TECHNISCHE UNIVERSITÄT DRESDEN

Artificial Intelligence, Computational Logic

# PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

## Lecture 11 Hypertree Decompositions

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Dresden, 24th June 2016

DRESDEN concept
Exzellenz aus Wissenschaft und Kultur

# Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction Problems (CSP)
7. Evolutionary Algorithms/ Genetic Algorithms
8. Structural Decomposition Techniques (Tree/Hypertree Decompositions)

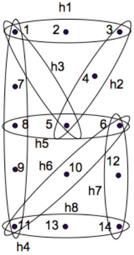
# Motivation

- The structure of a large number of problems is more faithfully described by a hypergraph than by a graph
- Several $NP$ complete problems become tractable if restricte to instances with acyclic hypergraphs
- An appropriate notion of hypergraph width should fulfil both of the following conditions
  1. Relevant hypergraph-based problems should be solvable in polynomial time for instances of bounded width
  2. For each constand $k$, one should be able to check in polynomial time whether a hypergraph is of width $k$, and, in the positive case, it should be possible to produce an associated decomposition of width $k$ of the given hypergraph
- The hypertree decomposition is the most general method leading to large tractable classes of important problems such as constraint satisfaction problems or conjunctive queries

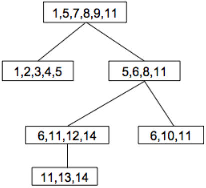
# Generalized Hypertree Decomposition

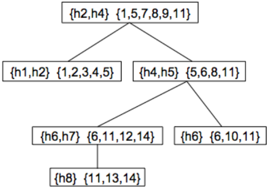
A generalized hypertree decomposition (GHD) of $H$ is a tree decomposition of $H$ with the following extension.

- GHD associates additionally to each node of the decomposition tree the set of hyperedges of $H$.
- The set of vertices associated to each node of the tree must be covered by the set of hyperedges associated to that node.
- The width of a generalized hypertree decomposition is the maximum number of hyperedges associated to a same node of the decomposition.




## Tree decomposition




## Generalized hypertree decomposition

# Hypertree

## Definition

A hypertree for a hypergraph $\mathcal{H} = (V(\mathcal{H}), H(\mathcal{H}))$ is a triple $\langle T, \chi, \lambda \rangle$, where $T = (N, E)$ is a rooted tree, and $\chi$ and $\lambda$ are labeling functions which associate to each vertex $p \in N$ two sets

- $\chi(p) \subseteq V(\mathcal{H})$ and
- $\lambda(p) \subseteq H(\mathcal{H})$.

If $T' = (N', E')$ is a subtree of T, we define $\chi(T') = \bigcup_{v \in N'} \chi(v)$. We denote the set of vertices $N$ of $T$ by $vertices(T)$, and the root of $T$ by $root(T)$. Moreover, for any $p \in N$, $T_p$ denotes the subtree of $T$ rooted at $p$.

# Hypertree Decomposition

## Definition ([Gottlob et al.(2002)])

Let $\mathcal{H} = (V(\mathcal{H}), H(\mathcal{H}))$ be a hypergraph. A hypertree decomposition of $\mathcal{H}$ is a hypertree $\langle T, \chi, \lambda \rangle$ for $\mathcal{H}$ which satisfies all the following conditions:

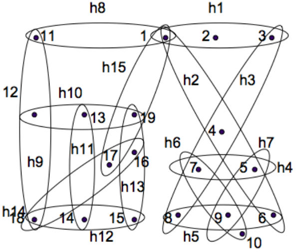
1. for each hyperedge $h \in H(\mathcal{H})$, there exists $p \in vertices(T)$ such that $vertices(h) \subseteq \chi_p$;
2. for each vertex $y \in V(\mathcal{H})$, the set $\{p \in vertices(T) \mid y \in \chi_p\}$ induces a (connected) subtree of $T$;
3. for each vertex $p \in vertices(T), \chi_p \subseteq vertices(\lambda_p)$;
4. for each vertex $p \in vertices(T), vertices(\lambda_p) \cap \chi(T_p) \subseteq \chi_p$.

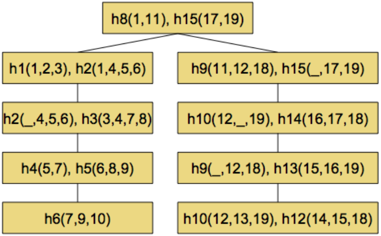
The width of the hypertree decomposition $\langle T, \chi, \lambda \rangle$ is $max_{p \in vertices(T)} |\lambda_p|$. The hypertree width, $hw(\mathcal{H})$, of $\mathcal{H}$ is the minimum width over all its hypertree decompositions.

Note: inclusion in Condition 4 is an equality, as Condition 3 implies the reverse inclusion!

# Generalized Hypertree Decomposition

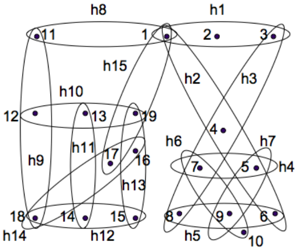
Generalized hypertree decomposition does not include condition 4) of hypertree decomposition.

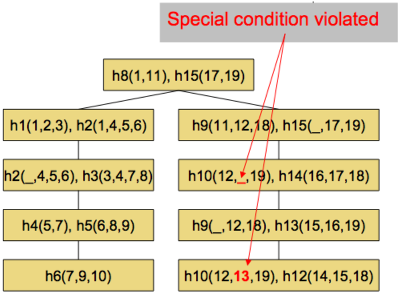



Generalized hypetree decomposition of width 2

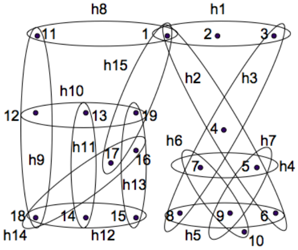
# Generalized Hypertree Decomposition

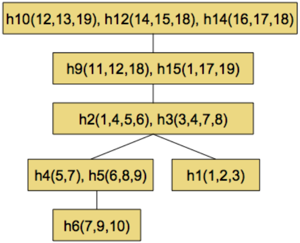



Generalized hypetree decomposition of width 2

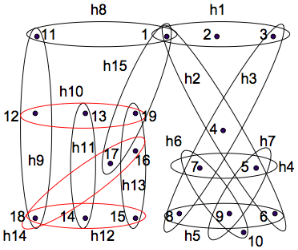
# Hypertree Decomposition

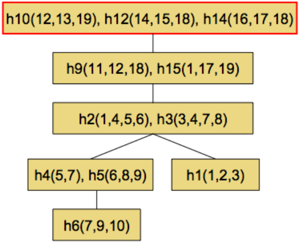



Hypertee decomposition of width 3

# Hypertree Decomposition

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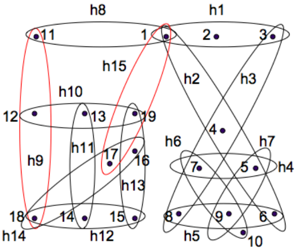


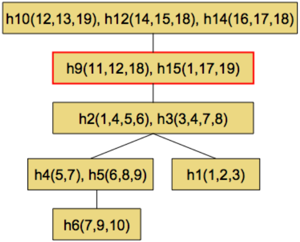


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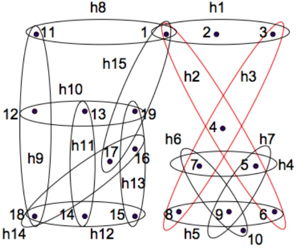


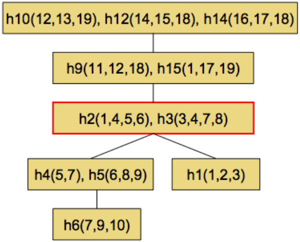


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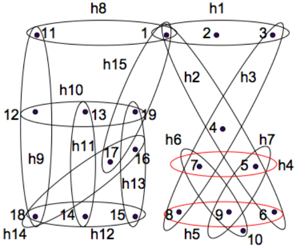


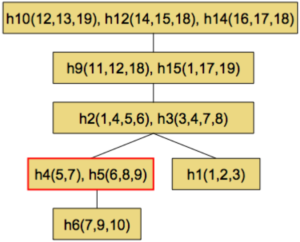


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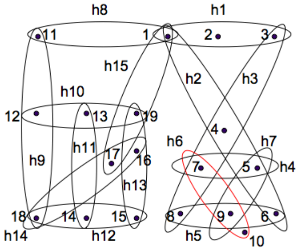


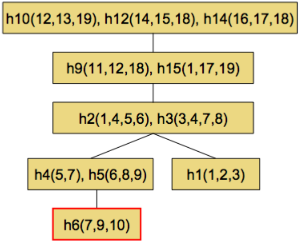


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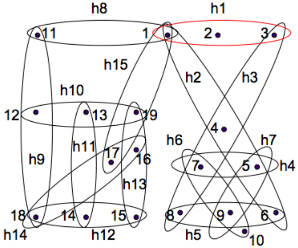
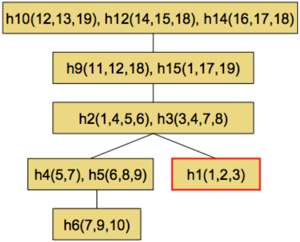



Hypertee decomposition of width 3

# Hypertree Width and CSPs

- The smaller the width of the obtained hypertree decompostion, the faster the corresponding CSP instance can be solved
- A CSP instance can be solved based on its hypertree decomposition as follows:
  - for each node $t$ of the hypertree, all constraints in $\lambda(t)$ are "joined" into a new constraint over the variables in $\chi(t)$
  - for bounded width, i.e., for bounded cardinality of $\lambda(t)$, this yields a polynomial time reduction to an equivalent acyclic CSP instance

# Boolean Conjuctive Query Problem BCQ

## Definition

A relational database is formalized as a finite relational structure $D$. A Boolean conjunctive query (BCQ) on $D$ is a sentence of first-order logic of the form:

$$\exists X_1, \ldots, X_r R_1(t_1^1, t_2^1, \ldots, t_{\alpha(1)}^1) \wedge \cdots \wedge R_k(t_1^k, t_2^k, \ldots, t_{\alpha(k)}^k),$$

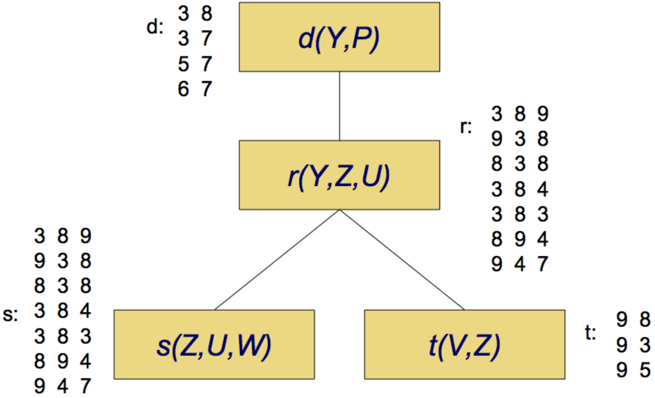
where, for $1 \leq i \leq k$ and $1 \leq j \leq \alpha(i)$, each $t_j^i$ is a term, i.e., either a variable from the list $X_i, \ldots, X_r$, or a constant element from $U_D$. The decision problem BCQ ist the problem of deciding for a pair $\langle D, Q \rangle$, where $D$ is a database and $Q$ is a Boolean conjunctive query, whether $Q$ evaluates to true over $D$, denoted by $D \models Q$.

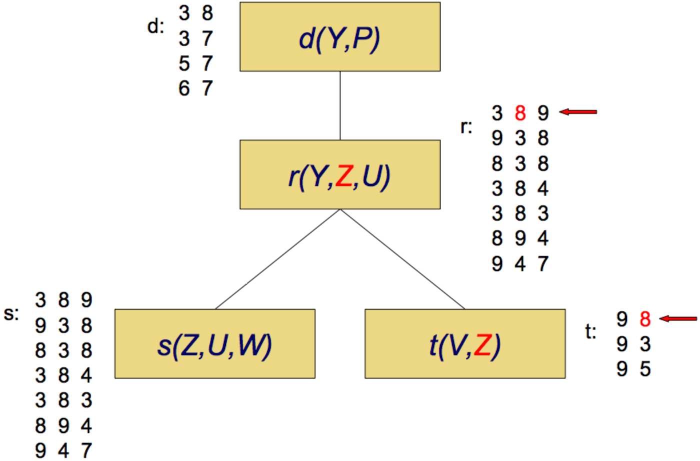
As all variables occuring in a BCQ are existentialy quantified, we usually omit the quantifier prefix and write a BCQ as a conjunction of query atoms.

## Example

Consider the BCQ $Q : d(Y, P) \wedge s(Z, U, W) \wedge t(V, Z) \wedge r(Y, Z, U)$.

# Solving problems based on hypertree decomposition




d(Y,P)
d:
3 8
3 7
5 7
6 7
r(Y,Z,U)
r:
3 8 9
9 3 8
8 3 8
3 8 4
3 8 3
8 9 4
9 4 7
s(Z,U,W)
s:
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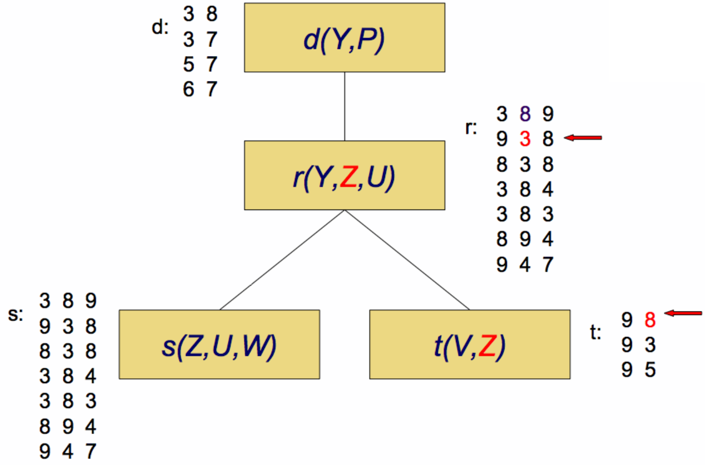
9 4 7
t(V,Z)
t:
9 8
9 3
9 5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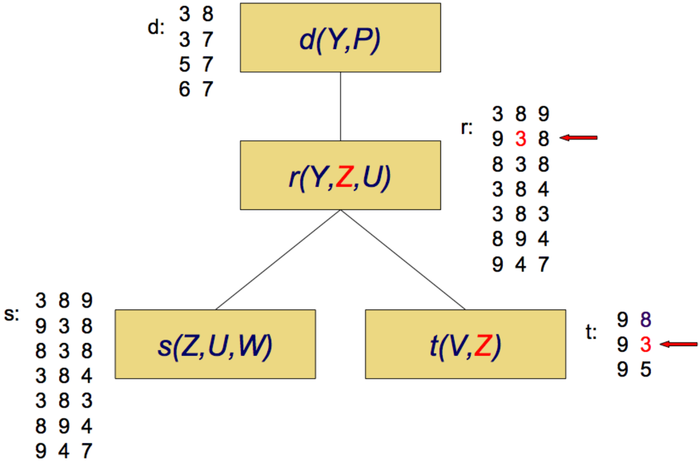
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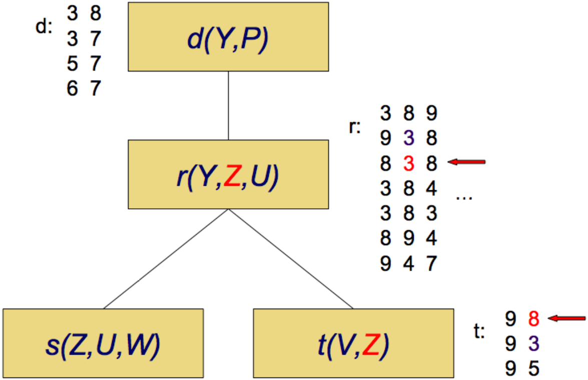


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| 9 | 4 | 7 |

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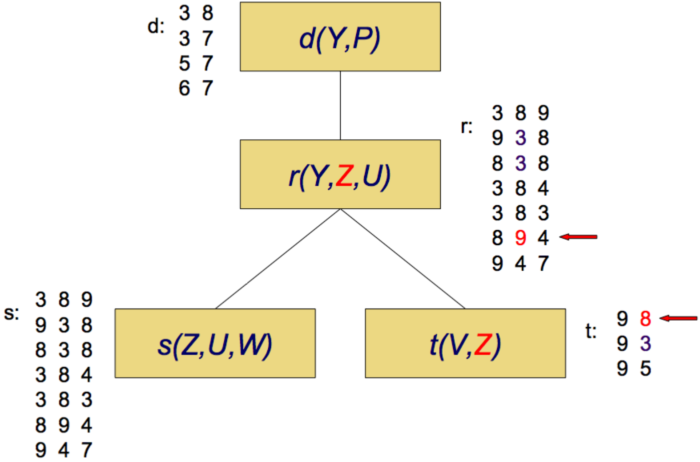
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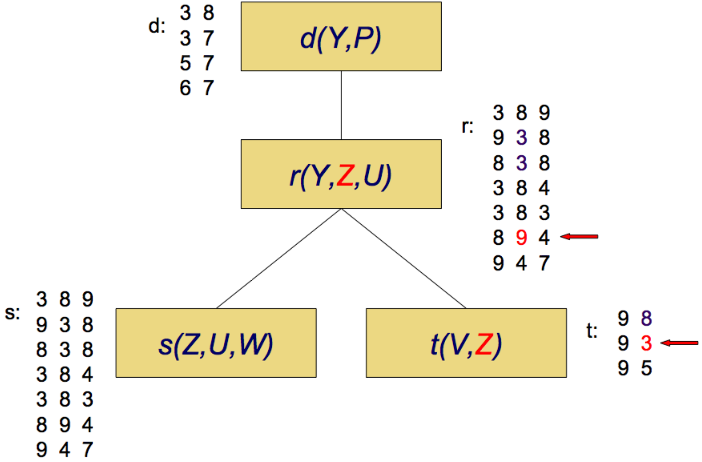
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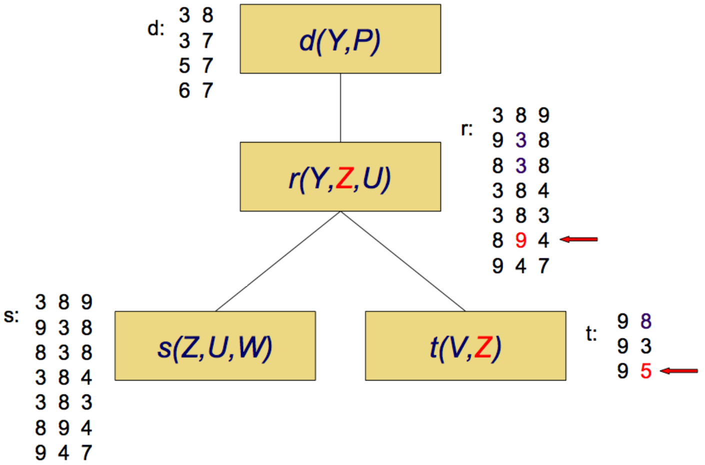
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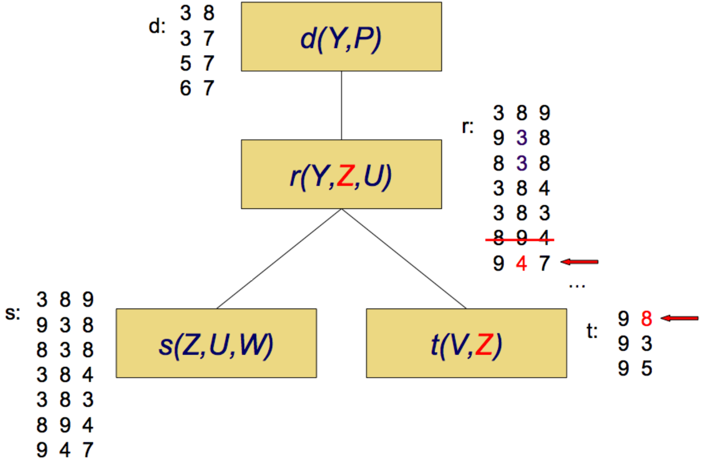
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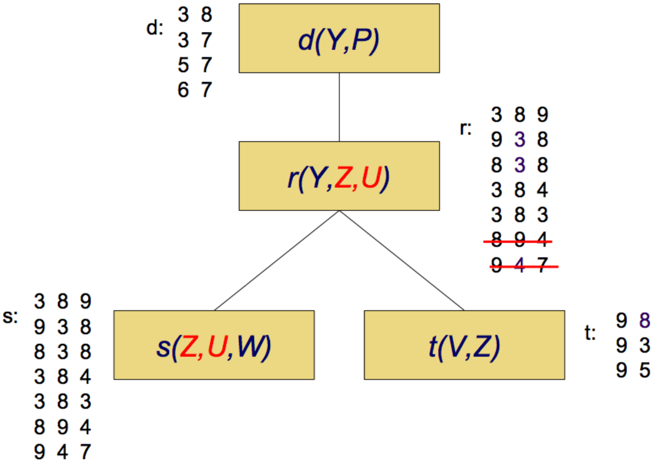
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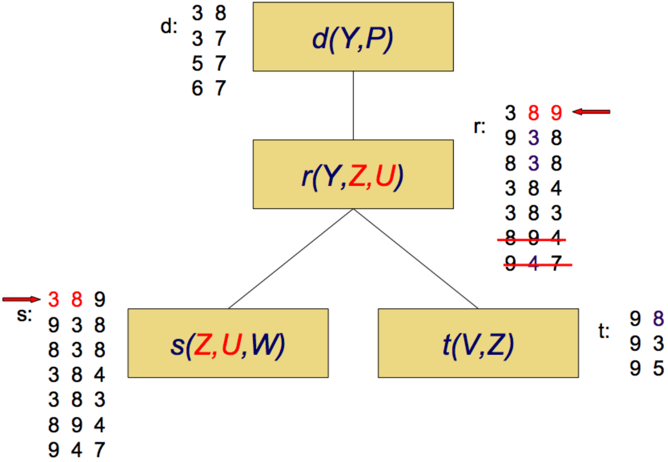
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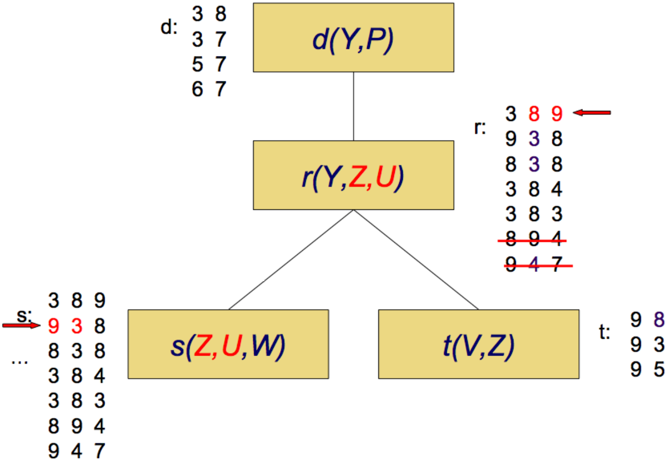
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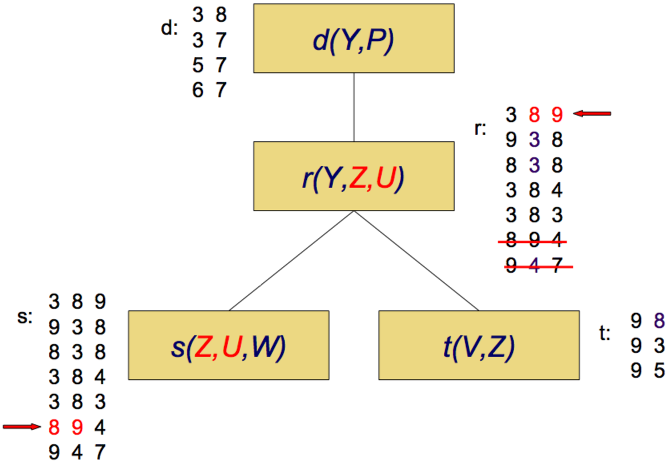
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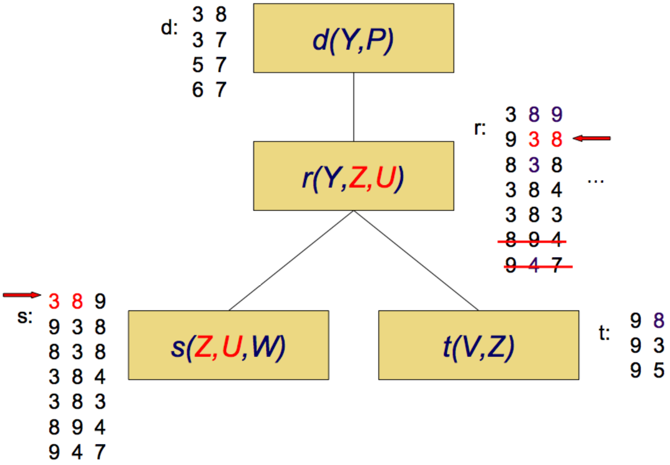
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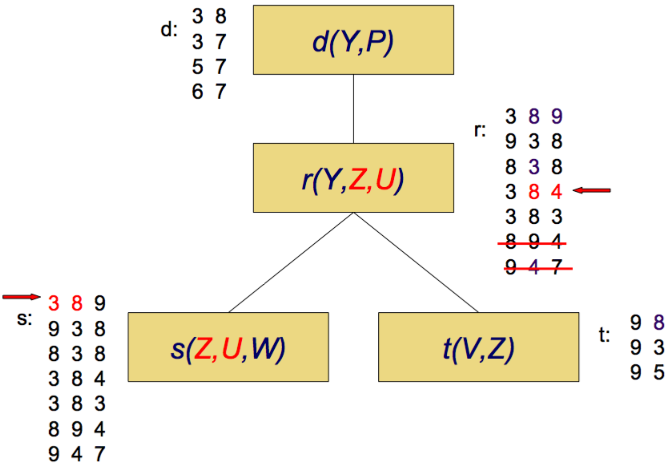
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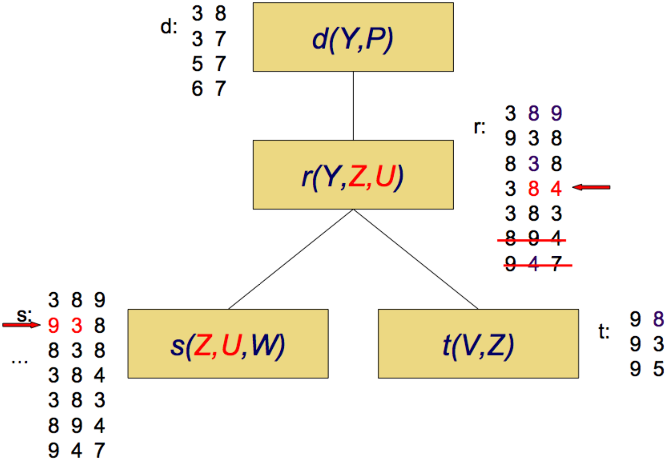
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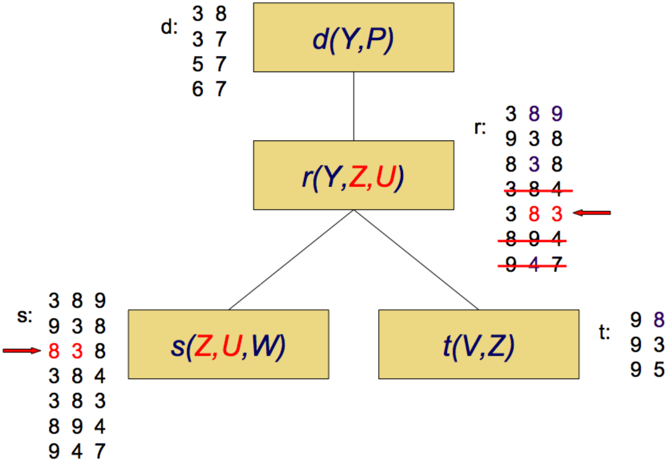
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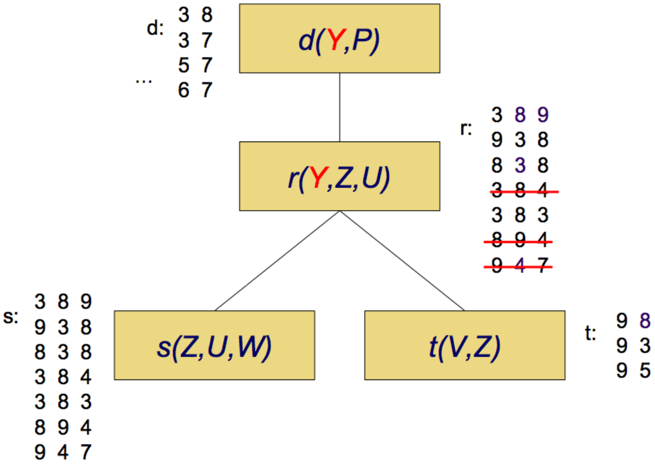
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9 4 7
r(Y,Z,U)
s:
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s(Z,U,W)
t(V,Z)
t:
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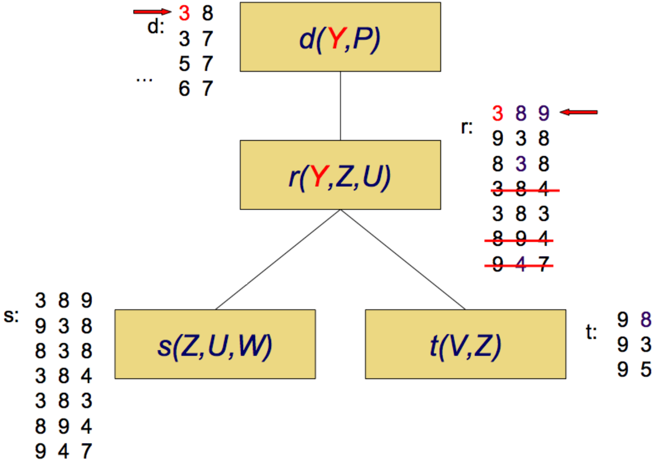
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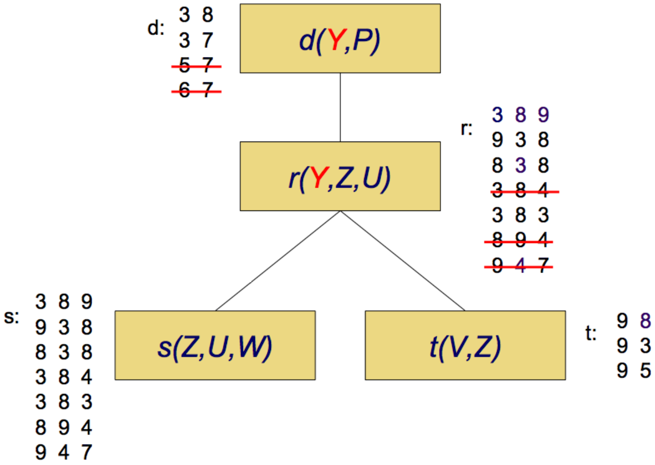
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9 4 7
s(Z,U,W)
t(V,Z)
t:
9 8
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# Algorithms for Generalized Hypertree Decomposition

- Methods based on tree decomposition
  - Generalized hypertree decomposition can be generated by algorithms for tree decomposition + Set Covering
- Hypertree decomposition based on hypergraph partitioning
- Exact methods
- Literature and benchmark instances for hypertree decomposition:
  `http://www.dbai.tuwien.ac.at/proj/hypertree/`
  `http://wwwinfo.deis.unical.it/~frank/Hypertrees/`

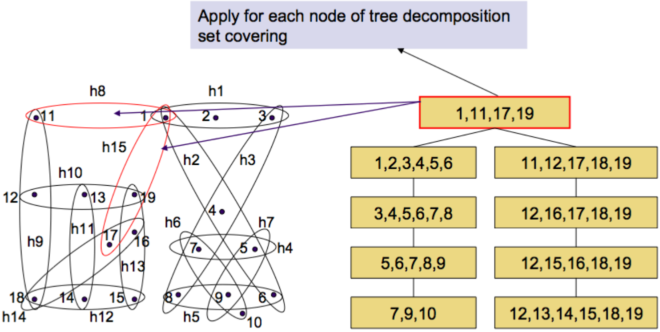
# Constructing Generalized Hypertree Decomposition from Tree Decomposition

Recall, a hypertree decompostion can be devided into two parts

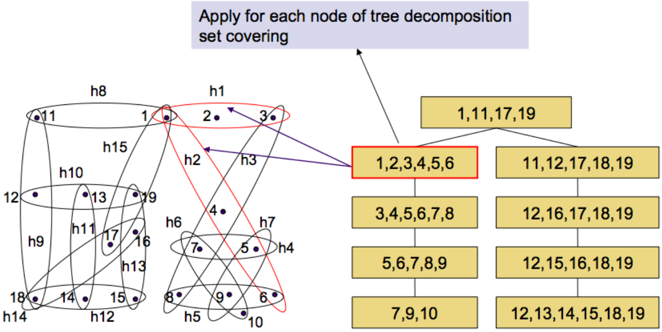
1. definition of a tree decomposition $(T, \chi)$
2. introduction of $\lambda$ such that $\chi(t) \subseteq \bigcup \lambda(t)$ for every node $t$.

$\chi$-labels contain vertices of the hypergraph and $\lambda$-labels contain hyperedges, i.e., sets of vertices, of the hypergraph (covering vertices in $\chi(t)$ by hyperedges in $\lambda(t)$).

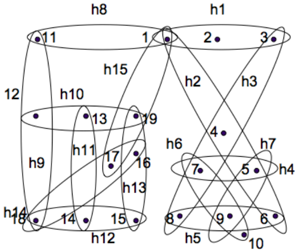
# Constructing Generalized Hypertree Decomposition from Tree Decomposition ctd.

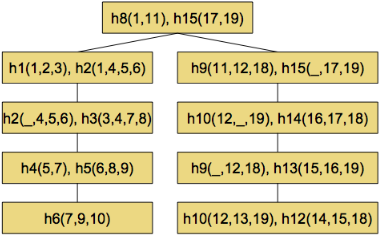


# Constructing Generalized Hypertree Decomposition from Tree Decomposition ctd.



# Generalized Hypertree Decomposition




Generalized hypetree decomposition of width 2

# Hypertree Decomposition Based on Hypergraph Partitioning

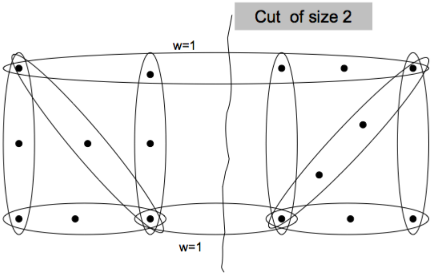
A method for generation of generalized hypertree decompositions based on recursive partitioning of the hypergraph [Dermaku et al.(2008)].

## Hypergraph Partitioning

Given a hypergraph $\mathcal{H}(V, H)$ with weighted vertices and hyperedges.

- Find a partition of set $V$ in two (or $k$) disjoint subsets such that the number of vertices in each set $V_i$ is bounded, and the function defined over hyperedges is optimized.
- Most commonly used objective is to minimize the sum of the weights of hyperedges connecting two or more subsets.

# Hypergraph Partitioning

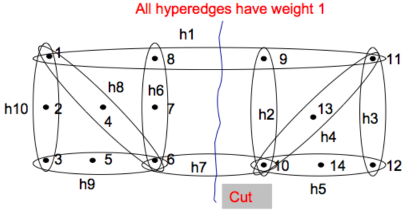



Hypergraph partitioning with constraint about the number of vertices in each partition is NP-Complete problem!

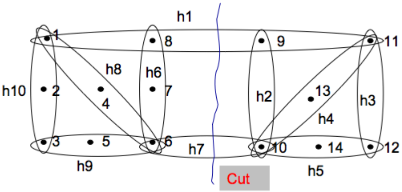
# Generation of Hypertree Decomposition by Hypergraph Partitioning

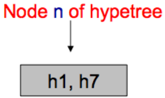
- Does recursive partitioning of hypergraph lead to "good" hypertree decomposition?
- Every cut in hypergraph partitioning can be considered as a node in a hypertree decomposition (called separator)
- Add a special hyperedge to each subgraph containing the vertices in the intersection between the subgraphs to enforce joint appearence in the $\chi$-label of a later generated node
- Connectedness condition for variables should be ensured!
- How to evaluate a cut whose separator contains such hyperedges?
  - associate weights to hyperedges
  - weight 1 for all ordinary hyperedges
  - (W+) weight of special hyperedge: number of ordinary hyperedges needed to cover the vertice of the special hyperedge
  - other weighting schemes associate different weights to special hyperedges (always weight 1 or weight 2)
  - cut evaluates as the sum of weights of all hyperedges in the separator
- Nodes of hypertree are connected at the end of partitioning

# From Partitioning to Hypertree

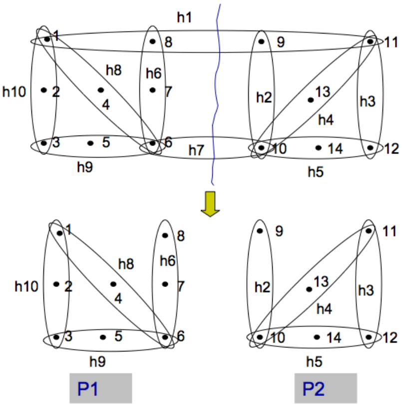


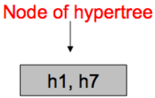
# From Partitioning to Hypertree



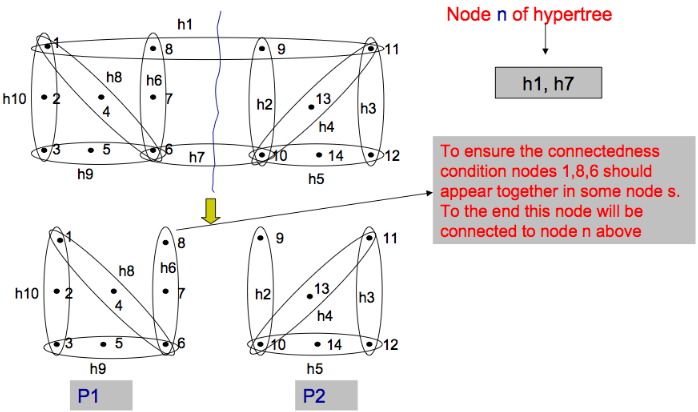



# From Partitioning to Hypertree



# From Partitioning to Hypertree



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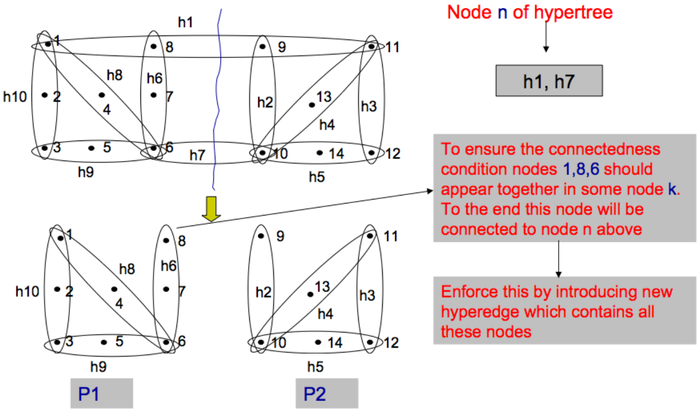

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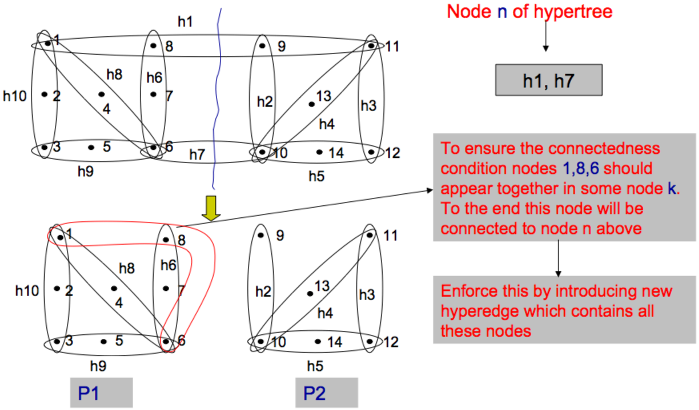

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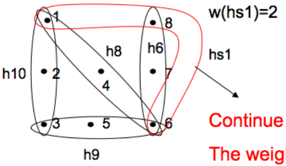
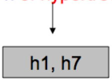


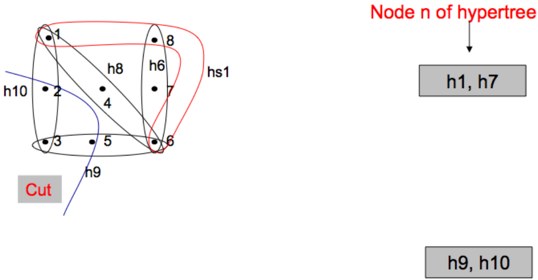

Node n of hypertree




Continue recursively the partitioning

The weight of hyperedge hs1 is the sum of weights of hyperedges which cover this hyperedge: in this case w(hs1)=2

# From Partitioning to Hypertree



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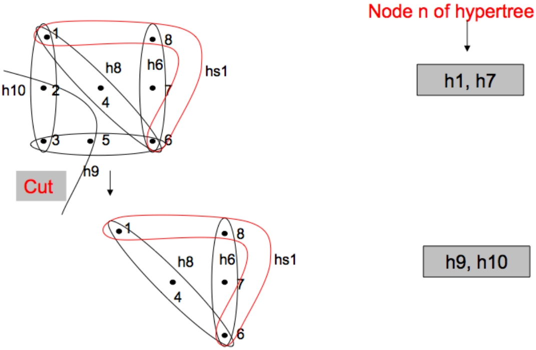

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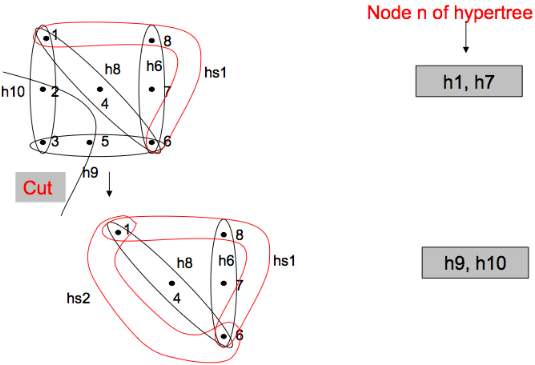

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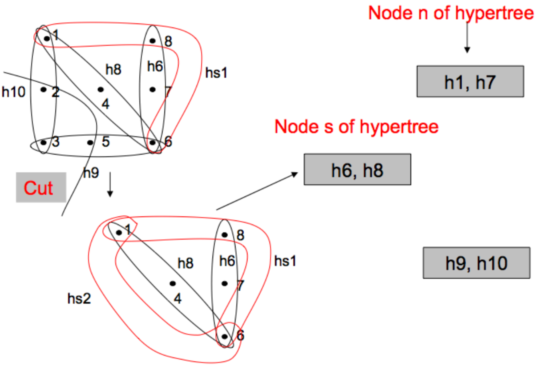

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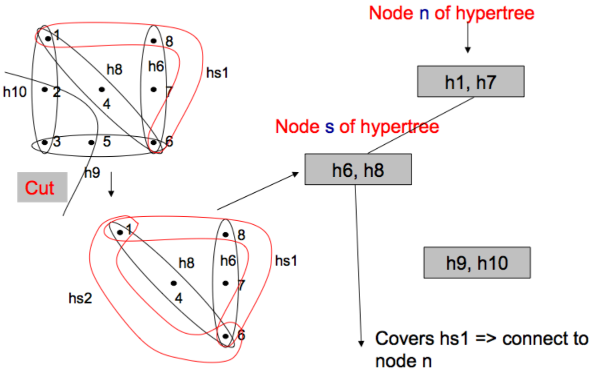

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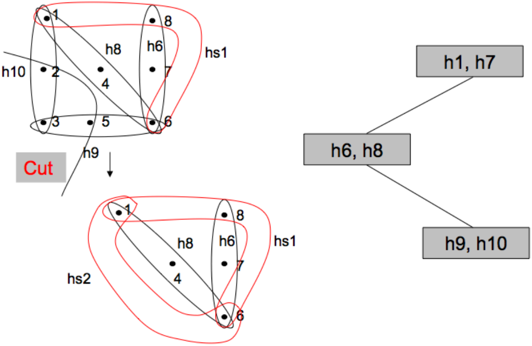

# Summary

- Hypertree decomposition is a method leading to a large class of tractable problems such as CSP or BCQ
- Computation of generalized hypertree decomposition based
  - on tree decompostion + Set Covering
  - hypergraph patitioning

# References


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**Heuristic methods for hypertree decomposition**, In Alexander Gelbukh and EduardoF. Morales, editors, MICAI 2008: Advances in Artificial Intelligence, volume 5317 of LNCS, pages 1–11. Springer Berlin Heidelberg, 2008. ISBN 978-3-540-88635-8.