TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

Introduction to Automatic Structures

Summer Semester 2016

Exercise Sheet 5 – Word Automatic Structures and First-Order-Logic

7th June 2016

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Hint. For two structures $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \leq_{\mathsf{FOL}} \mathcal{B}$ if $\mathcal{A}$ is FOL-definable in $\mathcal{B}$.

Exercise 5.1 Decide whether the following statements are true or false. Justify your answer.

(a) Each automata presentable structure has a countable domain.

(b) There is a finite structure which is not automata presentable.

(c) If $\mathcal{A} \cong \mathcal{B}$, and $\mathcal{A}$ is automata presentable, then $\mathcal{B}$ is automata presentable, too.

(d) If $\mathcal{A} \leq_{\mathsf{FOL}} \mathcal{B}$, and $\mathcal{A}$ is automata presentable, then $\mathcal{B}$ is automata presentable, too.

(e) If $\mathcal{A} \cong \mathcal{B}$, and $\mathcal{A}$ is universal, then $\mathcal{B}$ is universal, too.

(f) If $\mathcal{A} \leq_{\mathsf{FOL}} \mathcal{B}$, and $\mathcal{A}$ is universal, then $\mathcal{B}$ is universal, too.

(g) If $\mathcal{A}$ is a structure with a pairing function, i.e., a bijection $p\colon A \times A \to A$, then $\mathcal{A}$ is not automata presentable.

Exercise 5.2 Let $\mathcal{A} := (A; R_1, \ldots, R_n)$ be a structure such that k_i is the arity of R_i. A *congruence relation* $\approx$ on $\mathcal{A}$ is an equivalence relation on A that is compatible with each relation R_i of $\mathcal{A}$, i.e., for each index $i \in \{1, \ldots, n\}$, $(x_1, \ldots, x_{k_i}) \in R_i$ and $x_1 \approx y_1, \ldots, x_{k_i} \approx y_{k_i}$ imply $(y_1, \ldots, y_{k_i}) \in R_i$. Then, the *quotient* of $\mathcal{A}$ with respect to $\approx$ is defined as the structure

$$\mathcal{A}/_{\approx} := (A/_{\approx}; R_1/_{\approx}, \ldots, R_m/_{\approx}),$$

the domain of which is defined as the quotient

$$A/_{\approx} := \{\, [a]_{\approx} \mid a \in A \,\}$$

that consists of *congruence classes* $[a]_{\approx} := \{\, b \in A \mid a \approx b \,\}$, and furthermore for each index $i \in \{1, \ldots, m\}$,

$$R_i/_{\approx} := \{\, ([a_1]_{\approx}, \ldots, [a_{k_i}]_{\approx}) \mid (a_1, \ldots, a_{k_i}) \in R_i \,\}.$$

Show the following claim:
If $\mathcal{A}$ and $\approx$ are automatic, then their quotient $\mathcal{A}/_{\approx}$ is automata presentable.

Hint. Show that $\{\, a \in A \mid \forall b \in A \colon a \approx b \implies a \leq_{\mathsf{llex}} b \,\}$ is FOL-definable in $(A; R_1, \ldots, R_m, \approx, \leq_{\mathsf{llex}})$.

Exercise 5.3 Let $\mathcal{A} := (A; R_1, \ldots, R_m)$ be a structure such that k_i is the arity of R_i. Show that the following statements are equivalent:

(a) $\mathcal{A}$ is automata presentable.

(b) $\mathcal{A}$ is isomorphic to an automatic structure.

(c) There is a regular language $L_\delta \subseteq \Sigma^*$ and a surjective function $\nu\colon L_\delta \to A$ such that the following conditions are satisfied:

- The kernel $L_\epsilon := \ker \nu = \{\, (w_1, w_2) \in L_\delta \times L_\delta \mid \nu(w_1) = \nu(w_2) \,\}$ is automatic,
- for each $i \in \{1, \ldots, m\}$, the relation $L_{R_i} := \{\, (w_1, \ldots, w_{k_i}) \in (L_\delta)^{k_i} \mid (\nu(w_1), \ldots, \nu(w_{k_i})) \in R_i \,\}$ is automatic, and
- L_ϵ is a congruence relation of $(L_\delta; L_{R_1}, \ldots, L_{R_m})$.

Exercise 5.4 Let $\mathcal{A} := (A; R_1, \ldots, R_m)$ and $\mathcal{B} := (B; S_1, \ldots, S_n)$ be structures. Prove that the following statements are equivalent:

(a) $\mathcal{A}$ is isomorphic to a structure that is FOL-definable in $\mathcal{B}$.

(b) There is a *FOL-interpretation* of $\mathcal{A}$ in $\mathcal{B}$, i.e., there are FOL-formulae $\phi_\delta, \phi_\epsilon, \phi_{R_1}, \ldots, \phi_{R_m}$ over the signature of $\mathcal{B}$ and a surjective function $\nu\colon \phi_\delta^{\mathcal{B}} \to A$ such that the following conditions hold:

- $\phi_\epsilon^{\mathcal{B}} = \ker \nu$,
- for each $i \in \{1, \ldots, m\}$, it is true that $\phi_{R_i}^{\mathcal{B}} = \{\, (w_1, \ldots, w_{k_i}) \mid (\nu(w_1, \ldots, \nu(w_{k_i})) \in R_i \,\}$, and
- $\phi_\epsilon^{\mathcal{B}}$ is a congruence relation of $(\phi_\delta^{\mathcal{B}}; \phi_{R_1}^{\mathcal{B}}, \ldots, \phi_{R_m}^{\mathcal{B}})$.

Exercise 5.5 Consider the structure $(\mathbb{N}; +, |_p)$ where $+$ denotes the addition of two natural numbers, and $|_p$ is the *weak divisibility* for a $p \in \mathbb{N} \setminus \{0, 1\}$ defined by $x \mid_p y$ if $\exists k \in \mathbb{N}\colon x = p^k \mid y$.

Prove the following statements:

(a) $(\mathbb{N}; +, |_p)$ is automata presentable.

(b) $(\mathbb{N}; +, |_p)$ is universal.

Hint. The structures $(\mathbb{N}; +, |_p)$ and $(\{0, 1, \ldots, p-1\}^*; \preceq, S_0, S_1, \ldots, S_{p-1}, \mathrm{EqualLength})$ are mutually FOL-definable.