TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 3: Undecidability

Markus Krötzsch, Stephan Mennicke, Lukas Gerlach

Knowledge-Based Systems

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Decidability and Computability

Review: A language is

- recognisable (or semi-decidable, or recursively enumerable) if it is the language of all words recognised by some Turing machine
- decidable (or recursive) if it is the language of a Turing machine that allways halts, even on inputs that are not accepted
- undecidable if it is not decidable

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- undecidable if it is not decidable

Instead of acceptance of words, we can also consider other computations:

Definition 3.1: A TM $\mathcal{M}$ computes a partial function $f_{\mathcal{M}} : \Sigma^* \to \Sigma^*$ as follows. We have $f_{\mathcal{M}}(w) = v$ for a word $w \in \Sigma^*$ if $\mathcal{M}$ halts on input w with a tape that contains only the word $v \in \Sigma^*$ (followed by blanks).
In this case, the function $f_{\mathcal{M}}$ is called computable.
Total, computable functions are called recursive.

Functions may therefore be computable or uncomputable.

Undecidability is Real

A fundamental insight of computer science and mathematics is that there are undecidable languages:

Theorem 3.2: There are undecidable languages over every alphabet Σ.

Proof: See exercise. $\square$

Analogously, there are uncomputable functions.

Unknown $\neq$ Undecidable

How do we find concrete undecidable problems?

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It is not enough to not know how to solve a problem algorithmically!

Example 3.3: Let $\mathbf{L}_\pi$ be the set of all finite number sequences, that occur in the decimal representation of π. For example, `14159265` $\in \mathbf{L}_\pi$ and `41` $\in \mathbf{L}_\pi$.

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Example 3.3: Let $\mathbf{L}_\pi$ be the set of all finite number sequences, that occur in the decimal representation of π. For example, $\mathtt{14159265} \in \mathbf{L}_\pi$ and $\mathtt{41} \in \mathbf{L}_\pi$.

We do not know if the language $\mathbf{L}_\pi$ is decidable, but it might be (e.g., if every finite sequence of digits occurred in π, which, however, is not known to be true today).

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Example 3.4 (after Uwe Schöning): Let $\mathbf{L}_{\pi 7}$ be the set of all number sequences of the form 7^n that occur in the decimal representation of π.

$\mathbf{L}_{\pi 7}$ is decidable:

- Option 1: π contains sequences of arbitrary many 7. Then $\mathbf{L}_{\pi 7}$ is decided by a TM that accepts all words of the form 7^n.
- Option 2: π contains sequences of 7s only up to a certain maximal length ℓ. Then $\mathbf{L}_{\pi 7}$ is decided by a TM that accepts all words of the form 7^n with $n \leq \ell$.

In each possible case, we have a practical algorithm – we just don't know which one is correct.

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Answer: That depends on n ...

Definition 3.5: We define $S(n)$ as the largest number of steps that any DTM with n states and tape alphabet $\Gamma = \{\mathtt{x}, \text{␣}\}$ executes on the empty tape, before it eventually halts.

Observation: S is well defined.

- The number of TMs with at most n states is finite
- Among the relevant n-state TMs there must be a largest number of steps before halting (TMs that do not halt are ignored)

Busy Beaver

Tibor Radó, BB inventor

A small variation of the step counter function leads to the Busy-Beaver Problem:

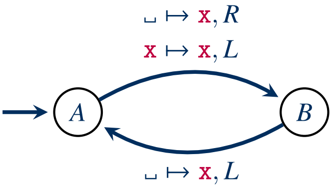
Definition 3.6: The Busy-Beaver function $\Sigma : \mathbb{N} \to \mathbb{N}$ is a total function, where $\Sigma(n)$ is the maximal number of $\mathtt{x}$ that a DTM with at most n states and tape alphabet $\Gamma = \{\mathtt{x}, \llcorner\!\lrcorner\}$ can write when starting on the empty tape and before it eventually halts.

Note: The exact value of $\Sigma(n)$ depends on details of the TM definition.

Most works in this area assume a two-sided infinite tape that can be extended to the left and to the right if necessary. We (often) assume a partial transition function in the following; such a TM halts if no transition is defined.

Example

The Busy-Beaver number $\Sigma(2)$ is 4 when using a two-way infinite tape.
The following TM (with partial transition function) implements this behaviour:



We obtain: $A\sqcup \vdash \mathtt{x}B\sqcup \vdash A\mathtt{xx} \vdash B\sqcup\mathtt{xx} \vdash A\sqcup\mathtt{xxx} \vdash \mathtt{x}B\mathtt{xxx}$

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- Then we can define a TM $\mathcal{M}_{\mathbf{\Sigma}}$ with tape alphabet $\{\mathtt{x}, \llcorner\!\lrcorner\}$ that computes $\mathtt{x}^n \mapsto \mathtt{x}^{\mathbf{\Sigma}(n)}$.

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- Let $\mathcal{M}_{+1}$ be a TM that computes $\mathtt{x}^n \mapsto \mathtt{x}^{n+1}$.
- Let $\mathcal{M}_{\times 2}$ be a TM that computes $\mathtt{x}^n \mapsto \mathtt{x}^{2n}$.
- Let k be the total number of states in $\mathcal{M}_\Sigma$, $\mathcal{M}_{+1}$, and $\mathcal{M}_{\times 2}$. There is a TM $\mathcal{I}_k$ with $k + 1$ states that writes the word $\mathtt{x}^k$ to the empty tape.

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- When executing $\mathcal{I}_k$, $\mathcal{M}_{\times 2}$, $\mathcal{M}_\Sigma$, and $\mathcal{M}_{+1}$ after another, the result is a TM with $\leq 2k$ states that writes $\Sigma(2k) + 1$ times $\mathtt{x}$ before halting.

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- Hence $\Sigma(2k) \geq \Sigma(2k) + 1$ – contradiction. $\square$

Proof Notes

Note 1: The proof involves an interesting idea of using TMs as "sub-routines" in other TMs. We will use this again later on.

Note 2: If a TM can compute $f : \mathbb{N} \to \mathbb{N}$ in the usual binary encoding, it is not hard to get a TM for $\mathtt{x}^n \mapsto \mathtt{x}^{f(n)}$ by just using unary encoding instead.

Note 3: Transforming an arbitrary TM into one that uses only symbols $\{\mathtt{x}, \llcorner\!\lrcorner\}$ on its tape is slightly more involved, but doable.

Note 4: To execute TMs after one another, we can assume w.l.o.g. that they terminate in a unique state that has no possible transitions. Then one can combine TMs by identifying this unique final state with the starting state of the next TM, which decreases the total number of states by merging states.

Note 5: Busy Beaver is increasing with its input, i.e., $\boldsymbol{\Sigma}(m) \leq \boldsymbol{\Sigma}(2k)$ for any $m < 2k$, so the proof works even if the composed machine has less than $2k$ states.

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n:	1	2
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For $n = 10$, one has found a lower bound of the form $\Sigma(10) > 3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}$, where the complete expression has more than 7.6×10^{12} occurrences of the number 3.

Universality

The Universal Machine

A first important observation of Turing was that TMs are powerful enough to simulate other TMs:

Step 1: Encode Turing Machines $\mathcal{M}$ as words $\langle \mathcal{M} \rangle$

Step 2: Construct a universal Turing Machine $\mathcal{U}$, which gets $\langle \mathcal{M} \rangle$ as input and then simulates $\mathcal{M}$

Step 1: encoding Turing Machines

Any reasonable encoding of a TM $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ is usable, e.g., the following (for DTMs):

- We use an alphabet $\{\mathtt{0}, \mathtt{1}, \#\}$
- States are enumerated in any order (beginning with q_0), and encoded in binary: $Q = \{q_0, \ldots, q_n\} \rightsquigarrow \langle Q \rangle = \text{bin}(0)\# \cdots \#\text{bin}(n)$
- We also encode Γ and the directions $\{R, L\}$ in binary
- A transition $\delta(q_i, \sigma_n) = \langle q_j, \sigma_m, D \rangle$ is encoded as 5-tuple: $\text{enc}(q_i, \sigma_n) = \text{bin}(i)\#\text{bin}(n)\#\text{bin}(j)\#\text{bin}(m)\#\text{bin}(D)$
- The transition function is encoded as a list of all these tuples, separated with #: $\langle \delta \rangle = (\text{enc}(q_i, \sigma_n)\#)_{q_i \in Q, \sigma_i \in \Gamma}$
- Combining everything, we set $\langle \mathcal{M} \rangle = \langle Q \rangle \#\# \langle \Sigma \rangle \#\# \langle \Gamma \rangle \#\# \langle \delta \rangle \#\# \langle q_{\text{accept}} \rangle \#\# \langle q_{\text{reject}} \rangle$

We can also encode arbitrary words to match this encoding:

- For a word $w = a_1 \cdots a_\ell$ we define $\langle w \rangle = \text{bin}(a_1)\# \cdots \#\text{bin}(a_\ell)$

Step 2: The Universal Turing Machine

We define the universal TM $\mathcal{U}$ as multi-tape TM:

Tape 1: Input tape of $\mathcal{U}$: contains $\langle\mathcal{M}\rangle\#\#\langle w\rangle$

Tape 2: Working tape of $\mathcal{U}$

Tape 3: Stores the state of the simulated TM

Tape 4: Working tape of the simulated TM

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The working principle of $\mathcal{U}$ is easily sketched:

- $\mathcal{U}$ validates the input, copies $\langle w\rangle$ to Tape 4, moves the head on Tape 4 to the start and initialises Tape 3 with $\mathrm{bin}(0)$ (i.e., $\langle q_0\rangle$).
- In each step $\mathcal{U}$ reads an (encoded) symbol from the head position on Tape 4, and searches for the simulated state (Tape 3) a matching transition in $\langle\mathcal{M}\rangle$ on Tape 1 (w.l.o.g. assume that the final states of the encoded TM have no transitions):
  - Transition found: update state on Tape 3; replace the encoded symbol on Tape 4 by the new symbol; move the head on Tape 4 accordingly
  - Transition not found: if the state on Tape 3 is q_{accept}, then go to the final accepting state; else go to the final rejecting state

The Theory of Software

Theorem 3.8: There is a universal Turing Machine $\mathcal{U}$, that, when given an input $\langle\mathcal{M}\rangle\#\#\langle w\rangle$, simulates the behaviour of a DTM $\mathcal{M}$ on w:

- If $\mathcal{M}$ halts on w, then $\mathcal{U}$ halts on $\langle\mathcal{M}\rangle\#\#\langle w\rangle$ with the same result
- If $\mathcal{M}$ does not halt on w, then $\mathcal{U}$ does not halt on $\langle\mathcal{M}\rangle\#\#\langle w\rangle$ either

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Our construction is for DTMs that recognise languages ("Turing acceptors") – DTMs that compute partial functions can be simulated in a similar fashion.

Practical consequences:

- Universal computers are possible
- We don't have to buy a new computer for every application
- Software exists

Undecidable Problems and Reductions

The Halting Problem

A classical undecidable problem:

Definition 3.9: The Halting Problem consists in the following question:

Given a TM $\mathcal{M}$ and a word w,
will $\mathcal{M}$ ever halt on input w?

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We can formulate the Halting Problem as a word problem by encoding $\mathcal{M}$ and w:

Definition 3.10: The Halting Problem is the word problem for the language

$$\mathbf{P}_{\mathsf{Halt}} = \{\langle\mathcal{M}\rangle\#\#\langle w\rangle \mid \mathcal{M} \text{ halts on input } w\},$$

where $\langle\mathcal{M}\rangle$ and $\langle w\rangle$ are suitable encodings of $\mathcal{M}$ and w, for which ## can be used as separator.

Remark: Wrongly encoded inputs are rejected.

“Proof” by Intuition

Theorem 3.11: The Halting Problem $\mathbf{P}_{\text{Halt}}$ is undecidable.

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Example 3.12: Goldbach’s Conjecture (Christian Goldbach, 1742) states that every even number $n \geq 4$ is the sum of two primes. For instance, $4 = 2 + 2$ and $100 = 47 + 53$.

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Example 3.12: Goldbach's Conjecture (Christian Goldbach, 1742) states that every even number $n \geq 4$ is the sum of two primes. For instance, $4 = 2 + 2$ and $100 = 47 + 53$.

One can easily give an algorithm $\mathcal{A}$ that verifies Goldbach's conjecture systematically by testing it for every even number starting with 4:

- Success: Test the next even number
- Failure: Terminate with output "Goldbach was wrong!"

The question "Will $\mathcal{A}$ halt?" therefore is equivalent ot the question
"Is Goldbach's conjecture wrong?"

Many other important open problems could be solved in this way.

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Then one can construct a TM $\mathcal{D}$ that does the following:

(1) Check if the given input is a TM encoding $\langle\mathcal{M}\rangle$

(2) Simulate $\mathcal{H}$ on input $\langle\mathcal{M}\rangle\#\#\langle\langle\mathcal{M}\rangle\rangle$, that is, check if $\mathcal{M}$ halts on $\langle\mathcal{M}\rangle$

(3) If yes, enter an infinite loop;
if no, halt and accept

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(3) If yes, enter an infinite loop;
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Will $\mathcal{D}$ accept the input $\langle\mathcal{D}\rangle$?

$\mathcal{D}$ halts and accepts   if and only if $\mathcal{D}$ does not halt

Contradiction. $\square$

Proof by Reduction

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An algorithm:

- Input: natural number k (in binary)
- Iterate over all Turing machines $\mathcal{M}$ that have k states and tape alphabet $\{\mathtt{x}, \sqcup\}$:
  - Decide if $\mathcal{M}$ halts on the empty input ε
    (possible if the Halting problem is decidable)
  - If yes, then simulate $\mathcal{M}$ on the empty input and, when $\mathcal{M}$ has halted, count the number of $\mathtt{x}$ on the tape
    (possible, since there are universal TMs)
- Output: the maximal number of $\mathtt{x}$ written.

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    (possible, since there are universal TMs)
- Output: the maximal number of $\mathtt{x}$ written.

This algorithm would compute the Busy-Beaver function $\mathbf{\Sigma} : \mathbb{N} \to \mathbb{N}$.

We have already shown that this is impossible – contradiction. $\square$

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Our previous proof constructs an algorithm for one task (Busy Beaver) by calling subroutines for another task (the Halting Problem)

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This idea can be generalised:

Informal Definition 3.13: A problem **P** is Turing reducible to a problem **Q** (in Symbols: **P** $\leq_T$ **Q**), if **P** can be solved by a program that may call **Q** as a sub-program.

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Our previous proof constructs an algorithm for one task (Busy Beaver) by calling subroutines for another task (the Halting Problem)

This idea can be generalised:

Informal Definition 3.13: A problem **P** is Turing reducible to a problem **Q** (in Symbols: **P** $\leq_T$ **Q**), if **P** can be solved by a program that may call **Q** as a sub-program.

Example 3.14: Our proof uses a reduction of the Busy-Beaver computation to the Halting problem. Note that the subroutine might be called exponentially many times here.

To make this more formal, we need oracles.

Oracles

Definition 3.15: An Oracle Turing Machine (OTM) is a Turing machine $\mathcal{M}$ with a special tape, called the oracle tape, and distinguished states $q_?$, q_{yes}, and q_{no}. For a language $\mathbf{O}$, the oracle machine $\mathcal{M}^{\mathbf{O}}$ can, in addition to the normal TM operations, do the following:

Whenever $\mathcal{M}^{\mathbf{O}}$ reaches $q_?$, its next state is q_{yes} if the content of the oracle tape is in $\mathbf{O}$, and q_{no} otherwise.

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Definition 3.16: A problem **P** is Turing reducible to a problem **Q** (in symbols: $\mathbf{P} \leq_T \mathbf{Q}$), if **P** is decided by an OTM $\mathcal{M}^{\mathbf{Q}}$ with oracle **Q**.

Undecidability via Turing Reductions

One can use Turing reductions to show undecidability:

Theorem 3.17: If **P** is undecidable and **P** $\leq_T$ **Q**, then **Q** is undecidable.

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Proof: Via contrapositive: If **P** $\leq_T$ **Q** and **Q** is decidable, then we can implement the OTM that shows **P** $\leq_T$ **Q** as a regular TM, which shows that **P** is decidable. □

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Here is a small application:

Theorem 3.18: The language $\mathbf{P}_{\overline{\text{Halt}}} = \{\langle \mathcal{M} \rangle \#\# \langle w \rangle \mid \mathcal{M} \text{ does not halt on } w\}$ (the "Non-Halting Problem") is undecidable.

Proof sketch: Decide Halting by using $\mathbf{P}_{\overline{\text{Halt}}}$ as an oracle and inverting the result. Check TM encoding first (wrong encodings are rejected by Halting and Non-Halting). $\square$

ε-Halting

Special cases of the Halting Problem are usually not simpler:

Definition 3.19: The ε-Halting Problem consists in the following question:
Given a TM $\mathcal{M}$,
will $\mathcal{M}$ ever halt on the empty input ε?

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Theorem 3.20: The ε-Halting Problem is undecidable.

Proof: We define an oracle machine for deciding Halting:

- Input: A Turing machine $\mathcal{M}$ and a word w.
- Construct a TM $\mathcal{M}_w$ that runs in two phases:
  (1) Delete the input tape and write the word w instead
  (2) Process the input like $\mathcal{M}$
- Solve the ε-Halting problem for $\mathcal{M}_w$ (oracle).
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- Output: output of the ε-Halting Problem

This Turing-reduces Halting to ε-halting, so the latter is also undecidable. □

Post's Correspondence Problem (PCP)

There are also undecidable problems that are not related to halting (in an obvious way).

Definition 3.21: Post's Correspondence Problem (PCP) is defined as follows.

Given: a finite sequence of pairs of words (over an alphabet Σ)

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \ldots \quad \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

Question: Is there a (non-empty), finite sequence of numbers $i_1, \ldots, i_\ell$ such that

$$x_{i_1} \cdots x_{i_\ell} = y_{i_1} \cdots y_{i_\ell},$$

where $\ell > 0$ and $1 \leq i_j \leq k$ for every $1 \leq j \leq \ell$?

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Attention: Even though the intuition with dominos is helpful, there is another problem actually called the domino problem!

Theorem 3.22: PCP is undecidable.

The Tiling Problem / The Domino Problem

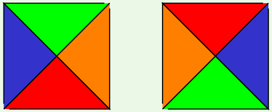
The actual domino problem is quite a bit different from PCP.

Definition 3.23: The Tiling Problem / Domino Problem is defined as follows.

Given: a finite set of tile types $\mathcal{T}$ with four colors each (taken from a countably infinite set of colors)

Question: Can an $\mathbb{N} \times \mathbb{N}$ grid be completely tiled with only tile types from $\mathcal{T}$ (without rotation or any other transformation)? That is, can tiles of types $\mathcal{T}$ be arranged in a way such that adjacent colors match?

Example 3.24: The following two tile types can tile an $\mathbb{N} \times \mathbb{N}$ grid.



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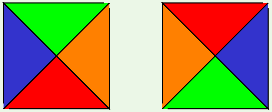

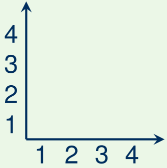

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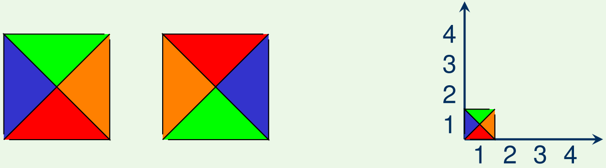

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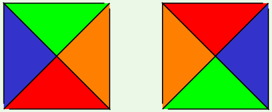

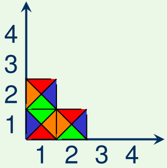

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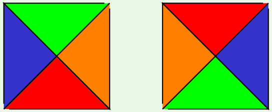

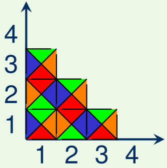

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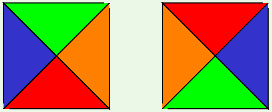

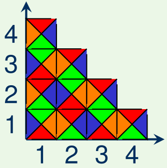

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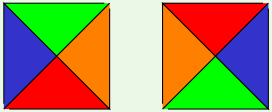

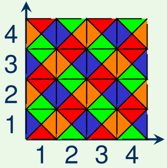

A Glimpse of the Arithmetical Hierarchy

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Definition 3.25: We define the following classes of languages:

- Σ_1^0 [Π_1^0] is the class of all [co-]semi-decidable languages.

  Note: A language L is co-semi-decidable if $\overline{L}$ is semi-decidable.

- Σ_{k+1}^0 [Π_{k+1}^0] with $k \geq 1$ is the class of all languages that can be [co-]semi-decided with OTMs with an oracle from Σ_k^0 [Π_k^0].

  If the oracle is from Σ_k^0 or Π_k^0 actually does not matter since its answer can be swapped.

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Definition 3.25: We define the following classes of languages:

- Σ^0_1 [Π^0_1] is the class of all [co-]semi-decidable languages.

  Note: A language L is co-semi-decidable if $\overline{L}$ is semi-decidable.

- Σ^0_{k+1} [Π^0_{k+1}] with $k \geq 1$ is the class of all languages that can be [co-]semi-decided with OTMs with an oracle from Σ^0_k [Π^0_k].

  If the oracle is from Σ^0_k or Π^0_k actually does not matter since its answer can be swapped.

Theorem 3.26: The tiling problem is Π^0_1-complete.

Given tile types as usual and one designated type t_0, deciding if tiling works such that t_0 occurs infinitely many often in the bottom row is not in Σ^0_n for any $n \geq 1$ but complete for Σ^1_1, the first level of the **analytical hierarchy** (not introduced here).

See `https://doi.org/10.1016/S0304-0208(08)73075-5` for a good tiling problem reference.

Summary and Outlook

Busy Beaver is uncomputable

Halting is undecidable (for many reasons)

Oracles and Turing reductions formalise the notion of a “subroutine” and help us to transfer our insights from one problem to another

What’s next?

- Some more undecidability
- Recursion and self-referentiality
- Actual complexity classes