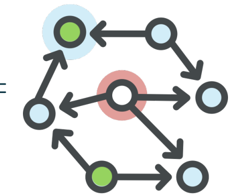




Goal of $\forall$: $\mathfrak{A}, \mathfrak{B}$ "look the same".

Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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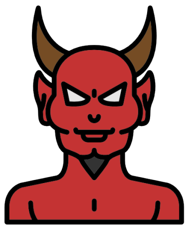
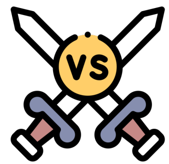
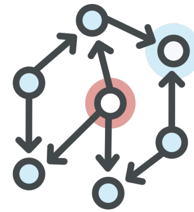


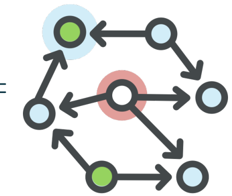


Goal of ∀: $\mathfrak{A}, \mathfrak{B}$ "look the same".

Goal of ∃: pinpoint the difference.

Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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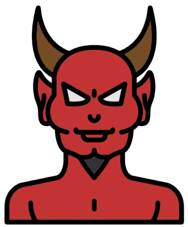

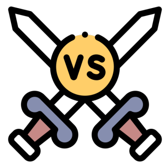

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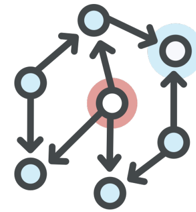
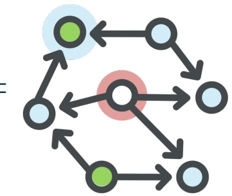
- During the i-th round:

Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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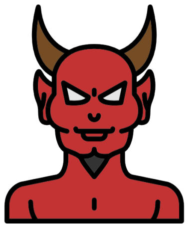 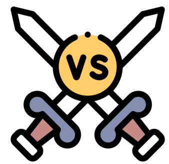

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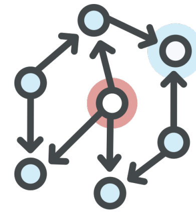
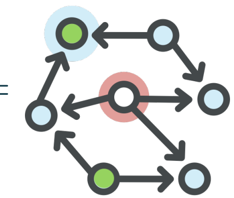
Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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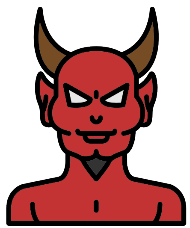

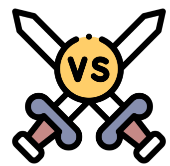

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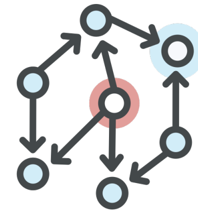
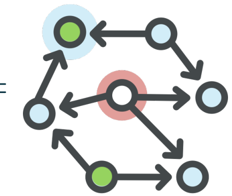
Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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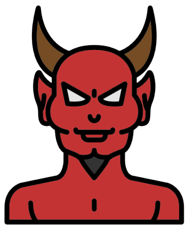

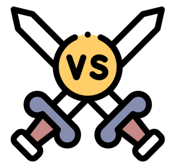

Goal of $\forall$: $\mathfrak{A}, \mathfrak{B}$ "look the same".

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- During the i-th round:

1. $\exists$ selects a structure (say $\mathfrak{A}$) and picks an element (say $a_i \in A$)

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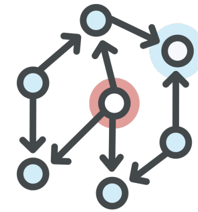
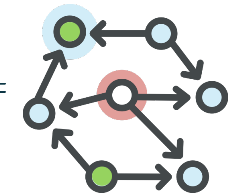
so that $(a_1 \mapsto b_1, \ldots, a_i \mapsto b_i)$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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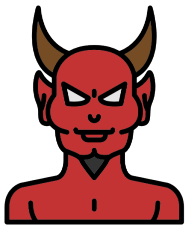

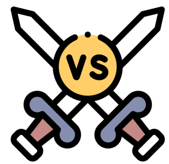

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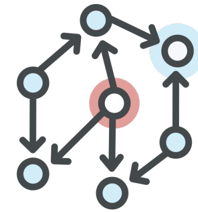
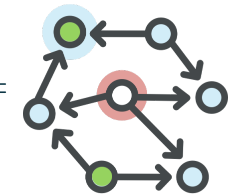
so that $(a_1 \mapsto b_1, \ldots, a_i \mapsto b_i)$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

- $\exists$ wins if $\forall$ cannot reply with a suitable element.

Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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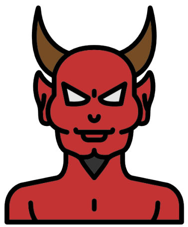

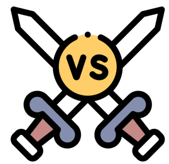

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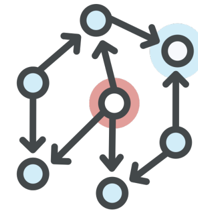
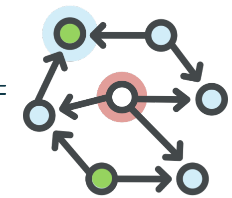
2. $\forall$ replies with an element (say $b_i \in B$) in the other structure (in this case $\mathfrak{B}$)

so that $(a_1 \mapsto b_1, \ldots, a_i \mapsto b_i)$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

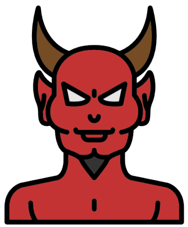
- $\exists$ wins if $\forall$ cannot reply with a suitable element. $\forall$ wins if he survives m rounds.

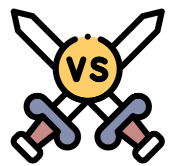
Ehrenfeucht-Fraïssé games

$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

- Duration: m rounds.
- Playground: two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$.
- Two players: Spoil∃r (D∃vil/∃loise/∃ve/Player I) vs Duplic∀tor (∀ngel/∀belard/∀dam/Player II)





Goal of $\forall$: $\mathfrak{A}, \mathfrak{B}$ "look the same".

Goal of $\exists$: pinpoint the difference.

- During the i-th round:

1. $\exists$ selects a structure (say $\mathfrak{A}$) and picks an element (say $a_i \in A$)

2. $\forall$ replies with an element (say $b_i \in B$) in the other structure (in this case $\mathfrak{B}$)

so that $(a_1 \mapsto b_1, \ldots, a_i \mapsto b_i)$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

- $\exists$ wins if $\forall$ cannot reply with a suitable element. $\forall$ wins if he survives m rounds.

Theorem (Fraïssé 1950 & Ehrenfeucht 1961)

$\forall$ has a winning strategy in m-round Ehrenfeucht-Fraïssé game on τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ iff $\mathfrak{A} \equiv_m^\tau \mathfrak{B}$.

Playing Ehrenfeucht-Fraïssé games on sets

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Consider an 3-round play of E-F game on sets $\mathfrak{A} := \{1, 2, 3\}$, $\mathfrak{B} := \{a, b, c, d\}$.

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$1 \mapsto d,$

$\mathfrak{B} :=$ ⓐ ⓑ ⓒ ⓓ

Playing Ehrenfeucht-Fraïssé games on sets

Consider an 3-round play of E-F game on sets $\mathfrak{A} := \{1, 2, 3\}$, $\mathfrak{B} := \{a, b, c, d\}$.

$1 \mapsto d, \; 2 \mapsto$

$\mathfrak{A} :=$ ① ② ③

$\mathfrak{B} :=$ ⓐ ⓑ ⓒ ⓓ

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$1 \mapsto d, \; 2 \mapsto b,$

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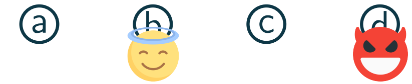
$\mathfrak{B} :=$ 

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$1 \mapsto d, \; 2 \mapsto b, \; 3 \mapsto$

$\mathfrak{B} :=$

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$1 \mapsto d,\ 2 \mapsto b,\ 3 \mapsto c$

Result: $\forall$ wins, so $\mathfrak{A} \equiv_3 \mathfrak{B}$.

$\mathfrak{B} :=$ 

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Following the strategy "always reply with a fresh element", $\forall$ wins any m-round game on sets of size $\geq m$.

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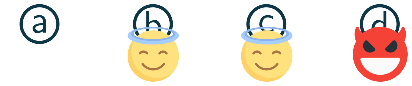

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Lemma (Even is not expressible in FO[$\emptyset$])

Playing Ehrenfeucht-Fraïssé games on sets

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Proof

ad absurdum φ exists

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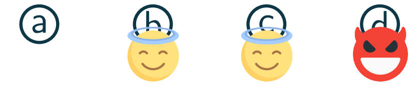

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Lemma (Even is not expressible in FO[$\emptyset$])

Proof Assume that such a φ exists. Let $m := \mathrm{qr}(\varphi)$.

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ad absurdum φ exists   q.r. of φ craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$

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Proof Assume that such a φ exists. Let $m := \mathrm{qr}(\varphi)$. Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be an $2m$ (resp. $2m+1$) element set.

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ad absurdum φ exists — q.r. of φ — craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$ — play qr(φ)-round game — E-F theorem

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ad absurdum φ exists | q.r. of φ | craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$ | play qr(φ)-round game | E-F theorem | infer $\mathfrak{B} \models \varphi$ contradiction!

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General proof scheme for showing that $\mathcal{P}$ is not FO[τ]-definable with Ehrenfeucht-Fraïssé games

ad absurdum φ exists | q.r. of φ | craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$ | play qr(φ)-round game | E-F theorem | infer $\mathfrak{B} \models \varphi$ contradiction!

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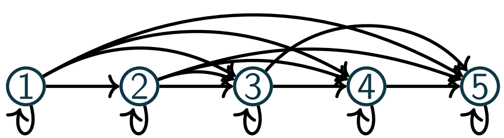
General proof scheme for showing that $\mathcal{P}$ is not $\mathrm{FO}[\tau]$-definable with Ehrenfeucht-Fraïssé games

ad absurdum φ exists $\rightsquigarrow$ q.r. of φ $\rightsquigarrow$ craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$ $\rightsquigarrow$ play $\mathrm{qr}(\varphi)$-round game $\rightsquigarrow$ E-F theorem $\rightsquigarrow$ infer $\mathfrak{B} \models \varphi$ contradiction!

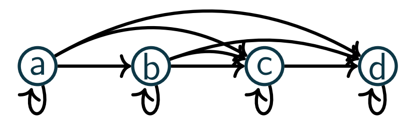
Playing Ehrenfeucht-Fraïssé games on linear orders

Playing Ehrenfeucht-Fraïssé games on linear orders

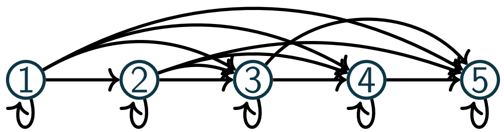
$\mathfrak{A} :=$

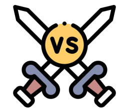


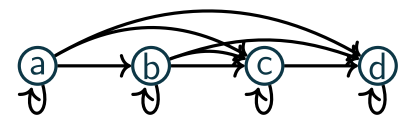
$\mathfrak{B} :=$



Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 

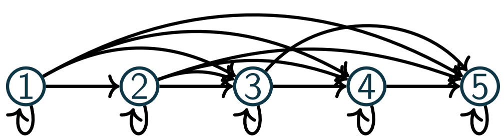


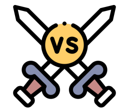
$\mathfrak{B} :=$ 

- Who has the winning strategy in 2 rounds?

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$





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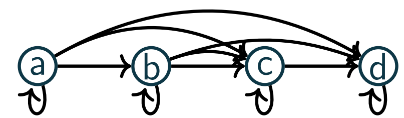

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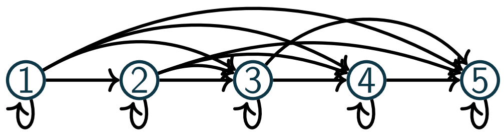
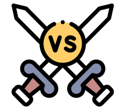
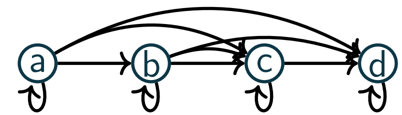
Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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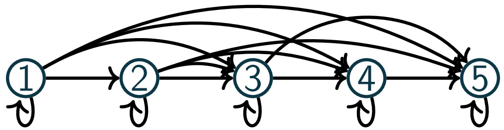
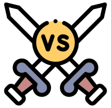
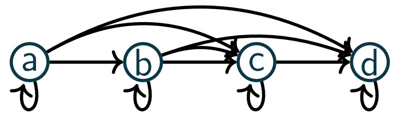
Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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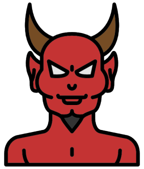
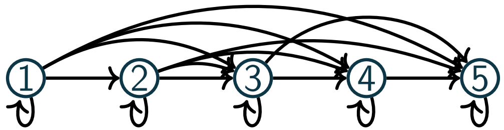
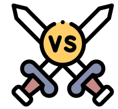
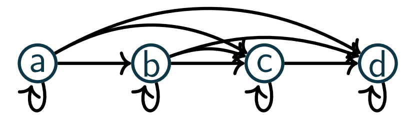


Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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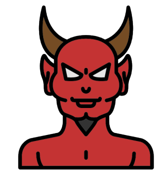
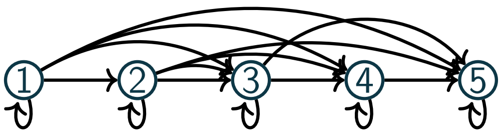
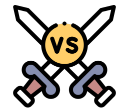


Lemma (Even is not expressible in FO[$\{\leq\}$])

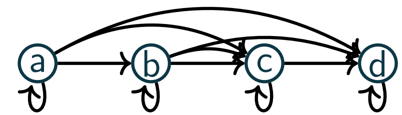
Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$





$\mathfrak{B} :=$



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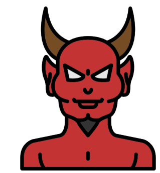

Lemma (Even is not expressible in FO[$\{\leq\}$])

Proof

Playing Ehrenfeucht-Fraïssé games on linear orders

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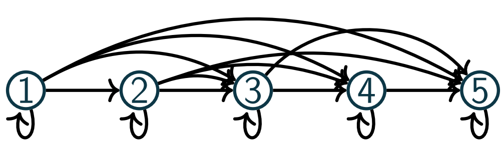
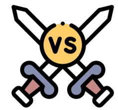
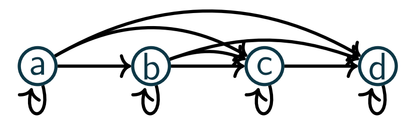




$\mathfrak{B} :=$



- Who has the winning strategy in 2 rounds?

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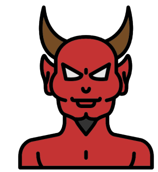
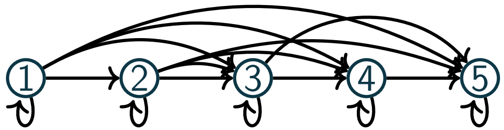


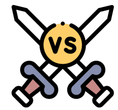
Lemma (Even is not expressible in FO[$\{\leq\}$])

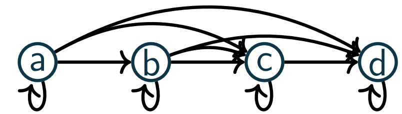
Proof

ad absurdum φ exists

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

- Who has the winning strategy in 2 rounds?

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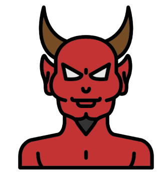


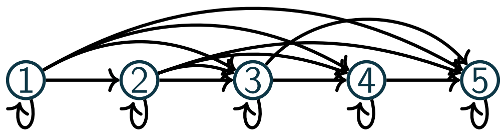
Lemma (Even is not expressible in FO[$\{\leq\}$])

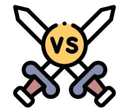
Proof Suppose that φ exists.

ad absurdum φ exists

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$





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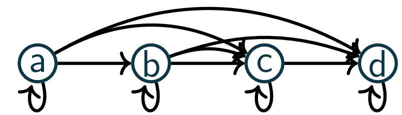

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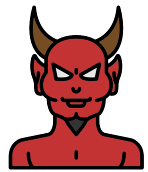


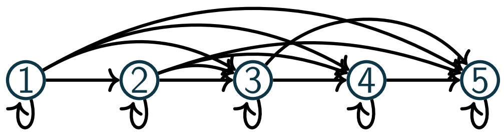
Lemma (Even is not expressible in $\mathrm{FO}[\{\leq\}]$)

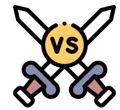
Proof Suppose that φ exists.

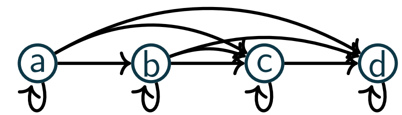
ad absurdum φ exists

q.r. of φ

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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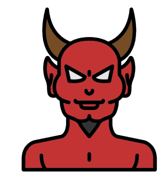
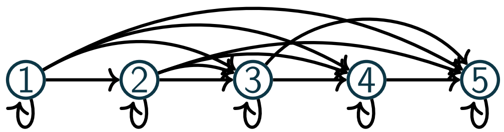


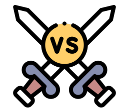
Lemma (Even is not expressible in FO[$\{\leq\}$])

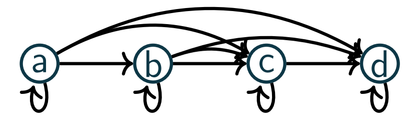
Proof Suppose that φ exists. Let $m := \mathrm{qr}(\varphi)$.

ad absurdum φ exists

q.r. of φ

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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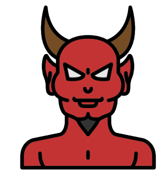

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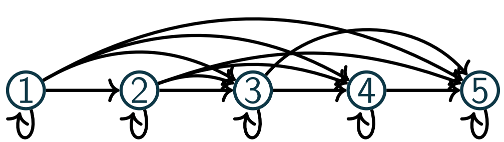
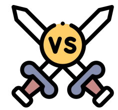
ad absurdum φ exists

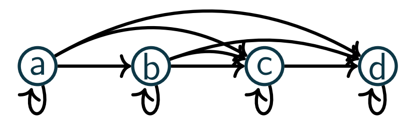
q.r. of φ

craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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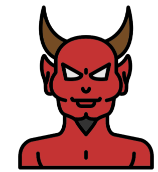

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ad absurdum φ exists

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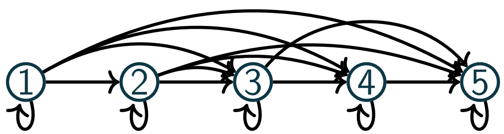
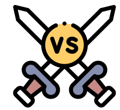
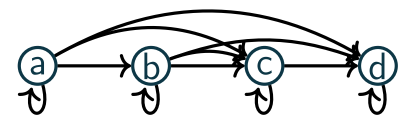
$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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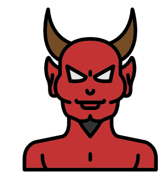

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ad absurdum φ exists

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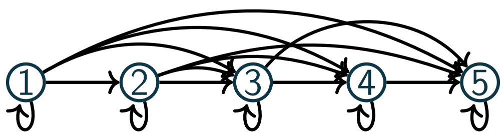
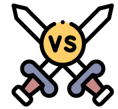
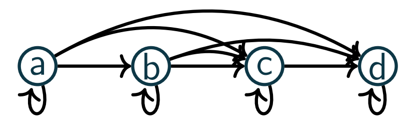
Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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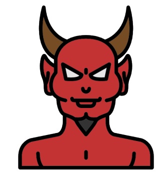

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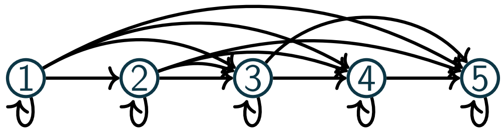
ad absurdum φ exists

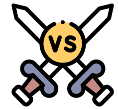
q.r. of φ

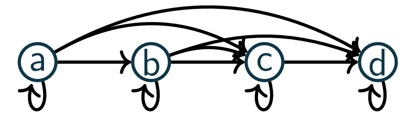
craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$

play qr(φ)-round game

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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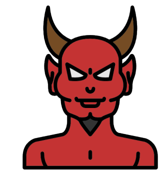

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What remains to be done is to show that

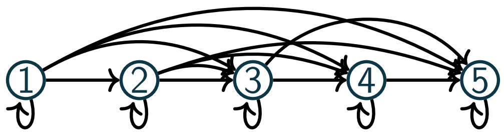
ad absurdum φ exists

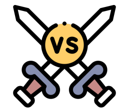
q.r. of φ

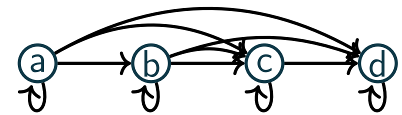
craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$

play $\mathrm{qr}(\varphi)$-round game

Playing Ehrenfeucht-Fraïssé games on linear orders

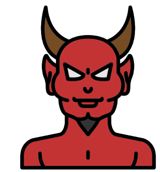
$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

- Who has the winning strategy in 2 rounds?

- In 3 rounds? more?



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By definition, we clearly have $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \not\models \varphi$.

What remains to be done is to show that $\forall$ has the winning strategy in any m-round E-F game.

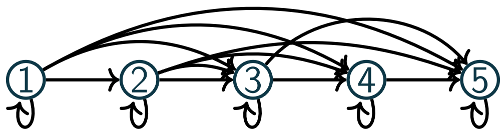
ad absurdum φ exists

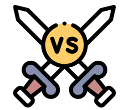
q.r. of φ

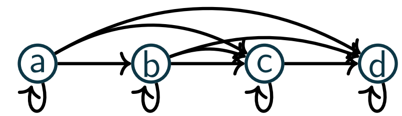
craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$

play qr(φ)-round game

Playing Ehrenfeucht-Fraïssé games on linear orders

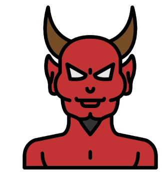
$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

- Who has the winning strategy in 2 rounds?

- In 3 rounds? more?



Lemma (Even is not expressible in FO[$\{\leq\}$])

Proof Suppose that φ exists. Let $m := \text{qr}(\varphi)$. Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be linear orders of size 2^m (resp. 2^m+1).

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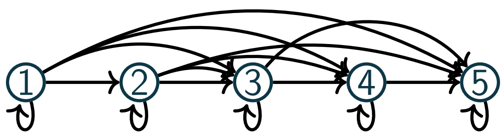
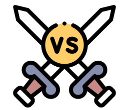
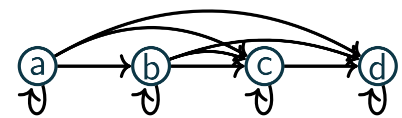
E-F theorem

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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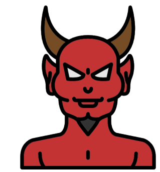

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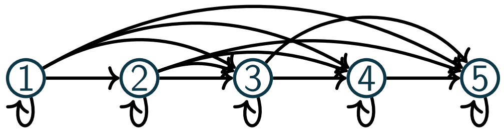
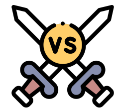
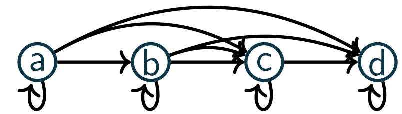
ad absurdum φ exists | q.r. of φ | craft τ-structures $\mathfrak{A} \models \varphi$, $\mathfrak{B} \not\models \varphi$ | play $\mathrm{qr}(\varphi)$-round game | E-F theorem

Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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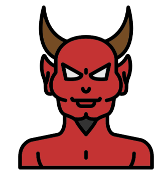

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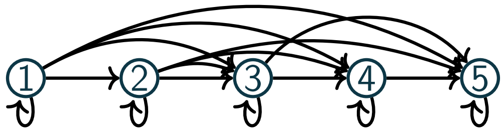
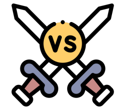
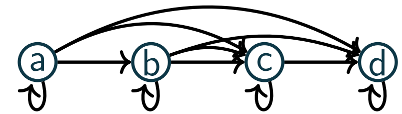
Playing Ehrenfeucht-Fraïssé games on linear orders

$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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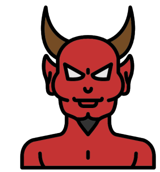

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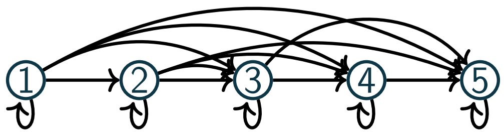
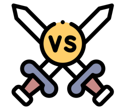
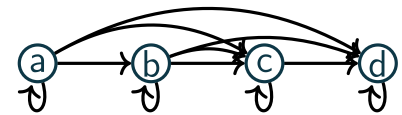
$\mathfrak{A} :=$ 



$\mathfrak{B} :=$ 

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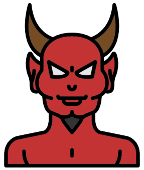

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Super Lemma About Linear Orders: I

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Any linearly ordered[a] $\{\leq\}$-structures $\mathfrak{A}, \mathfrak{B}$ of cardinality $\geq 2^m$ satisfy $\mathfrak{A} \equiv_m^{\{\leq\}} \mathfrak{B}$.

[a] We assume that $\mathfrak{A}, \mathfrak{B}$ interpret $\leq$ as a linear order over the domain

induction

- Let $\bar{a} := (a_{-1}, a_0, \ldots, a_i)$ and $\bar{b} := (b_{-1}, b_0, \ldots, b_i)$ be the history of the play after i-rounds.
- Dummy (-1)-th and 0-th rounds of the game: select min/max elements of $\mathfrak{A}, \mathfrak{B}$.

This establishes an invariant that any freshly selected element is between some previously selected ones.

- We play as $\forall$: we want to guarantee that after the i-th round we have for all $l, k \leq i$:

1. $a_k \leq^{\mathfrak{A}} a_l$ iff $b_k \leq^{\mathfrak{B}} b_l$ (maintain the partial isomorphism).
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$\forall$ should preserve these conditions

- Assume $\exists$ picks $a_{i+1} \in A$. Let a_l, a_k be the closest such that $a_l \leq^{\mathfrak{A}} a_{i+1} \leq^{\mathfrak{A}} a_k$.

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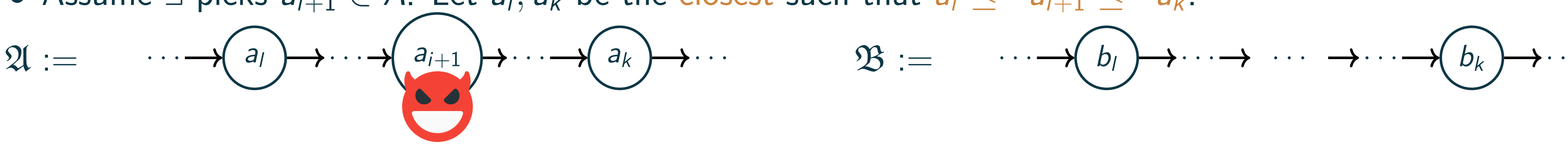

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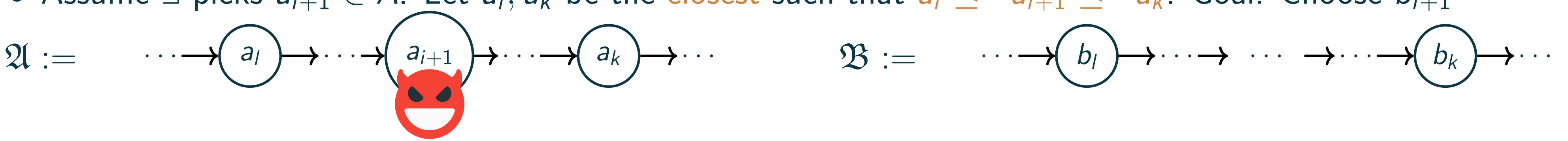
- Assume $\exists$ picks $a_{i+1} \in A$. Let a_l, a_k be the closest such that $a_l \leq^{\mathfrak{A}} a_{i+1} \leq^{\mathfrak{A}} a_k$. Goal: Choose b_{i+1}



Super Lemma About Linear Orders: II

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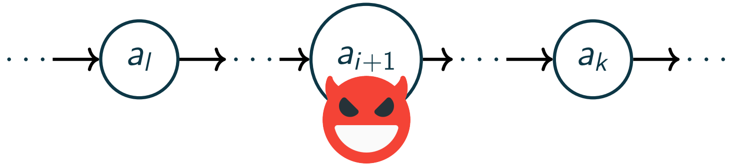
$\mathfrak{A} :=$ 

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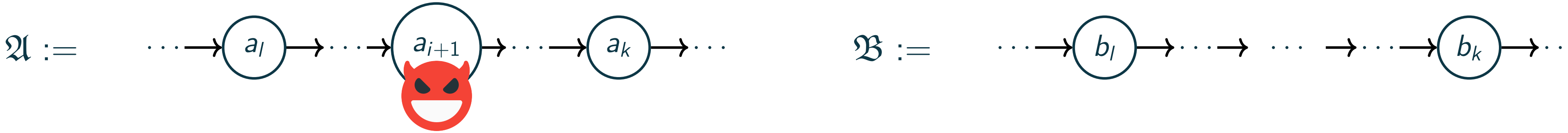

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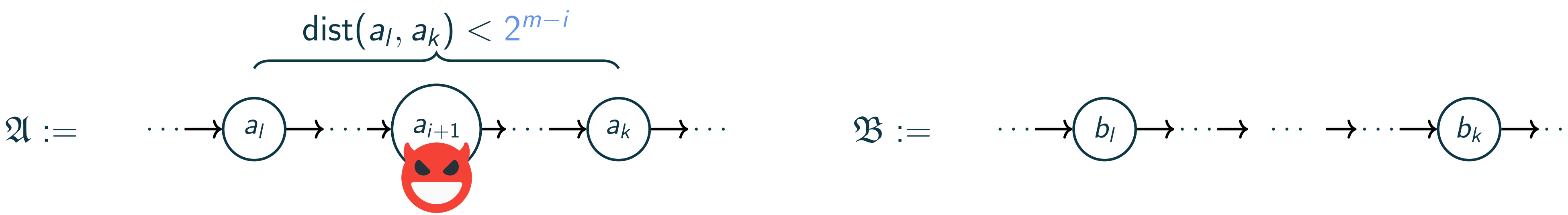

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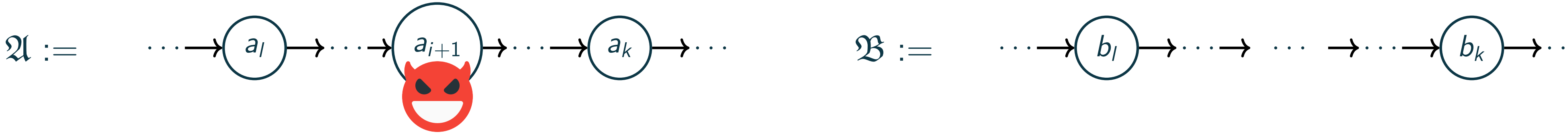

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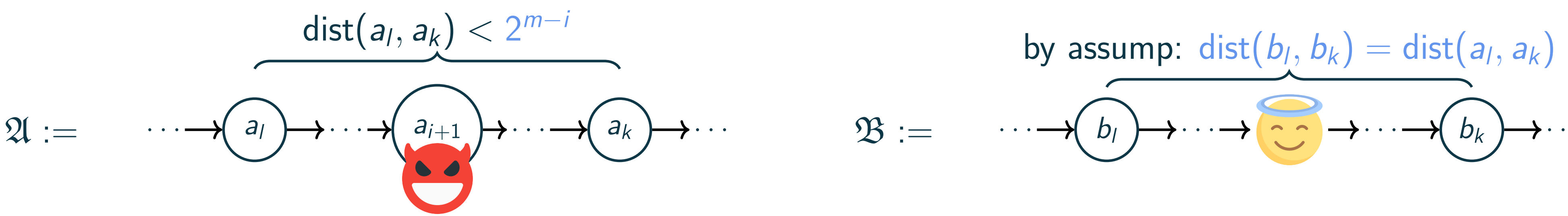

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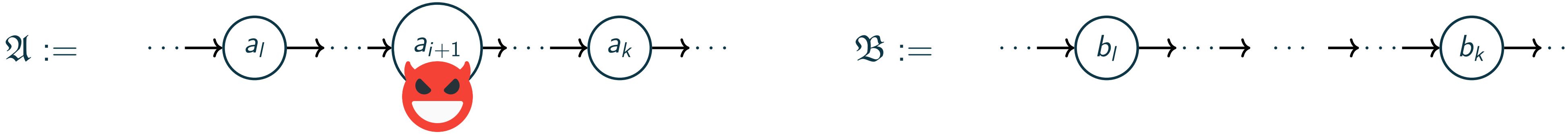

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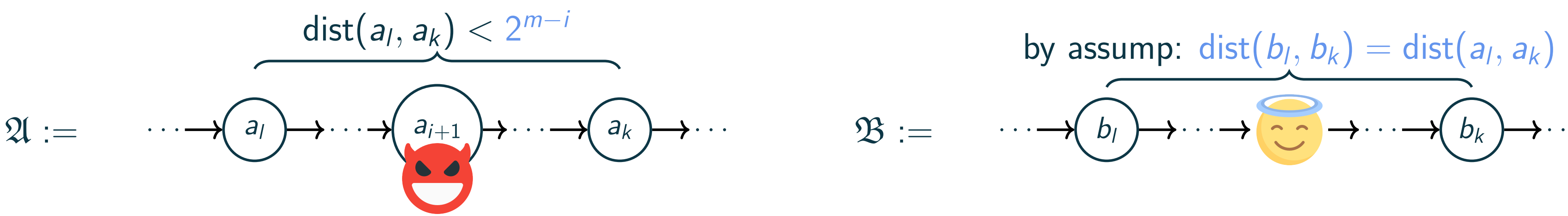

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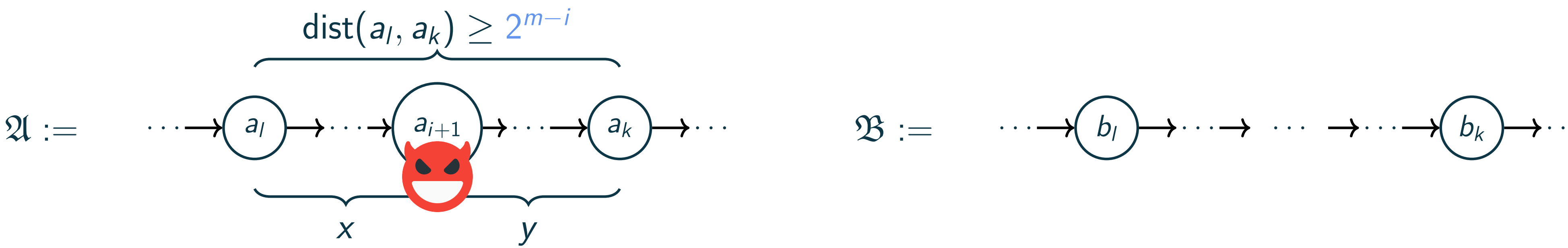

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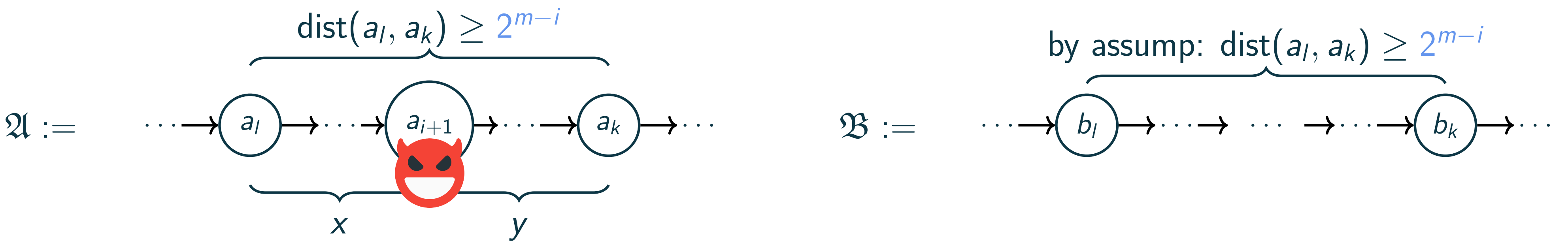

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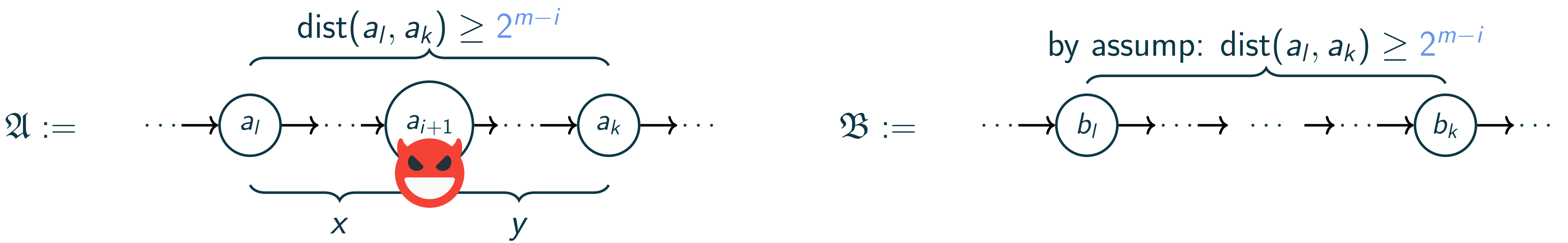

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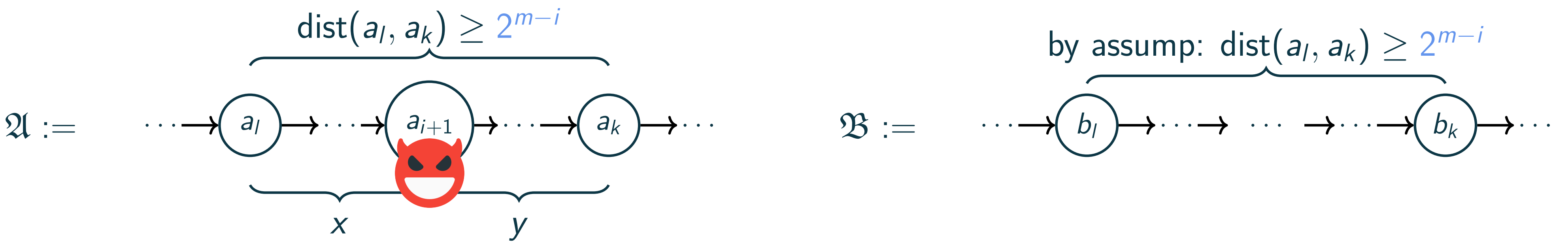

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- $x \geq 2^{m-i-1}$ and $y \geq 2^{m-i-1}$

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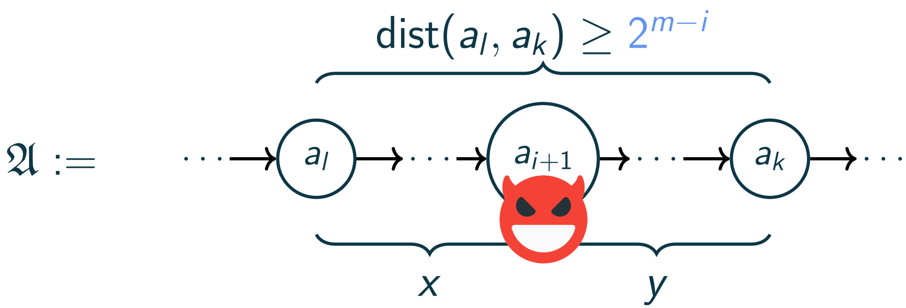

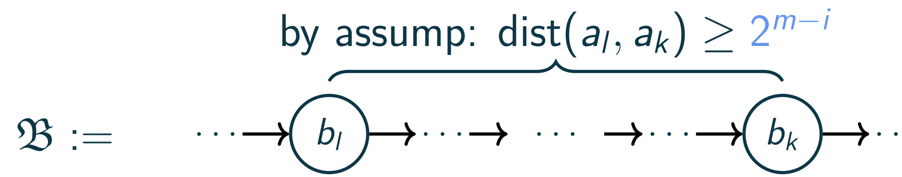

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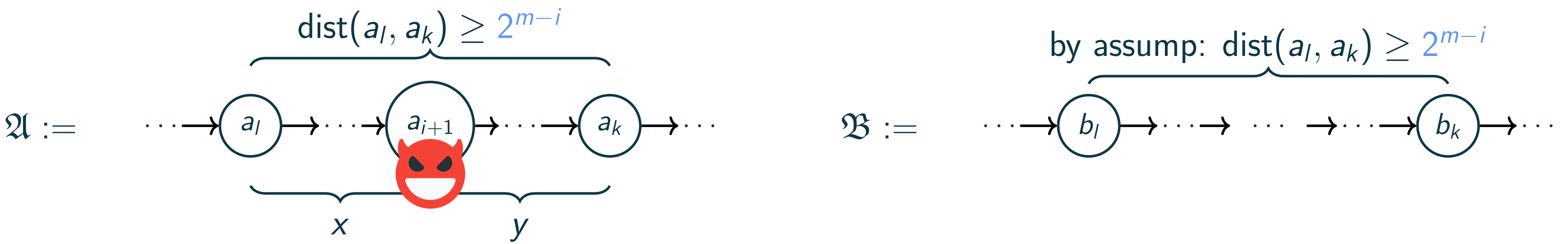

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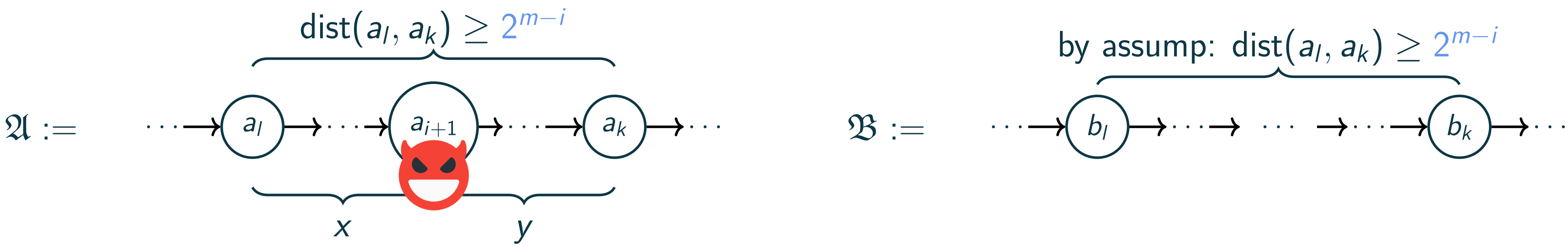

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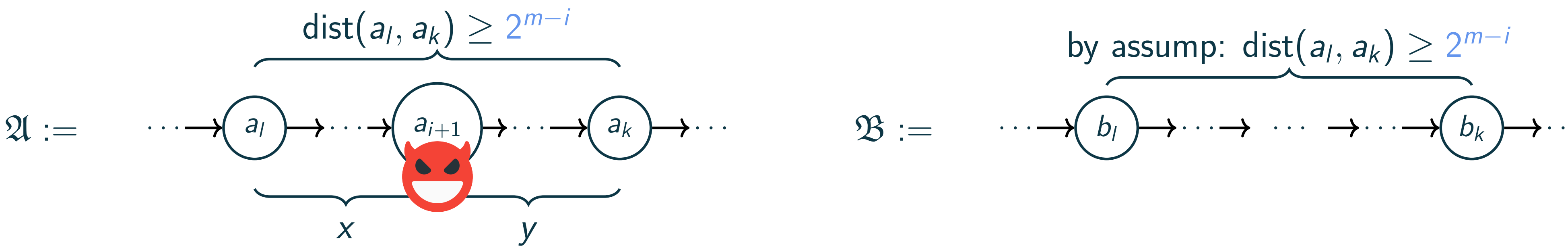

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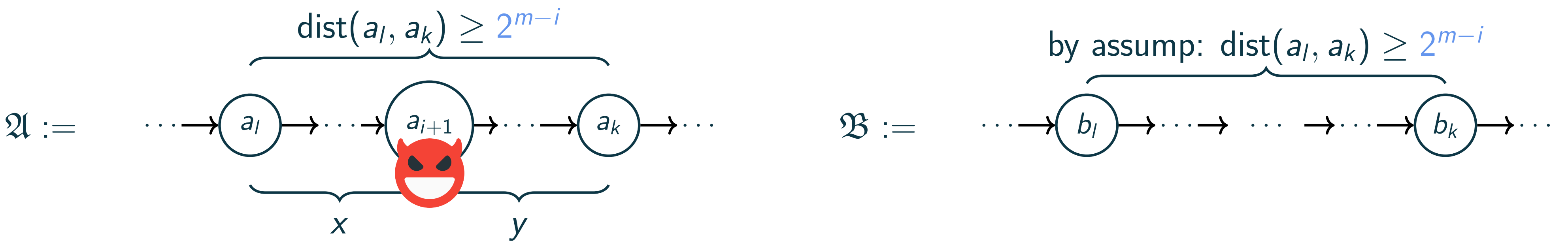

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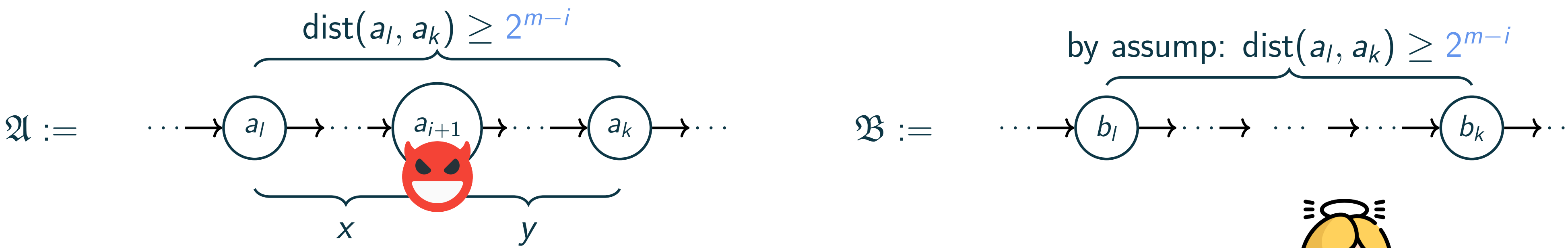

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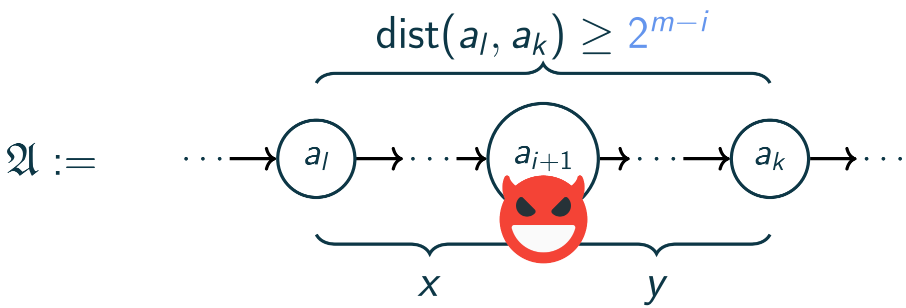

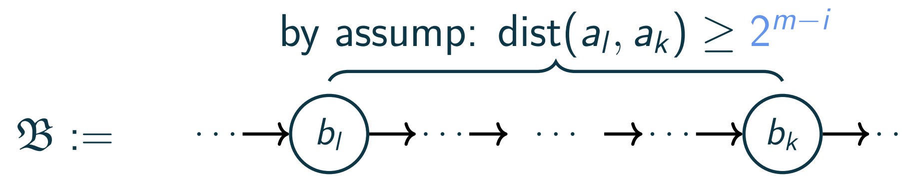

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More about Ehrenfeucht-Fraïssé games

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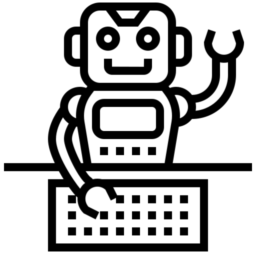

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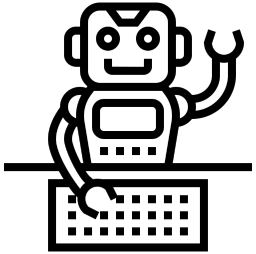

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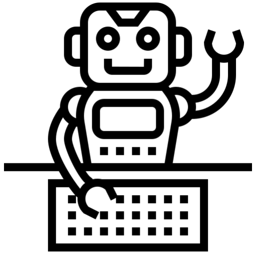

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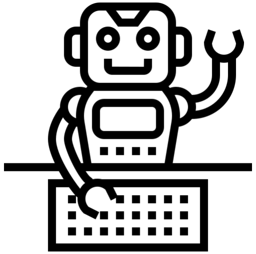

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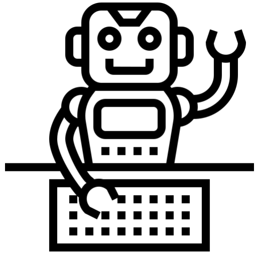

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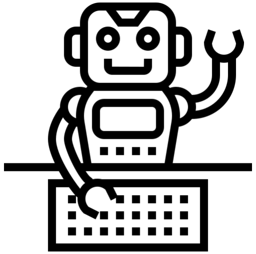

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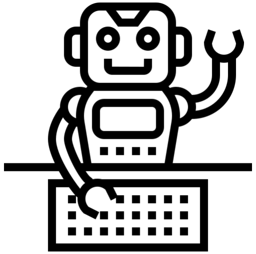

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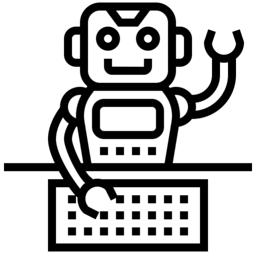

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Consult excellent slides by [Angelo Montanari] for more!

Logical Reductions

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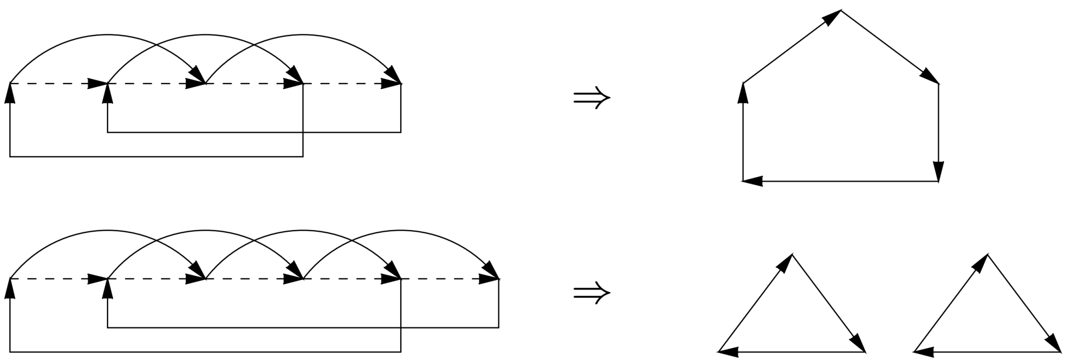

Reduction of parity to connectivity

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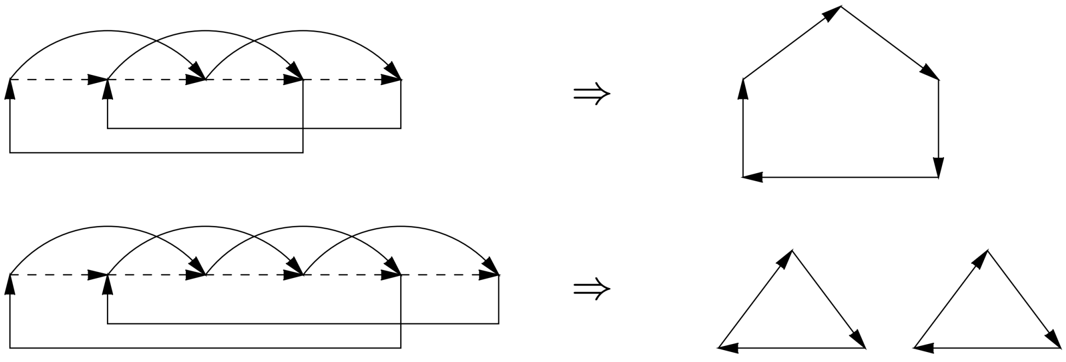

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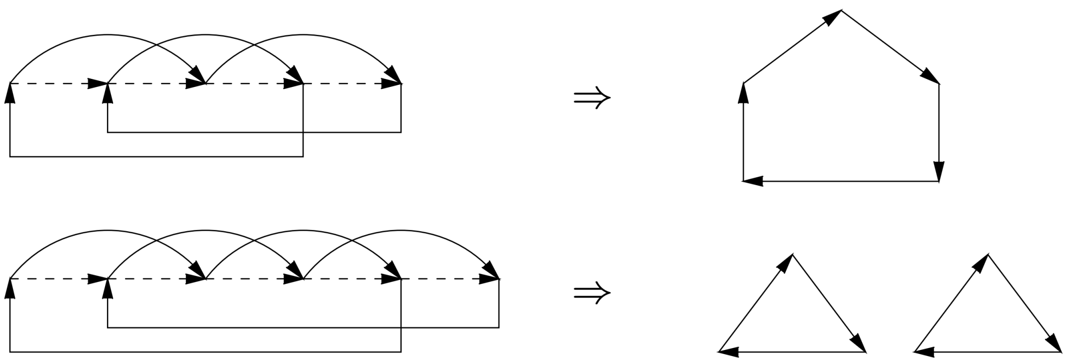

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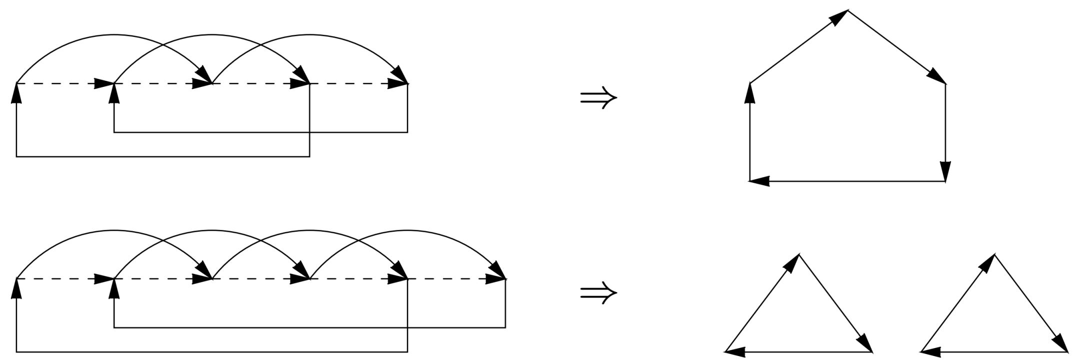

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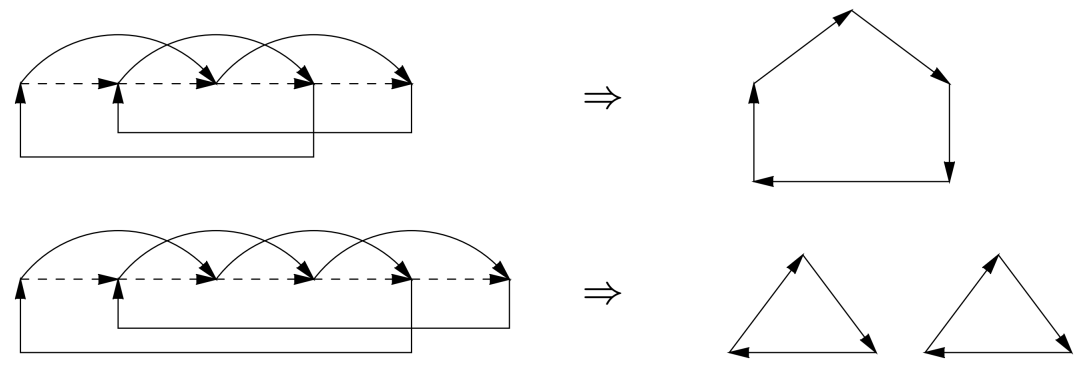

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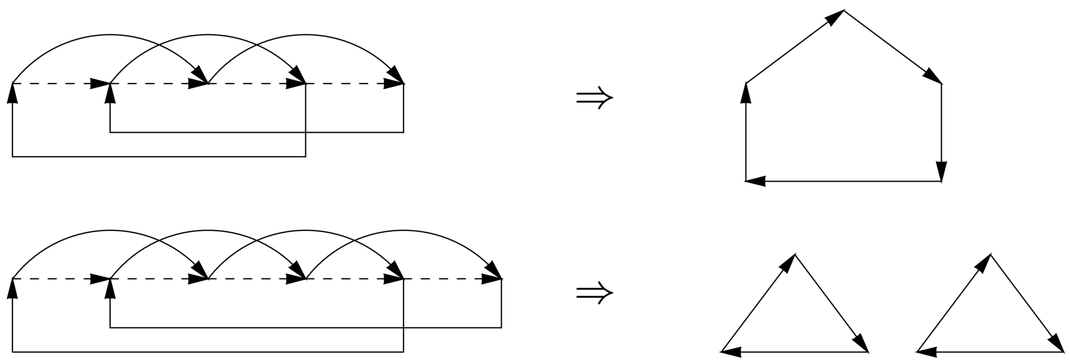

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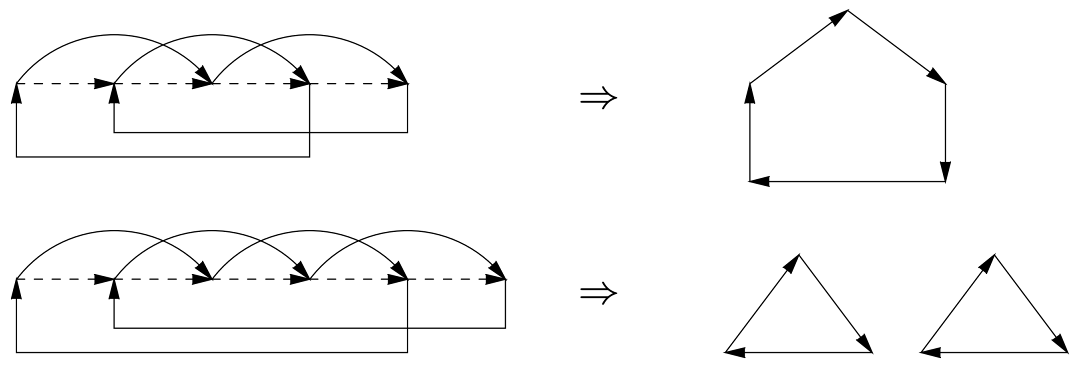

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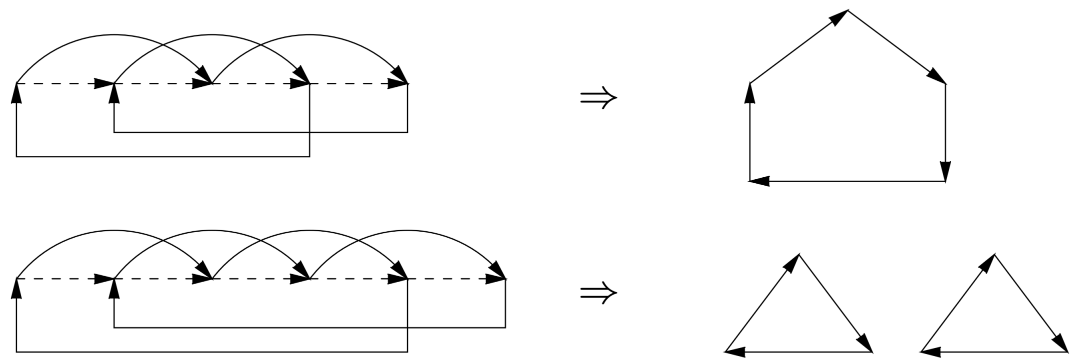

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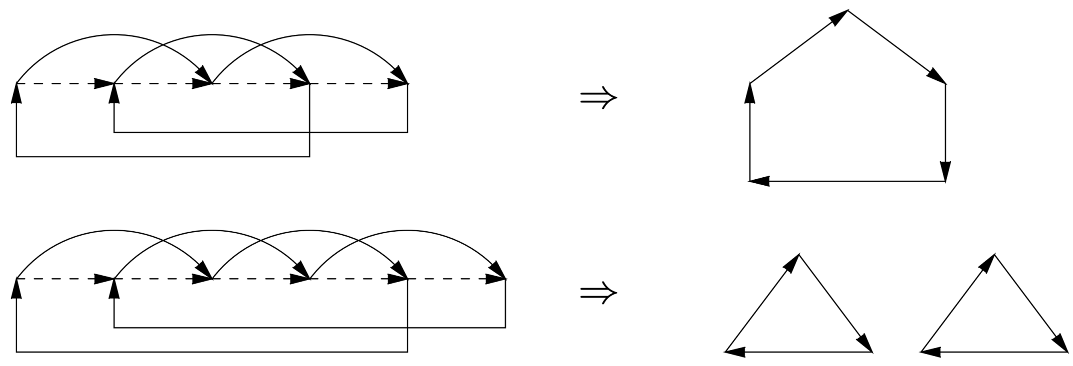

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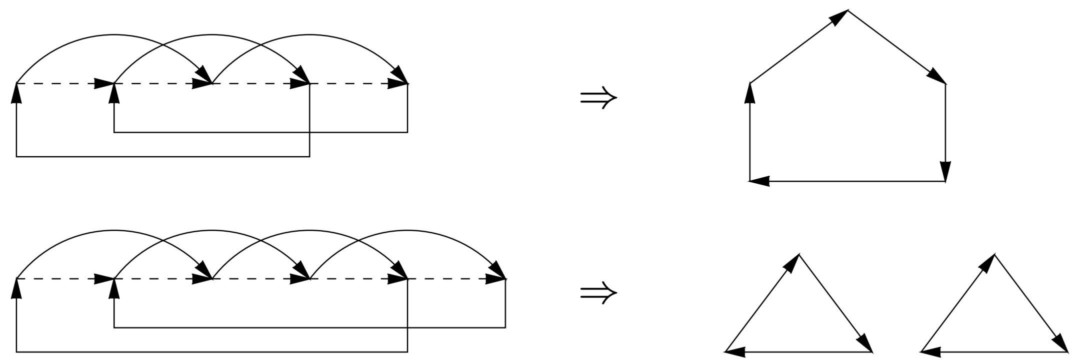

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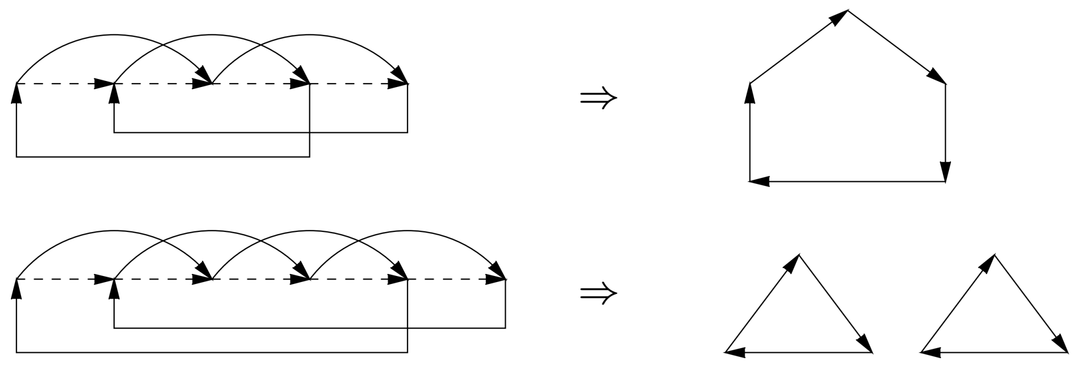

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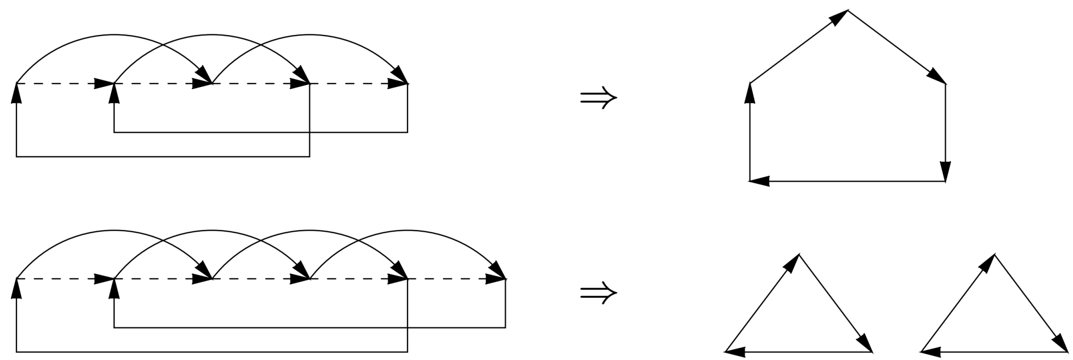

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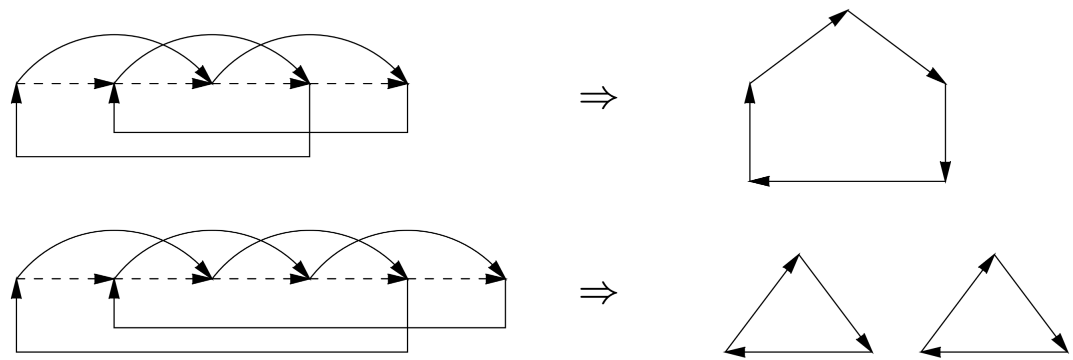

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Yes, with a notion of locality. Next 2–3 lectures!

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