

Complexity Theory

**Exercise 2: Undecidability and Rice's Theorem**

26th October 2022

**Exercise 2.1.** Using an oracle that decides the halting problem, construct a decider for the language  $\{ \langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts } w \}$ .

**Exercise 2.2.** A *useless state* in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Show that this language is undecidable.

**Exercise 2.3.** Show the following: “If a language  $\mathbf{L}$  is Turing-recognisable and  $\bar{\mathbf{L}}$  is many-one reducible to  $\mathbf{L}$ , then  $\mathbf{L}$  is decidable.”

**Exercise 2.4.** For this task assume an alphabet  $\Sigma$  with  $|\Sigma| > 1$ . Let

$$\mathbf{L} = \{ \langle \mathcal{M} \rangle \mid \mathcal{M} \text{ a TM that accepts } w^r \text{ whenever it accepts } w \text{ (for all } w \in \Sigma^*) \},$$

where  $w^r$  is the word  $w$  reversed. Show that  $\mathbf{L}$  is undecidable.

**Exercise 2.5.** Consider the following languages  $\mathbf{L}$  and  $\mathbf{L}'$ :

$$\begin{aligned} \mathbf{L} &= \{ \langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts } w \} \\ \mathbf{L}' &= \{ \langle \mathcal{M} \rangle \mid \mathcal{M} \text{ is a TM that does not accept any word} \} \end{aligned}$$

Show that there cannot exist a many-one reduction from  $\mathbf{L}$  to  $\mathbf{L}'$ .

**Exercise 2.6.** Show that every Turing-recognisable language can be mapping-reduced to the following language.

$$\{ \langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts the word } w \}$$