Connecting many-sorted structures and theories through adjoint functions

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Abstract. In a previous paper, we have introduced a general approach for connecting two many-sorted theories through connection functions that behave like homomorphisms on the shared signature, and have shown that, under appropriate algebraic conditions, decidability of the validity of universal formulae in the component theories transfers to their connection. This work generalizes decidability transfer results for so-called $\mathcal{E}$-connections of modal logics. However, in this general algebraic setting, only the most basic type of $\mathcal{E}$-connections could be handled. In the present paper, we overcome this restriction by looking at pairs of connection functions that are adjoint pairs for partial orders defined in the component theories.

1 Introduction

Transfer of decidability from component theories/logics to their combination have been investigated independently in different areas of computer science and logic, and only recently it has turned out that there are close connections between different such transfer results. For example, in modal logics it was shown that in many cases decidability of (relativized) validity transfers from two modal logics to their fusion [14, 21, 23, 3]. In automated deduction, the Nelson-Oppen combination procedure [18, 17] and combination procedures for the word problem [20, 19, 4] were generalized to the case of the union of theories over non-disjoint signatures [7, 22, 5, 8, 11, 2], and it could be shown that some of these approaches [11, 2] actually generalize decidability transfer results for fusions of modal logics from equational theories induced by modal logics to more general first-order theories satisfying certain model-theoretic restrictions. In particular, these generalizations no longer require the shared theory to be the theory of Boolean algebras.

The purpose of this work is to develop similar algebraic generalizations of decidability transfer results for so-called $\mathcal{E}$-connections [15] of modal logics. Intuitively, the difference between fusion and $\mathcal{E}$-connection can be explained as follows. A model of the fusion is obtained from two models of the component logics by identifying their domains. In contrast, a model of the $\mathcal{E}$-connection consists of two separate models of the component logics together with certain connecting relations between their domains. There are also differences in the syntax of the combined logic. In the case of the fusion, the Boolean operators are shared, and all operators can be applied to each other without restrictions. In the case of the $\mathcal{E}$-connection, there are two copies of the Boolean operators, and operators of the different logics cannot be mixed; the only connection between the logics are new modal operators that are induced by the connecting relations.

In [1], this connection approach was generalized to the more general setting of connecting many-sorted first-order theories. The use of many-sorted theories allowed us to keep the domains separate and to restrict the way function symbols can be applied to each other. To be more precise, let T_1, T_2 be two many-sorted theories that may share some sorts as well as function and relation symbols. We first build the disjoint union $T_1 \uplus T_2$ of these two theories (by using disjoint copies of the shared parts), and then connect them by introducing *connection functions* between the shared sorts. These connection functions must behave like homomorphisms for the shared function and predicate symbols, i.e., the axioms stating this are added to $T_1 \uplus T_2$. This corresponds to the fact that the new modal operators in the $\mathcal{E}$-connection approach interact with the Boolean operators of the component logics. In [1], we started with the simplest case where there is just one connection function, and showed that decidability of the universal fragments of T_1, T_2 transfers to their connection whenever certain model-theoretic conditions are satisfied. The approach was then extended to the case of several connection functions, and to variants of the general combination scheme where the connection function must satisfy additional properties (like being surjective, an embedding, or an isomorphism).

However, in the $\mathcal{E}$-connection approach introduced in [15], one usually considers not only the modal operator induced by a connecting relation, but also the modal operator induced by its inverse. It is not adequate to express these two modal operators by independent connection function going in different directions since this does not capture the relationships that must hold between them. For example, if $\Diamond$ is the diamond operator induced by the connecting relation E, and $\Box^-$ is the box operator induced by its inverse E^-, then the formulae $x \to \Box^-\Diamond x$ and $\Diamond\Box^- y \to y$ are valid in the $\mathcal{E}$-connection. In order to express these relationships in the algebraic setting without assuming the presence of the Boolean operators in the shared theory, we replace the logical implication $\to$ by a partial order $\leq$,[1] and require that $x \leq r(\ell(x))$ and $\ell(r(y)) \leq y$ holds for the corresponding connection functions. If ℓ, r are also order preserving, then this means that ℓ, r is a pair of *adjoint functions* for the partial order $\leq$. We call the connection of two theories obtained this way an *adjoint theory connection.*

In this paper we give an abstract algebraic condition under which the decidability of the universal fragment transfers from the component theories to their adjoint theory connection. In contrast to the conditions in [1], which are compatibility conditions between a shared theory and the component theories, this is a condition that requires the existence of certain subtheories of the component

[1] In the presence of (some of) the Boolean operators, this partial order is obtained in the usual way, e.g., by defining $x \leq y$ iff $x \sqcup y = y$, where $\sqcup$ is the join (disjunction) operator. Note that the applications of $\Diamond$ and $\Box^-$ preserve this order.

theories, but these subtheories need not be the same for different components. We then give sufficient conditions under which our new condition is satisfied. In particular, this shows that the decidability transfer results for $\mathcal{E}$-connection with inverse connection modalities follow from our more general algebraic result.

2 Notation and definitions

In this section, we fix the notation and give some important definitions, in particular a formal definition of the adjoint connection of two theories. In addition, we show some simple results regarding adjoint functions in partially ordered set.

Basic model theory We use standard *many-sorted first-order logic* (see, e.g., [9]), but try to avoid the notational overhead caused by the presence of sorts as much as possible. Thus, a *signature* Ω consists of a non-empty set of sorts $\mathcal{S}$ together with a set of function symbols $\mathcal{F}$ and a set of predicate symbols $\mathcal{P}$. The function and predicate symbols are equipped with arities from $\mathcal{S}^*$ in the usual way. For example, if the arity of $f \in \mathcal{F}$ is $S_1S_2S_3$, then this means that the function f takes tuples consisting of an element of sort S_1 and an element of sort S_2 as input, and produces an element of sort S_3. We consider logic with equality, i.e., the set of predicate symbols contains a symbol $\approx_S$ for equality in every sort S. Usually, we will just use $\approx$ without explicitly specifying the sort.

Terms and first-order formulae over Ω are defined in the usual way, i.e., they must respect the arities of function and predicate symbols, and the variables occurring in them are also equipped with sorts. An *Ω-atom* is a predicate symbol applied to (sort-conforming) terms, and an *Ω-literal* is an atom or a negated atom. A *ground* literal is a literal that does not contain variables. We use the notation $\phi(\underline{x})$ to express that ϕ is a formula whose free variables are among the ones in the tuple of variables $\underline{x}$. An *Ω-sentence* is a formula over Ω without free variables. An *Ω-theory* T is a set of Ω-sentences (called the axioms of T). If T, T' are Ω-theories, then we write (by a slight abuse of notation) $T \subseteq T'$ to express that all the axioms of T are logical consequences of the axioms of T'. The formula ϕ is called *open* iff it does not contain quantifiers; it is called *universal* iff it is obtained from an open formula by adding a prefix of universal quantifiers. The theory T is a *universal theory* iff its axioms are universal sentences.

From the semantic side, we have the standard notion of an *Ω-structure* $\mathcal{A}$, which consists of non-empty and pairwise disjoint domains A_S for every sort S, and interprets function symbols f and predicate symbols P by functions $f^{\mathcal{A}}$ and predicates $P^{\mathcal{A}}$ according to their arities. By A we denote the union of all domains A_S. Validity of a formula ϕ in an Ω-structure $\mathcal{A}$ ($\mathcal{A} \models \phi$), satisfiability, and logical consequence are defined in the usual way. The Ω-structure $\mathcal{A}$ is a *model* of the Ω-theory T iff all axioms of T are valid in $\mathcal{A}$. The class of all models of T is denoted by $Mod(T)$.

If $\phi(\underline{x})$ is a formula with free variables $\underline{x} = x_1, \ldots, x_n$ and $\underline{a} = a_1, \ldots, a_n$ is a (sort-conforming) tuple of elements of A, then we write $\mathcal{A} \models \phi(\underline{a})$ to express that $\phi(\underline{x})$ is valid in $\mathcal{A}$ under the assignment $\{x_1 \mapsto a_1, \ldots, x_n \mapsto a_n\}$. Note that $\phi(\underline{x})$ is valid in $\mathcal{A}$ iff it is valid under all assignments iff its universal closure is valid in $\mathcal{A}$. An *Ω-homomorphism* between two Ω-structures $\mathcal{A}$ and $\mathcal{B}$ is a mapping $\mu : A \to B$ that is sort-conforming (i.e., maps elements of sort S in $\mathcal{A}$ to elements of sort S in $\mathcal{B}$), and satisfies the condition

$$(*) \qquad \mathcal{A} \models A(a_1, \ldots, a_n) \quad \text{implies} \quad \mathcal{B} \models A(\mu(a_1), \ldots, \mu(a_n))$$

for all Ω-atoms $A(x_1, \ldots, x_n)$ and (sort-conforming) elements $a_1, \ldots, a_n$ of A. In case the converse of $(*)$ holds too, μ is called an *embedding.* Note that an embedding is something more than just an injective homomorphism since the stronger condition must hold not only for the equality predicate, but for all predicate symbols. If the embedding μ is the identity on A, then we say that $\mathcal{A}$ is an *Ω-substructure* of $\mathcal{B}$. An important property of universal theories is that their classes of models are *closed under building substructures*, i.e., if T is a universal Ω-theory and $\mathcal{A}$ is an Ω-substructure of $\mathcal{M}$, then $\mathcal{M} \in Mod(T)$ implies $\mathcal{A} \in Mod(T)$ (see, e.g. [6]).

We say that Σ is a subsignature of Ω (written $\Sigma \subseteq \Omega$) iff Σ is a signature that can be obtained from Ω by removing some of its sorts and function and predicate symbols. If $\Sigma \subseteq \Omega$ and $\mathcal{A}$ is an Ω-structure, then the *Σ-reduct* of $\mathcal{A}$ is the Σ-structure $\mathcal{A}|_\Sigma$ obtained from $\mathcal{A}$ by forgetting the interpretations of sorts, function and predicate symbols from Ω that do not belong to Σ. Conversely, $\mathcal{A}$ is called an *expansion* of the Σ-structure $\mathcal{A}_{|\Sigma}$ to the larger signature Ω.

Given a set X of constant symbols not belonging to the signature Ω, but each equipped with a sort from Ω, we denote by Ω^X the extension of Ω by these new constants. If $\mathcal{A}$ is an Ω-structure, then we can view the elements of A as a set of new constants, where $a \in A_S$ has sort S. By interpreting each $a \in A$ by itself, $\mathcal{A}$ can also be viewed as an Ω^A-structure. The *diagram* $\Delta_\Omega(\mathcal{A})$ of $\mathcal{A}$ is the set of all ground Ω^A-literals that are true in $\mathcal{A}$. *Robinson's diagram theorem* [6] says that there is an embedding between the Ω-structures $\mathcal{A}$ and $\mathcal{B}$ iff there is an expansion of $\mathcal{B}$ to an Ω^A-structure that is a model of the diagram of $\mathcal{A}$.

Adjoint functions in posets We recall some basic facts about adjoints among posets (see, e.g., [12] for more details). A partially ordered set (*poset*, for short) is a set P equipped with a reflexive, transitive, and antisymmetric binary relation $\leq$. Such a poset is called *complete* if the *meet* $\bigwedge_i a_i \in P$ and the *join* $\bigvee_i a_i \in P$ of a family $\{a_i\}_{i \in I}$ of elements of P always exist. In case I is empty, the meet is the greatest and the join the least element of P.

Let P, Q be posets. A pair of maps $f^* : P \to Q$ and $f_* : Q \to P$ is said to be an *adjoint pair* (written $f^* \dashv f_*$) iff the condition

$$f^*(a) \leq b \qquad \text{iff} \qquad a \leq f_*(b) \tag{1}$$

is satisfied for all $a \in P, b \in Q$. In this case, f^* is called the *left adjoint* to f_*, and f_* is called the *right adjoint* to f^*. The left (right) adjoint to a given map $f : P \to Q$ may not exist, but if it does, then it is unique.

Condition (1) implies that f^*, f_* are *order preserving.* For example, assume that $a_1, a_2 \in P$ are such that $a_1 \leq a_2$. Now, $f^*(a_2) \leq f^*(a_2)$ implies $a_2 \leq$

$f_*(f^*(a_2))$ by (1), and thus by transitivity $a_1 \leq f_*(f^*(a_2))$. By (1), this implies $f^*(a_1) \leq f^*(a_2)$.

Instead of condition (1), we may equivalently require that f^*, f_* are order preserving and satisfy, for all $a \in P, b \in Q$, the conditions

$$a \leq f_*(f^*(a)) \qquad \text{and} \qquad f^*(f_*(b) \leq b. \tag{2}$$

If $f^* \dashv f_*$ is an adjoint pair, then the mappings f^*, f_* are inverse to each other on their images, i.e., for all $a \in P, b \in Q$

$$f^*(a) = f^*(f_*(f^*(a))) \qquad \text{and} \qquad f_*(f^*(f_*(b))) = f_*(b). \tag{3}$$

Adjoint pairs compose in the following sense: if $f^* : P \to Q, f_* : Q \to P$ and $g^* : Q \to R, g_* : R \to Q$ are such that $f^* \dashv f_*$ and $g^* \dashv g_*$, then we also have that $g^* \circ f^* \dashv f_* \circ g_*$ (where composition should be read from right to left).

If P, Q are complete posets, then any pair of adjoints $f^* \dashv f_*$ between P and Q *preserves meet and join* in the following sense: the left adjoint preserves join and the right adjoint preserves meet. The latter can, e.g., be seen as follows:

$$a \leq f_*(\bigwedge b_i) \text{ iff } f^*(a) \leq \bigwedge b_i \text{ iff } \forall i. f^*(a) \leq b_i \text{ iff } \forall i. a \leq f_*(b_i) \text{ iff } a \leq \bigwedge f_*(b_i).$$

Since a is arbitrary, this shows that $f_*(\bigwedge b_i) = \bigwedge f_*(b_i)$.

Given a mapping $f : P \to Q$ between the posets P, Q, we may ask under what conditions it has a left (right) adjoint. As we have seen above, order preserving is a necessary condition, but it is easy to see that it is not sufficient.

If P, Q are complete, then meet preserving is a necessary condition for f to have a left adjoint f^*, and join preserving is a necessary condition for f to have a right adjoint f_*. These conditions are also sufficient: if f preserves join (meet), then the following mapping f_* (f^*) is a right (left) adjoint to f:

$$f_*(b) := \bigvee_{f(a) \leq b} a \qquad \text{and} \qquad f^*(b) := \bigwedge_{b \leq f(a)} a.$$

Example 1. Let W_1, W_2 be sets, and consider the posets induced by the subset relation on their powersets $\wp(W_1)$ and $\wp(W_2)$. Obviously, these posets are complete, where set union is the join and set intersection is the meet operation. Any binary relation $E \subseteq W_2 \times W_1$ yields a join-preserving diamond operator $\Diamond_E : \wp(W_1) \to \wp(W_2)$ by defining for all $a \in \wp(W_1)$:

$$\Diamond_E a := \{w_2 \in W_2 \mid \exists w_1 \in W_1.\ (w_2, w_1) \in E \wedge w_1 \in a\}.$$

The right adjoint to this diamond operator is the box operator $\Box_E^- : \wp(W_2) \to \wp(W_1)$, which can be defined as the map taking $b \in \wp(W_2)$ to

$$\Box_E^- b := \{w_1 \in W_1 \mid \forall w_2 \in W_2.\ (w_2, w_1) \in E \to w_2 \in b\}.$$

It is easy to see that these two maps indeed form an adjoint pair for set inclusion, i.e., we have $\Diamond_E \dashv \Box_E^-$. Conversely, for any adjoint pair $f^* \dashv f_*$ with

$$f^* : \wp(W_1) \to \wp(W_2) \qquad \text{and} \qquad f_* : \wp(W_2) \to \wp(W_1),$$

there is a *unique* relation $E \subseteq W_2 \times W_1$ such that $f^* = \Diamond_E$ and $f_* = \Box_E^-$. To show this, just take E to consist of the pairs (w_2, w_1) such that $w_2 \in f^*(\{w_1\})$). This shows that the adjoint pairs among powerset Boolean algebras coincide with the pairs of inverse modal operators on the powersets defined above.

Adjoint connections We define adjoint connections first on the semantic side, where we connect classes of structures, and then on the syntactic side, where we connect theories.

Let Ω_1, Ω_2 be two disjoint (many-sorted) signatures.[2] We assume that Ω_1 contains a binary predicate symbol $\sqsubseteq^1$ of arity S^1S^1, and Ω_2 contains a binary predicate symbol $\sqsubseteq^2$ of arity S^2S^2. The combined signature $\Omega_1 +^* \Omega_2$ contains the union $\Omega_1 \cup \Omega_2$ of the signatures Ω_1 and Ω_2. In addition $\Omega_1 +^* \Omega_2$ contains *two new function symbols* ℓ, r of arity S^1S^2 and S^2S^1. Since the signatures Ω_1 and Ω_2 are sorted and disjoint, it is easy to see that $(\Omega_1 +^* \Omega_2)$-structures are formed by 4-tuples of the form $(\mathcal{M}^1, \mathcal{M}^2, \ell^{\mathcal{M}}, r^{\mathcal{M}})$, where $\mathcal{M}^1$ is an Ω_1-structure, $\mathcal{M}^2$ is an Ω_2-structure, and

$$\ell^{\mathcal{M}} : \mathcal{S}^1 \to \mathcal{S}^2 \qquad \text{and} \qquad r^{\mathcal{M}} : \mathcal{S}^2 \to \mathcal{S}^1$$

are functions between the interpretations $\mathcal{S}^1, \mathcal{S}^2$ of the sorts S^1, S^2 in $\mathcal{M}^1, \mathcal{M}^2$.

Let $\mathcal{K}_1$ be a class of Ω_1-structures and $\mathcal{K}_2$ a class of Ω_2-structures such that each of the structures in $\mathcal{K}_i$ interprets $\sqsubseteq^i$ as a partial order on the interpretation $\mathcal{S}^i$ of the sort S^i $(i = 1, 2)$. The combined class of structures $\mathcal{K}_1 +^* \mathcal{K}_2$, called the *adjoint connection* of $\mathcal{K}_1$ and $\mathcal{K}_2$, consists of those $(\Omega_1 +^* \Omega_2)$-structures $(\mathcal{M}^1, \mathcal{M}^2, \ell^{\mathcal{M}}, r^{\mathcal{M}})$ for which $\mathcal{M}^1 \in \mathcal{K}_1$, $\mathcal{M}^2 \in \mathcal{K}_2$, and $\ell^{\mathcal{M}}, r^{\mathcal{M}}$ is an adjoint pair for the posets given by $\mathcal{S}^1, \mathcal{S}^2$ and the interpretations of the predicate symbols $\sqsubseteq^1, \sqsubseteq^2$ in $\mathcal{M}^1, \mathcal{M}^2$, respectively.

Let T_1 be an Ω_1-theory and T_2 an Ω_2-theory such that the axioms of T_i $(i = 1, 2)$ entail the reflexivity, transitivity, and antisymmetry axioms for $\sqsubseteq^i$. The combined theory $T_1 +^* T_2$, called the *adjoint theory connection* of T_1 and T_2, has $\Omega_1 +^* \Omega_2$ as its signature, and the following *axioms*:

$$T_1 \quad \cup \quad T_2 \quad \cup \quad \{\forall x, y. (\ell(x) \sqsubseteq^2 y \leftrightarrow x \sqsubseteq^1 r(y))\}.$$

In the sequel, superscripts 1 and 2 for the partial orders $\sqsubseteq^1, \sqsubseteq^2$ are sometimes omitted. It is easy to see that the adjoint theory connection corresponds to building the adjoint connection of the corresponding classes of models.

Proposition 2. $Mod(T_1 +^* T_2) = Mod(T_1) +^* Mod(T_2)$.

Example 3. We show that basic $\mathcal{E}$-connections of abstract description systems, as introduced in [15], are instances of our approach for connecting classes of structures. A *Boolean-based signature* is a signature Ω including the signature Ω_{BA} of Boolean algebras. Boolean-based signatures correspond to the *abstract*

[2] If Ω_1, Ω_2 are not disjoint, we can make them disjoint by appropriately renaming the shared sorts and the shared function and predicate symbols.

description languages (ADL) introduced in [15], with the exception that we do not consider object variables and relation symbols.[3]

An *algebraic Ω-model* is an Ω-structure whose Ω_{BA}-reduct is a Boolean algebra. As a special case we consider *Ω-frames*, which are algebraic Ω-models $\mathcal{F}(W)$ whose Ω_{BA}-reduct is the Boolean algebra $\wp(W)$, where W is a set (called the set of possible worlds). Ω-frames are the same as the *abstract description models* (ADM) introduced in [15]. An *abstract description system* (ADS) is determined by an ADL together with a class of ADMs for this ADL. Thus, in our setting, an ADS is given by a Boolean-based signature Ω together with a class of Ω-frames.

Let Ω_1, Ω_2 be Boolean-based signatures, and $\mathcal{K}_1, \mathcal{K}_2$ be classes of Ω_1- and Ω_2-frames, respectively. Any element of their adjoint connection $\mathcal{K}_1 +^* \mathcal{K}_2$ is of the form $(\mathcal{F}(W_1), \mathcal{F}(W_2), \ell^{\mathcal{M}}, r^{\mathcal{M}})$, where $\mathcal{F}(W_1) \in \mathcal{K}_1$, $\mathcal{F}(W_2) \in \mathcal{K}_2$, and $\ell^{\mathcal{M}}, r^{\mathcal{M}}$ is an adjoint pair between the powersets $\wp(W_1)$ and $\wp(W_2)$. The considerations in Example 1 show that there is a relation $E \subseteq W_2 \times W_1$ such that $\ell^{\mathcal{M}} = \Diamond_E$ and $r^{\mathcal{M}} = \Box^-_E$. We call such a relation a *connecting relation*. Conversely, assume that $\mathcal{F}(W_1) \in \mathcal{K}_1$, $\mathcal{F}(W_2) \in \mathcal{K}_2$. If $E \subseteq W_2 \times W_1$ is a connecting relation, then $\Diamond_E, \Box^-_E$ is an adjoint pair, and thus $(\mathcal{F}(W_1), \mathcal{F}(W_2), \Diamond_E, \Box^-_E)$ belongs to the adjoint connection $\mathcal{K}_1 +^* \mathcal{K}_2$.

Let $\mathcal{ADS}_1, \mathcal{ADS}_2$ be the ADSs induced by Ω_1, Ω_2 and $\mathcal{K}_1, \mathcal{K}_2$. The above argument shows that the basic $\mathcal{E}$-connection of $\mathcal{ADS}_1$ and $\mathcal{ADS}_2$ (with just one connecting relation) is given by $\Omega_1 +^* \Omega_2$ and the frame class $\mathcal{K}_1 +^* \mathcal{K}_2$.

This example shows that the adjoint connection of frame classes really captures the basic $\mathcal{E}$-connection approach introduced in [15]. On the one hand, our approach is more general in that it can also deal with arbitrary classes of algebraic models (and not just frame classes), and even more generally with signatures that are not Boolean based. On the other hand, in [15], also more general types of $\mathcal{E}$-connections are considered. First, there may be more than one connecting relation in $\mathcal{E}$. In our algebraic setting this means that more than one pair of adjoints is considered. Though we do not treat this case here, it is straightforward to extend our approach to several (independent) pairs of adjoints. Second, in [15] $n \geq 2$ rather than just 2 ADSs are connected. We will show later on how our approach can be extended to deal with this case. Third, [15] considers extensions of the basic connection approach such as applying Boolean operations to connecting relations. These kinds of extensions can currently not be handled by our algebraic approach.

3 The decidability transfer result

We are interested in deciding universal fragments i.e., validity of universal formulae (or, equivalently open formulae) in a theory T or a class of structures $\mathcal{K}$.

[3] This means that our approach cannot treat the *relational* object assertions of [15] (see Example 9 below for more details). These object assertions correspond to role assertions of description logic ABoxes, and are usually not considered in modal logic.

The formula ϕ is *valid* in the class of structures $\mathcal{K}$ iff ϕ is valid in each element of $\mathcal{K}$. It is valid in the theory T iff it is valid in $Mod(T)$. It is well known that the validity problem for universal formulae is equivalent to the problem of deciding whether a set of literals is satisfiable in some element of $\mathcal{K}$ (some model of T). We call such a set of literals a *constraint*.

By introducing new free constants (i.e., constants not occurring in the axioms of the theory), we can assume without loss of generality that such constraints contain no variables. In addition, we can transform any ground constraint into an equi-satisfiable set of *ground flat literals*, i.e., literals of the form

$$a \approx f(a_1, \ldots, a_n), \quad P(a_1, \ldots, a_n), \quad \text{or} \quad \neg P(a_1, \ldots, a_n),$$

where $a, a_1, \ldots, a_n$ are (sort-conforming) free constants, f is a function symbol, and P is a predicate symbol (possibly also equality).

Before we can formulate the decidability transfer result, we must first define the conditions under which it holds. These conditions are conditions regarding the existence of certain subtheories T_0 of the component theories. Let Ω_0 be a single-sorted signature containing (possibly among other symbols) a binary predicate symbol $\sqsubseteq$, and let T_0 be a universal Ω_0-theory that entails reflexivity, transitivity, and antisymmetry of $\sqsubseteq$.

The *first condition* is that T_0 must be locally finite, i.e., all finitely generated models of T_0 are finite. To be more precise, we need the following restricted version of the effective variant of local finiteness defined in [11, 2]. The theory T_0 is called *locally finite with an effective bound* iff there is a computable function B_{T_0} from the non-negative integers into the non-negative integers with the following property: if the model $\mathcal{A}$ of T_0 is generated by a set of generators of size n, then the cardinality of $\mathcal{A}$ is bounded by $B_{T_0}(n)$.

The *second condition* requires the existence of certain adjoint functions. We say that T_0 *guarantees adjoints* iff every Ω_0-embedding $e : \mathcal{A} \to \mathcal{M}$ of a finitely generated model $\mathcal{A}$ of T_0 into a model $\mathcal{M}$ of T_0 has both a left adjoint e^* and a right adjoint e_* for the posets induced by the interpretations of $\sqsubseteq$ in $\mathcal{A}$ and $\mathcal{M}$.

Definition 4. *Let Ω be a (many-sorted) signature, and $\mathcal{K}$ be a class of Ω-structures. We say that $\mathcal{K}$ is* adjoint combinable *iff there exist a finite single-sorted subsignature Ω_0 of Ω containing the binary predicate symbol $\sqsubseteq$, and a universal Ω_0-theory T_0 such that*

1. *every axiom of T_0 is valid in $\mathcal{K}$;*
2. *the axioms of T_0 entail reflexivity, transitivity, and antisymmetry for $\sqsubseteq$;*
3. *T_0 is locally finite with an effective bound;*
4. *T_0 guarantees adjoints.*

Let T be an Ω-theory. We say that T is adjoint combinable *iff the corresponding class of models $Mod(T)$ is adjoint combinable.*

For adjoint combinable classes of structures, decidability of the universal fragment transfers from the components to their adjoint connection. It should be noted that the universal theory T_0 ensuring adjoint combinability need not be the same for the component theories.

Theorem 5. *Let $\mathcal{K}_1, \mathcal{K}_2$ be adjoint combinable classes of structures over the respective signatures Ω_1, Ω_2. Then the decidability of the universal fragments of $\mathcal{K}_1$ and $\mathcal{K}_2$ entails the decidability of the universal fragment of $\mathcal{K}_1 +^* \mathcal{K}_2$.*

Proof. Let $T_0^{(1)}$ and $T_0^{(2)}$ be the universal theories over the signatures $\Omega_0^{(1)}$ and $\Omega_0^{(2)}$ ensuring adjoint combinability of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. To prove the theorem, we consider a finite set Γ of ground flat literals over the signature $\Omega_1 +^* \Omega_2$ (with additional free constants), and show how it can be tested for satisfiability in $\mathcal{K}_1 +^* \mathcal{K}_2$. Since all literals in Γ are flat, we can divide Γ into three disjoint sets $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_i $(i = 1, 2)$ is a set of literals over Ω_i (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{\ell(a_1) \approx b_1, \ldots, \ell(a_n) \approx b_n, r(b'_1) \approx a'_1, \ldots, r(b'_m) \approx a'_m\}$$

for free constants a_j, b_j, a'_i, b'_i.

The following procedure decides satisfiability of Γ in $\mathcal{K}_1 +^* \mathcal{K}_2$:

1. Guess a 4-tuple $\mathcal{A}, \mathcal{B}, \mu, \nu$, where:
   (a) $\mathcal{A}$ is a finite $\Omega_0^{(1)}$-structure generated by $\{a_1, \ldots, a_n, a'_1, \ldots, a'_m\}$ such that $|A| \leq B_{T_0^{(1)}}(n+m)$ and $\sqsubseteq$ is interpreted as a partial order, and $\mathcal{B}$ is a finite $\Omega_0^{(2)}$-structure generated by $\{b_1, \ldots, b_n, b'_1, \ldots, b'_m\}$ such that $|B| \leq B_{T_0^{(2)}}(n+m)$ and $\sqsubseteq$ is interpreted as a partial order.
   (b) $\mu : \mathcal{A} \longrightarrow \mathcal{B}$ and $\nu : \mathcal{B} \longrightarrow \mathcal{A}$ is an adjoint pair for the partial orders induced by the interpretations of $\sqsubseteq$ in $\mathcal{A}, \mathcal{B}$ such that
   $$\mu(a_j) = b_j \ (j = 1, \ldots, n) \quad \text{and} \quad \nu(b'_i) = a'_i \ (i = 1, \ldots, m).$$
2. Check whether $\Gamma_1 \cup \Delta_{\Omega_0^{(1)}}(\mathcal{A})$ is satisfiable in $\mathcal{K}_1$ (if not, go back to Step 1).
3. Check whether $\Gamma_2 \cup \Delta_{\Omega_0^{(2)}}(\mathcal{B})$ is satisfiable in $\mathcal{K}_2$ (if not, go back to Step 1). If it is satisfiable, return 'satisfiable'.
4. If all guesses fail, return 'unsatisfiable'.

Local finiteness with an effective bound of the theories $T_0^{(i)}$ entails that the functions $B_{T_0^{(i)}}$ are computable. Since the signatures $\Omega_0^{(i)}$ are finite, there are only finitely many guesses in Step 1, and we can effectively generate all of them. Steps 2 and 3 are effective since satisfiability of a finite set of literals in $\mathcal{K}_i$ $(i = 1, 2)$ is decidable by our assumption that the universal fragments of $\mathcal{K}_1$ and $\mathcal{K}_2$ are decidable. Thus, it is sufficient to show that the procedure is sound and complete.

To show *completeness*, suppose that the constraint Γ is satisfiable in $\mathcal{K}_1 +^* \mathcal{K}_2$. Thus, there is a structure $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \ell^{\mathcal{M}}, r^{\mathcal{M}}) \in \mathcal{K}_1 +^* \mathcal{K}_2$ satisfying Γ. In particular, $\mathcal{M}_1 \in \mathcal{K}_1, \mathcal{M}_2 \in \mathcal{K}_2$, and $\ell^{\mathcal{M}} \dashv r^{\mathcal{M}}$ is an adjoint pair such that

$$\ell^{\mathcal{M}}(a_j) = b_j \quad \text{and} \quad r^{\mathcal{M}}(b'_i) = a'_i.[4]$$

[4] Here we identify (for the sake of simplicity) the constants a_j, a'_i, b_j, b'_i with their interpretations in $\mathcal{M}_1, \mathcal{M}_2$.

Let $\mathcal{A}$ be the $\Omega_0^{(1)}$-substructure of $\mathcal{M}_1|_{\Omega_0^{(1)}}$ generated by $\{a_1, \dots, a_n, a'_1, \dots, a'_m\}$, and $\mathcal{B}$ be the $\Omega_0^{(2)}$-substructure of $\mathcal{M}_2|_{\Omega_0^{(2)}}$ generated by $\{b_1, \dots, b_n, b'_1, \dots, b'_m\}$. The $\Omega_0^{(i)}$-reduct $\mathcal{M}_i|_{\Omega_0^{(i)}}$ of $\mathcal{M}_i$ $(i = 1, 2)$ is a model of $T_0^{(i)}$. Since $T_0^{(i)}$ is universal, the substructures $\mathcal{A}, \mathcal{B}$ are also models of $T_0^{(1)}, T_0^{(2)}$, respectively. In particular, this implies that $\sqsubseteq$ is interpreted as a partial order in $\mathcal{A}$ and $\mathcal{B}$. Since the theories $T_0^{(i)}$ are locally finite with an effective bound, the cardinalities of these substructures are bounded by the respective functions $B_{T_0^{(i)}}$.

We know $\mathcal{M}_1 \in \mathcal{K}_1$ satisfies Γ_1. In addition, since $\mathcal{A}$ is an $\Omega_0^{(1)}$-substructure of $\mathcal{M}_1$, Robinson's diagram theorem entails that $\mathcal{M}_1$ satisfies $\Delta_{\Omega_0^{(1)}}(\mathcal{A})$. Thus, $\Gamma_1 \cup \Delta_{\Omega_0^{(1)}}(\mathcal{A})$ is satisfiable in $\mathcal{K}_1$. The fact that $\Gamma_2 \cup \Delta_{\Omega_0^{(2)}}(\mathcal{B})$ is satisfiable in $\mathcal{K}_2$ can be shown in the same way.

To construct the adjoint pair $\mu \dashv \nu$, we consider the $\Omega_0^{(1)}$-embedding e and the $\Omega_0^{(2)}$-embedding f, where

$$e : \mathcal{A} \to \mathcal{M}_1|_{\Omega_0^{(1)}} \quad \text{and} \quad f : \mathcal{B} \to \mathcal{M}_2|_{\Omega_0^{(2)}}$$

are given by the inclusion maps. Since the theories $T_0^{(i)}$ guarantee adjoints, these embeddings have both left and right adjoints. Let us call f^* the left adjoint to f and e_* the right adjoint to e. We define

$$\mu := f^* \circ \ell^{\mathcal{M}} \circ e \quad \text{and} \quad \nu := e_* \circ r^{\mathcal{M}} \circ f.$$

Since adjoints compose, we have indeed $\mu \dashv \nu$. It remains to be shown that $\mu(a_j) = b_j$ and $\nu(b'_i) = a'_i$. We restrict the attention to the first identity (as the second one can be proved symmetrically). We know that $\ell^{\mathcal{M}}(a_j) = b_j$, and since e is the inclusion map we have $e(a_j) = a_j$. Thus

$$\mu(a_j) = f^*(\ell^{\mathcal{M}}(e(a_j))) = f^*(\ell^{\mathcal{M}}(a_j)) = f^*(b_j).$$

Since f is the inclusion map, we have $f^*(b_j) = f(f^*(f(b_j)))$ and because $f^* \dashv f$ we know by (3) that $f(f^*(f(b_j))) = f(b_j) = b_j$. If we put all these identities together, we obtain $\mu(a_j) = b_j$.

To show *soundness*, we argue as follows. If $\Gamma_1 \cup \Delta_{\Omega_0^{(1)}}(\mathcal{A})$ is satisfiable in $\mathcal{K}_1$, then there is a structure $\mathcal{M}_1 \in \mathcal{K}_1$ that satisfies Γ_1 and has $\mathcal{A}$ as $\Omega_0^{(1)}$-substructure. The $\Omega_0^{(1)}$-reduct of $\mathcal{M}_1$ is a model of $T_0^{(1)}$, and since $T_0^{(1)}$ is universal this implies that the substructure $\mathcal{A}$ is also a model of $T_0^{(1)}$. Analogously, if $\Gamma_2 \cup \Delta_{\Omega_0^{(2)}}(\mathcal{B})$ is satisfiable in $\mathcal{K}_2$, then there is a structure $\mathcal{M}_2 \in \mathcal{K}_2$ that satisfies Γ_2 and has the model $\mathcal{B}$ of $T_0^{(2)}$ as $\Omega_0^{(2)}$-substructure.

In order to construct a structure $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \ell^{\mathcal{M}}, r^{\mathcal{M}}) \in \mathcal{K}_1 +^* \mathcal{K}_2$ satisfying $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, it is enough to construct the adjoint pair $\ell^{\mathcal{M}}, r^{\mathcal{M}}$ such that it extends the pair μ, ν provided by Step 2b of the procedure. Let

$$e : \mathcal{A} \to \mathcal{M}_1|_{\Omega_0^{(1)}} \quad \text{and} \quad f : \mathcal{B} \to \mathcal{M}_2|_{\Omega_0^{(2)}}$$

be the $\Omega_0^{(1)}$- and $\Omega_0^{(2)}$-embeddings of $\mathcal{A}, \mathcal{B}$ into the reducts of $\mathcal{M}_1, \mathcal{M}_2$, respectively. Without loss of generality we can assume that e, f are inclusion maps. Since the theories $T_0^{(i)}$ guarantee adjoints, these embeddings have both left and right adjoints. Let us call e^* the left adjoint to e and f_* the right adjoint to f. We define

$$\ell^{\mathcal{M}} := f \circ \mu \circ e^* \quad \text{and} \quad r^{\mathcal{M}} := e \circ \nu \circ f_*.$$

Since adjoints compose, we have again $\ell^{\mathcal{M}} \dashv r^{\mathcal{M}}$. It remains to be shown that $\mathcal{M} := (\mathcal{M}_1, \mathcal{M}_2, \ell^{\mathcal{M}}, r^{\mathcal{M}})$ satisfies Γ_0, i.e., $\ell^{\mathcal{M}}(a_j) = b_j$ and $r^{\mathcal{M}}(b'_i) = a'_i$.[5]

Again, we restrict the attention to the first identity (as the second one can be proved symmetrically). We have $\mu(a_j) = b_j$, and $\ell^{\mathcal{M}}(a_j) = f(\mu(e^*(a_j))) = \mu(e^*(a_j))$ since f is the inclusion map. Thus, it is enough to show that $e^*(a_j) = a_j$. Since e is the inclusion map, we have $e^*(a_j) = e(e^*(e(a_j)))$ and because $e^* \dashv e$ we know by (3) that $e(e^*(e(a_j))) = e(a_j) = a_j$. □

Proposition 2 and the above theorem yield the following transfer result for adjoint theory connections.

Corollary 6. *Let T_1, T_2 be adjoint combinable theories over the respective signatures Ω_1, Ω_2. Then the decidability of the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 +^* T_2$.*

4 Applications of the transfer result

In order to apply Theorem 5, we must find universal theories that extend the theory of posets, guarantee adjoints, and are locally finite with an effective bound. Given such theories $T_0^{(1)}, T_0^{(2)}$, every pair $\mathcal{K}_1, \mathcal{K}_2$ of classes of Ω_1- and Ω_2-structures whose members are models of $T_0^{(1)}, T_0^{(2)}$, respectively, satisfy the conditions of Theorem 5, and hence allow transfer of decidability (of the universal fragment) from $\mathcal{K}_1$ and $\mathcal{K}_2$ to $\mathcal{K}_1 +^* \mathcal{K}_2$.

In order to ensure the existence of adjoints for embeddings, it is enough that meets and joins exist and embeddings preserve them. For this reason, we start with the theory of bounded lattices since it provides us with meet and join. Recall that the theory T_L of *bounded lattices* is the theory of posets endowed with binary meet and join, and a least and a greatest element. In the following, we assume that the signature Ω_L of this theory contains the function symbols $\sqcup, 0$ for the join and the least element, the function symbols $\sqcap, 1$ for the meet and the greatest element, and the relation symbol $\sqsubseteq$ for the partial order. Note, however, that is not really necessary to have $\sqsubseteq$ explicitly in the signature since it can be expressed using meet or join (e.g., $x \sqsubseteq y$ iff $x \sqcup y = y$).

The theory T_L is not locally finite, but we can make it locally finite by adding as extra axioms all the identities that are true in a fixed *finite* lattice $\mathcal{A}$. The theory $T_{\mathcal{A}}$ obtained this way is locally finite: two n-variable terms cannot

[5] As before, we identify (for the sake of simplicity) the constants a_j, a'_i, b_j, b'_i with their interpretations in $\mathcal{M}_1, \mathcal{M}_2$.

be distinct modulo $T_{\mathcal{A}}$ in case they are interpreted in $\mathcal{A}$ by the same n-ary function $A^n \to A$, and there are only finitely many such functions. This argument also yields an effective bound: if $|A| = c$, then $B_{T_{\mathcal{A}}}(n) = c^{c^n}$. In addition, $T_{\mathcal{A}}$ guarantees adjoints. To show this, consider an Ω_L-embedding $e : \mathcal{B} \to \mathcal{M}$ of a finitely generated model $\mathcal{B}$ of $T_{\mathcal{A}}$ into a model $\mathcal{M}$ of $T_{\mathcal{A}}$. Since $\mathcal{B}$ is a finite, it is a complete lattice, and the preservation of binary joins, meets, as well as the least and greatest element by e implies that e preserves all joins and meets. Thus, it has both a left and a right adjoint.

If we take as $\mathcal{A}$ the two element bounded lattice, then it is well known (see, e.g., [13]) that the theory $T_{\mathcal{A}}$ coincides with the theory T_D of *distributive* lattices, i.e., the extension of T_L by the distributivity axiom $x \sqcup (y \sqcap z) \approx (x \sqcup y) \sqcap (x \sqcup z)$.

Corollary 7. *Let $\mathcal{K}_1, \mathcal{K}_2$ be classes of Ω_1- and Ω_2-structures whose members are models of the theory T_D of distributive lattices. Then the decidability of the universal fragments of $\mathcal{K}_1, \mathcal{K}_2$ implies the decidability of the universal fragment of $\mathcal{K}_1 +^* \mathcal{K}_2$.*

Obviously, any pair of classes of frames over two Boolean-based signatures (see Example 3) satisfies the precondition of the above corollary.

Corollary 8. *Let Ω_1, Ω_2 be Boolean-based signatures, and $\mathcal{K}_1, \mathcal{K}_2$ be classes of Ω_1- and Ω_2-frames. Then the decidability of the universal fragments of $\mathcal{K}_1, \mathcal{K}_2$ implies the decidability of the universal fragment of $\mathcal{K}_1 +^* \mathcal{K}_2$.*

As shown in Example 3, a Boolean-based signature together with a class of frames corresponds to an ADS in the sense of [15]. To show the connection between Corollary 8 and the decidability transfer result proved in [15], we must relate the problem of deciding the universal fragment of a class of frames to the decision problem considered in [15].

Example 9. Consider a Boolean-based signature Ω and a class $\mathcal{K}$ of Ω-frames. Taking into account the Boolean structure and the (implicit or explicit) presence of the partial order $\sqsubseteq$, an Ω-constraint can be represented in the form

$$t_1 \sqsubseteq u_1, \ \ldots, \ t_n \sqsubseteq u_n, \ v_1 \not\approx 0, \ \ldots, \ v_m \not\approx 0.$$

We call such a constraint a *modal constraint*. It is satisfiable in $\mathcal{K}$ whenever there are $\mathcal{F}(W) \in \mathcal{K}$ and $w_1, \ldots, w_m \in W$ such that

$$t_1^{\mathcal{F}(W)} \subseteq u_1^{\mathcal{F}(W)}, \ \ldots, \ t_n^{\mathcal{F}(W)} \subseteq u_n^{\mathcal{F}(W)}, \ w_1 \in v_1^{\mathcal{F}(W)}, \ \ldots, \ w_m \in v_m^{\mathcal{F}(W)}.$$

If one restricts the attention to modal constraints with just one negated equation (i.e., if $m = 1$), then one obtains the traditional relativized satisfiability problem in modal logic. The satisfiability problem introduced in [15] is slightly more general since the set of constraints considered there can also contain object assertions involving relation symbols. As mentioned in Example 3, such assertions can currently not be handled by our approach. Consequently, our transfer result applies to a slightly more restricted satisfiability problem than the one considered in [15]. On the other hand, our result holds for more general theories and classes of structures, i.e., also ones that are not given by classes of frames.

Complexity considerations The complexity of the combination algorithm described in the proof of Theorem 5 can be quite high. It is non-deterministic since it guesses finitely generated structures up to a given bound, which may itself be quite large. In addition, the possibly large diagrams of these structures are part of the input for the decision procedures of the component theories.

Depending on the theories $T_0^{(i)}$, specific features of these theories may allow for sensible improvements, due for instance to the possibility of more succinct representations of the diagrams of models of $T_0^{(i)}$. We illustrate this phenomenon by showing how our combination algorithm can be improved in the case of adjoint connections of Boolean-based equational theories, as treated in Corollary 8. With this modified algorithm, we obtain complexity bounds that coincide with the ones shown in [15]. Actually, the algorithm obtained this way is also similar to the one described in [15]. It should be noted, however, that the correctness of this modified algorithm still follows from the proof of our general Theorem 5.

Thus, let Ω_1, Ω_2 be Boolean-based signatures, and $\mathcal{K}_1, \mathcal{K}_2$ be classes of Ω_1- and Ω_2-frames, respectively. As theories $T_0^{(1)}, T_0^{(2)}$ we can then take the theory BA of Boolean algebras. Let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ be a constraint, where Γ_i $(i = 1, 2)$ is a set of literals over Ω_i (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{\ell(a_1) \approx b_1, \ldots, \ell(a_n) \approx b_n, r(b'_1) \approx a'_1, \ldots, r(b'_m) \approx a'_m\}$$

for free constants a_j, b_j, a'_i, b'_i. If we follow the instructions in the proof of Theorem 5 literally, in order to guarantee the satisfiability of Γ, we must find:

1. a finite Boolean algebra $\mathcal{A}$ generated by $G_1 := \{a_1, \ldots, a_n, a'_1, \ldots, a'_m\}$ such that $\Gamma_1 \cup \Delta_{\Omega_{BA}}(\mathcal{A})$ is satisfiable in $\mathcal{K}_1$;
2. a finite Boolean algebra $\mathcal{B}$ generated by $G_2 := \{b_1, \ldots, b_n, b'_1, \ldots, b'_m\}$ such that $\Gamma_2 \cup \Delta_{\Omega_{BA}}(\mathcal{B})$ is satisfiable in $\mathcal{K}_2$;
3. an adjoint pair $\mu : \mathcal{A} \longrightarrow \mathcal{B}$ and $\nu : \mathcal{B} \longrightarrow \mathcal{A}$, such that

$$\mu(a_j) = b_j \ (j = 1, \ldots, n) \quad \text{and} \quad \nu(b'_i) = a'_i \ (i = 1, \ldots, m). \tag{4}$$

It is well known that a Boolean algebra generated by $n + m$ elements can have cardinality $2^{2^{n+m}}$, and hence its diagram may also be of doubly-exponential size. However, we will show that exponential space is sufficient to represent all the relevant information contained in such a diagram.

Let us call G_1-*minterm* a term τ that is of the form

$$\bigsqcap_{g \in G_1} \sigma_\tau(g),$$

where $\sigma_\tau(g)$ is either g or $\overline{g}$. Notice that the G_1-minterm τ is uniquely determined (up to associativity and commutativity of conjunction) by the function σ_τ, and hence there are as many G_1-minterms as there are subsets of G_1. We associate with every finite Boolean algebra $\mathcal{A}$ generated by G_1 the set $W_\mathcal{A}$ of the G_1-minterms τ such that $\mathcal{A} \models \tau \neq 0$. The following is not difficult to show:

(i) the map associating with $g \in G_1$ the set $\{\tau \in W_\mathcal{A} \mid \sigma_\tau(g) = g\}$ extends to an isomorphism $\iota_\mathcal{A} : \mathcal{A} \longrightarrow \wp(W_\mathcal{A})$;

(ii) $BA \models \Delta_{\Omega_{BA}}(\mathcal{A}) \Leftrightarrow \delta(\mathcal{A})$, where $\delta(\mathcal{A})$ is the conjunction of the formulas $\tau = 0$ for $\tau \notin W_{\mathcal{A}}$.

Fact (ii) means that $\delta(\mathcal{A})$ can replace $\Delta_{\Omega_{BA}}(\mathcal{A})$ in the consistency test of Step 1 above, and the same consideration obviously applies to $\mathcal{B}$ in Step 2. The size of $\delta(\mathcal{A})$ is singly-exponential, and to guess $\delta(\mathcal{A})$ it is sufficient to guess the set $W_{\mathcal{A}}$ (and not the whole $\mathcal{A}$).

A similar technique can be applied to Step 3. By Fact (i) above, we have $\mathcal{A} \simeq \wp(W_{\mathcal{A}})$ and $\mathcal{B} \simeq \wp(W_{\mathcal{B}})$. Hence, the considerations in Example 1 show that the adjoint pair of Step 3 is uniquely determined by a relation $E \subseteq W_{\mathcal{B}} \times W_{\mathcal{A}}$.

To sum up, the data that we are required to guess are simply a set $W_{\mathcal{A}}$ of G_1-minterms, a set $W_{\mathcal{B}}$ of G_2-minterms, and a relation E among them. All this is an exponential size guess, and thus can be done in non-deterministic exponential time. The decision procedures for the component theories receive exponential size instances of their constraint satisfiability problems as inputs. Finally, Condition (4) can be checked in exponential time. From the considerations in Example 1 and from Fact (ii) above, it follows that $\mu(a_j) = b_j$ is equivalent to the following statement:

$$\forall \tau \in W_{\mathcal{B}}.\ (\tau \in \iota_{\mathcal{B}}(b_j) \ \text{ iff } \ \exists \tau' \in W_{\mathcal{A}}.\ (\tau' \in \iota_{\mathcal{A}}(a_j) \wedge (\tau, \tau') \in E)).$$

Since $W_{\mathcal{A}}$ and $W_{\mathcal{B}}$ are of exponential size, this condition can be tested in exponential time. The same approach can be used to test the conditions $\nu(b'_i) = a'_i$.

Overall, the improved combined decision procedure has the following complexity. Its starts with a non-deterministic exponential step that guesses the sets $\delta(\mathcal{A})$ and $\delta(\mathcal{B})$. Then it tests satisfiability in $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\Gamma_1 \cup \delta(\mathcal{A})$ and $\Gamma_2 \cup \delta(\mathcal{B})$, respectively. The complexity of these tests is one exponential higher than the complexity of the decision procedures for $\mathcal{K}_1$ and $\mathcal{K}_2$. Testing Condition (4) needs exponential time. This shows that our combination procedure has the same complexity as the one for $\mathcal{E}$-connections described in [15].

Let us consider the complexity increase caused by the combination procedure in more detail for the complexity class ExpTime, which is often encountered when considering the relativized satisfiability problem in modal logic. Thus, assume that the decision procedures for $\mathcal{K}_1$ and $\mathcal{K}_2$ are in ExpTime. The combined decision procedure then generates doubly-exponentially many decision problems of exponential size for the component procedures. Each of these component decision problems can be decided in doubly-exponential time. This majorizes the exponential complexity of testing Condition (4). Thus, in this case the overall complexity of the combined decision procedure is 2ExpTime, i.e, one exponential higher than the complexity of the component procedures.

5 N-ary adjoint connections

We sketch how our results can be extended to the case of n-ary connections by using parametrized notions of adjoints, as suggested in [10]. For simplicity, we limit ourselves to the case $n = 3$, and use a notation inspired by Lambek's syntactic calculus [16]. Let P_1, P_2, P_3 be posets. A triple $(\cdot, /, \backslash)$ of functions

$$\cdot : P_1 \times P_2 \to P_3, \qquad \backslash : P_1 \times P_3 \to P_2, \qquad / : P_3 \times P_2 \to P_1$$

is an *adjoint triple* iff the following holds for all $a_1 \in P_1, a_2 \in P_2, a_3 \in P_3$:

$$a_1 \cdot a_2 \leq a_3 \qquad \text{iff} \qquad a_2 \leq a_1 \backslash a_3 \qquad \text{iff} \qquad a_1 \leq a_3 / a_2.$$

To illustrate this definition, we consider a ternary variant of Example 1.

Example 10. Suppose we are given three sets W_1, W_2, W_3 and a ternary relation $E \subseteq W_3 \times W_2 \times W_1$. With two given subsets $a_1 \subseteq W_1, a_2 \subseteq W_2$, we can associate a subset $a_1 \cdot_E a_2 \subseteq W_3$ as follows:

$$a_1 \cdot_E a_2 := \{w_3 \mid \exists (w_2, w_1) \in W_2 \times W_1.\ (w_3, w_2, w_1) \in E \wedge w_1 \in a_1 \wedge w_2 \in a_2\}.$$

If we fix a_1, the function $a_1 \cdot_E (-) : \wp(W_2) \to \wp(W_3)$ preserves all joins, and hence has a right adjoint $a_1 \backslash_E (-) : \wp(W_3) \to \wp(W_2)$, which can be described as follows: for every $a_3 \subseteq W_3$, the subset $a_1 \backslash_E a_3 \subseteq W_2$ is defined as

$$a_1 \backslash_E a_3 := \{w_2 \mid \forall (w_3, w_1) \in W_3 \times W_1.\ (w_3, w_2, w_1) \in E \wedge w_1 \in a_1 \Rightarrow w_3 \in a_3\}.$$

Similarly, if we fix a_2, the function $(-) \cdot_E a_2 : \wp(W_1) \to \wp(W_3)$ preserves all joins, and hence has a right adjoint $(-)/_E a_2 : \wp(W_3) \to \wp(W_1)$, which can be described as follows: for every $a_3 \subseteq W_3$, the subset $a_3 /_E a_2 \subseteq W_1$ is defined as

$$a_3 /_E a_2 := \{w_1 \mid \forall (w_3, w_2) \in W_3 \times W_2.\ (w_3, w_2, w_1) \in E \wedge w_2 \in a_2 \Rightarrow w_3 \in a_3\}.$$

It is easy to see that the three binary operators $(\cdot_E, /_E, \backslash_E)$ fulfill the definition of an adjoint triple (with set inclusion as partial order). Conversely every adjoint triple (for set inclusion) is induced in this way by a unique ternary relation E.

Using the notion of an adjoint triple, we can now define a ternary variant of the notion of an adjoint connection. Let $\Omega_1, \Omega_2, \Omega_3$, be three disjoint signatures containing binary predicate symbols $\sqsubseteq^i$ of arity $S^i S^i$ $(i = 1, 2, 3)$. The combined signature $+^*(\Omega_1, \Omega_2, \Omega_3)$ contains the union $\Omega_1 \cup \Omega_2 \cup \Omega_3$ of the signatures $\Omega_1, \Omega_2, \Omega_3$ and, in addition, three new function symbols $\cdot, \backslash, /$ of arity $S^1 S^2 S^3$, $S^1 S^3 S^2$ and $S^3 S^2 S^1$, respectively. For $i = 1, 2, 3$, let $\mathcal{K}_i$ be a class of Ω_i-structures such that each of the structures in $\mathcal{K}_i$ interprets $\sqsubseteq^i$ as a partial order on the interpretation of S^i. The *ternary adjoint connection* $+^*(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ of $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ consists of those $+^*(\Omega_1, \Omega_2, \Omega_3)$-structures $(\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3, \cdot^{\mathcal{M}}, \backslash^{\mathcal{M}}, /^{\mathcal{M}})$ for which $\mathcal{M}^1 \in \mathcal{K}_1$, $\mathcal{M}^2 \in \mathcal{K}_2$, $\mathcal{M}^3 \in \mathcal{K}_3$, and $(\cdot^{\mathcal{M}}, \backslash^{\mathcal{M}}, /^{\mathcal{M}})$ is an adjoint triple for the underlying posets.

Using the observations made in Example 10, it is easy to see that the ternary adjoint connection corresponds to the basic $\mathcal{E}$-connection of three ADSs. Under the same conditions as in Theorem 5, and with a very similar proof, we can show that decidability of the universal fragment also transfers to ternary adjoint connections.

Theorem 11. *Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ be adjoint combinable classes of structures over the respective signatures $\Omega_1, \Omega_2, \Omega_3$. Then decidability of the universal fragments of $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ entails decidability of the universal fragment of $+^*(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$.*

6 Conclusion

The main motivation of this work was to develop an algebraic generalization of the decidability transfer results for $\mathcal{E}$-connections shown in [15]. On the one hand, our approach is more general than the one in [15] since it also applies to theories and classes of structures that are not given by ADSs (i.e., classes of frames). More generally, since the theories $T_0^{(1)}, T_0^{(2)}$ need not be the theory of Boolean algebras and since $T_0^{(1)}$ need not coincide with $T_0^{(2)}$, we do not require the underlying logic to be classical propositional logic, and the components may even be based on different logics. On the other hand, we currently cannot handle the relational object assertions considered in [15], and we cannot deal with extensions of the basic $\mathcal{E}$-connection approach such as applying Boolean operations to connecting relations. It is the topic of future research to find out whether such extensions and relational object assertions can be expressed in our algebraic setting.

The paper [1] has the same motivation, but follows a different route towards generalizing $\mathcal{E}$-connections. In the present paper, we used as our starting point the observation that the pair $(\Diamond, \Box^-)$ consisting of the diamond operator induced by the connecting relation E, and the box operator induced by its inverse E^- is an adjoint pair for the partial order $\leq$ defined as $x \leq y$ iff $x \sqcup y = y$, where $\sqcup$ is the Boolean disjunction operator. In [1] we used instead the fact that the diamond operator behaves like a homomorphism for $\sqcup$, i.e., $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$. This was generalized to the case of connection functions that behave like homomorphisms for an arbitrary shared subsignature of the theories to be combined. The conditions required in [1] for the transfer of decidability are model-theoretic conditions on a shared subtheory T_0 and its algebraic compatibility with the component theories T_1, T_2. There are examples of theories T_0, T_1, T_2 satisfying these requirements that are quite different from theories induced by (modal) logics. However, there is a price to be payed for this generality: since no partial order is required, it is not possible to model pairs of connection functions that are induced by a connecting relation and its inverse. In contrast, the conditions considered in the present paper are abstract algebraic conditions, which do not look at the structure of models. They require the existence of certain adjoint functions for embeddings between models of subtheories of the component theories. These subtheories need not be identical for different component theories, and there is no additional compatibility requirement between the subtheories and the component theories. In order to require adjoints, we must, however, assume that the models are equipped with a partial order. In addition, one possibility to guarantee the existence of adjoints is to assume that the subtheories provide us with meets and joins. In this case, the theories that we obtain are quite close to theories induced by logics, though not necessarily classical propositional logic.

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