Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is $\mathbf{t} \in \mathbf{Q}(\mathbf{db})$?

⇝ How hard is it to retrieve data?

Based on slides by D. Figueira, G. Puppis, A.Dawar

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Emptiness problem: Given a query **Q**, is there a database instance **db** so that **Q**(**db**) $\neq \emptyset$?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

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Emptiness problem: Given a query **Q**, is there a database instance **db** so that **Q**(**db**) ≠ ∅ ?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

Equivalence problem: Given queries $\mathbf{Q}_1$, $\mathbf{Q}_2$, is

$$\mathbf{Q}_1(\mathbf{db}) = \mathbf{Q}_2(\mathbf{db})$$

for all database instances **db**?

⇝ Can we safely replace a query with another? (Query optimization)

Complexity theory

What can be **mechanized**? $\rightsquigarrow$ decidable/undecidable

How **hard** is it to mechanise? $\rightsquigarrow$ complexity classes

Complexity theory

What can be **mechanized**? $\leadsto$ decidable/undecidable

Domino

H's 10th

PCP

K

$\vdots$

How **hard** is it to mechanise? $\leadsto$ complexity classes

Domino, H's 10th, PCP, K, ⋮

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usage of resources:
- time
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Algorithm **Alg** is TIME-bounded
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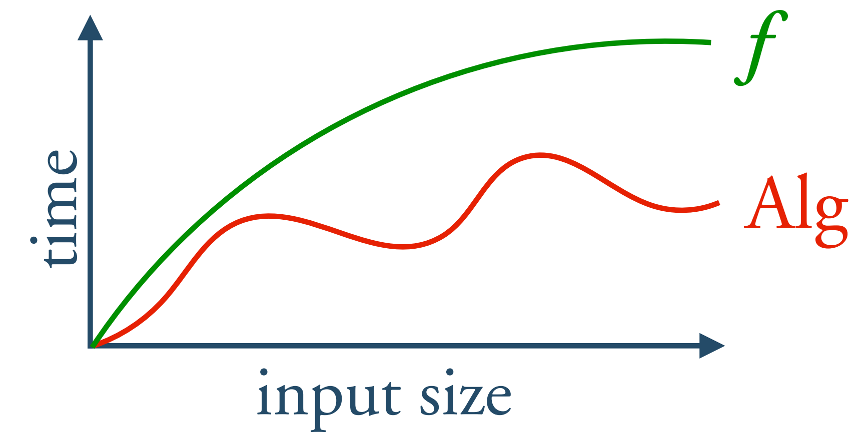

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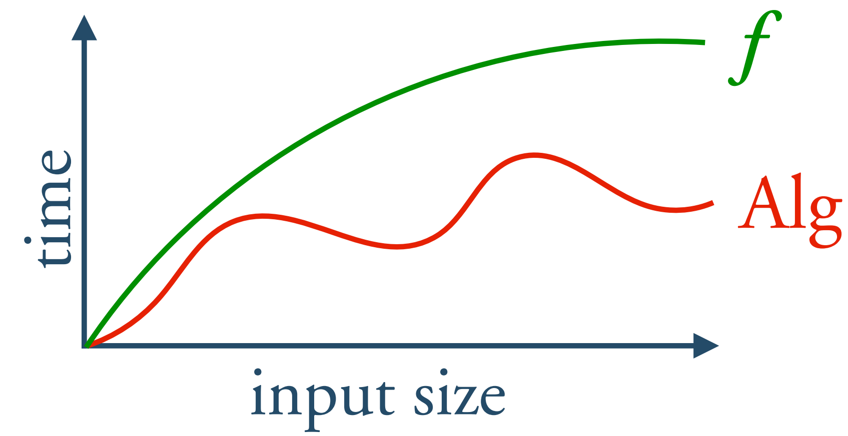

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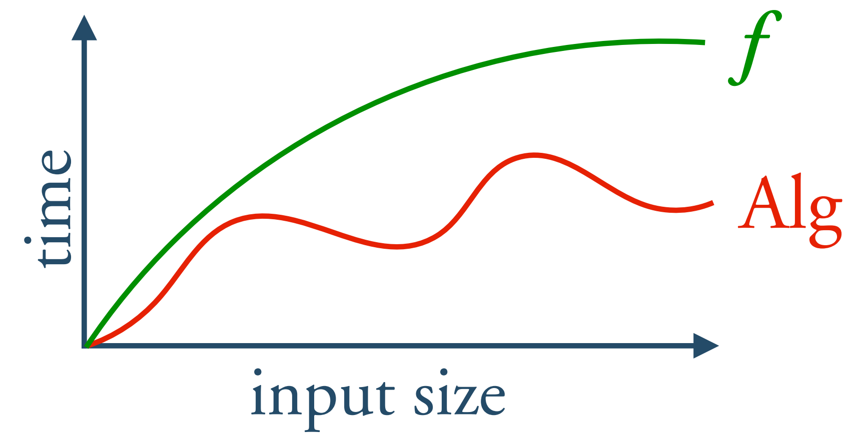

LOGSPACE $\subsetneq$ PTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq \cdots$

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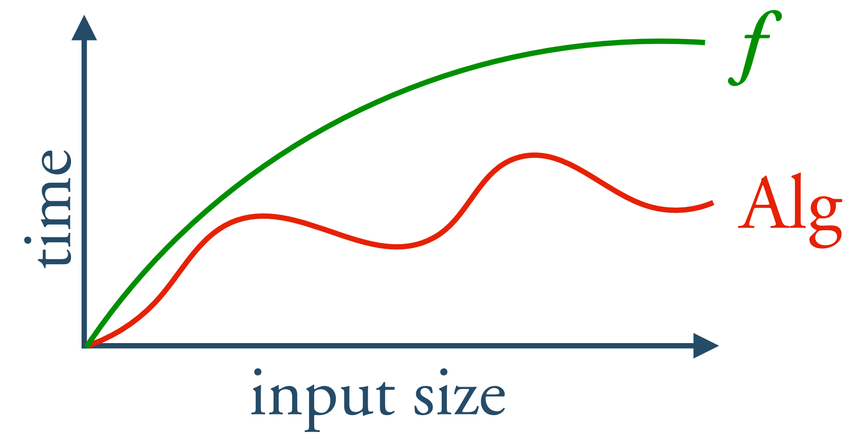

LOGSPACE $\subsetneq$ PTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq \cdots$

- PTIME: TIME-bounded by a polynomial
- PSPACE: SPACE-bounded by a polynomial
- LOGSPACE: SPACE-bounded by log(n)

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

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$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

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DECIDABLE ⇝ foundations of the database industry

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💀 UNDECIDABLE $\rightsquigarrow$ by reduction to the satisfiability problem

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Proof: By reduction from the Domino (aka Tiling) problem.

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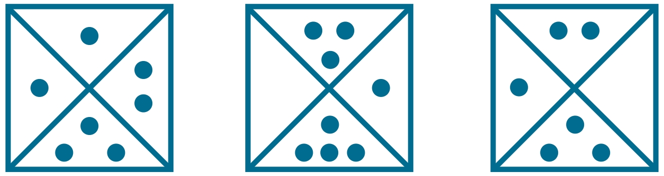
Proof: By reduction from the Domino (aka Tiling) problem.

Reduction from P to P': Algorithm that solves P using a O(1) procedure
" P'(x) "
that returns the truth value of P'(x).

The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

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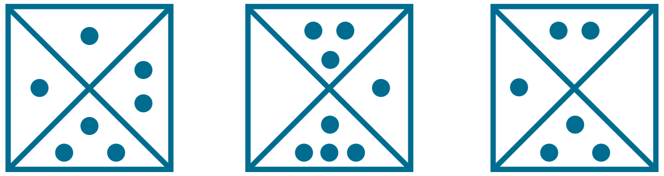
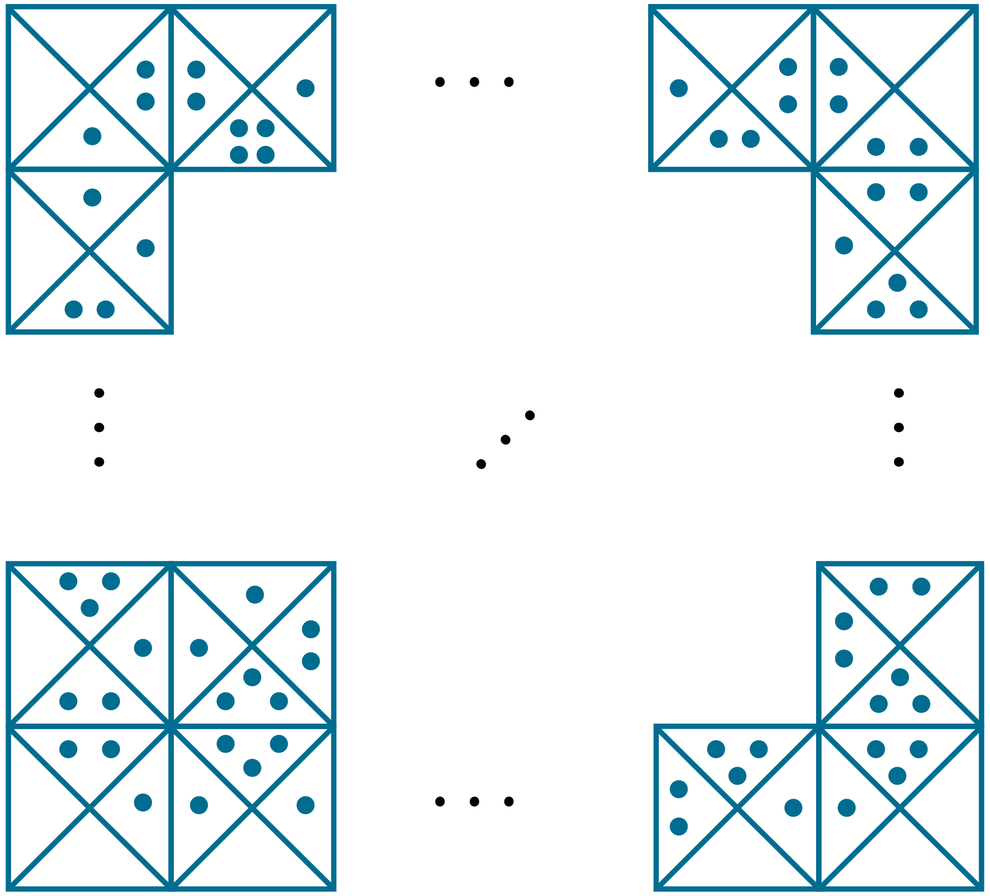
Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)



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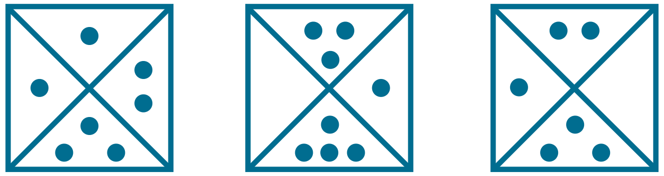
Input: 4-sided dominos: 

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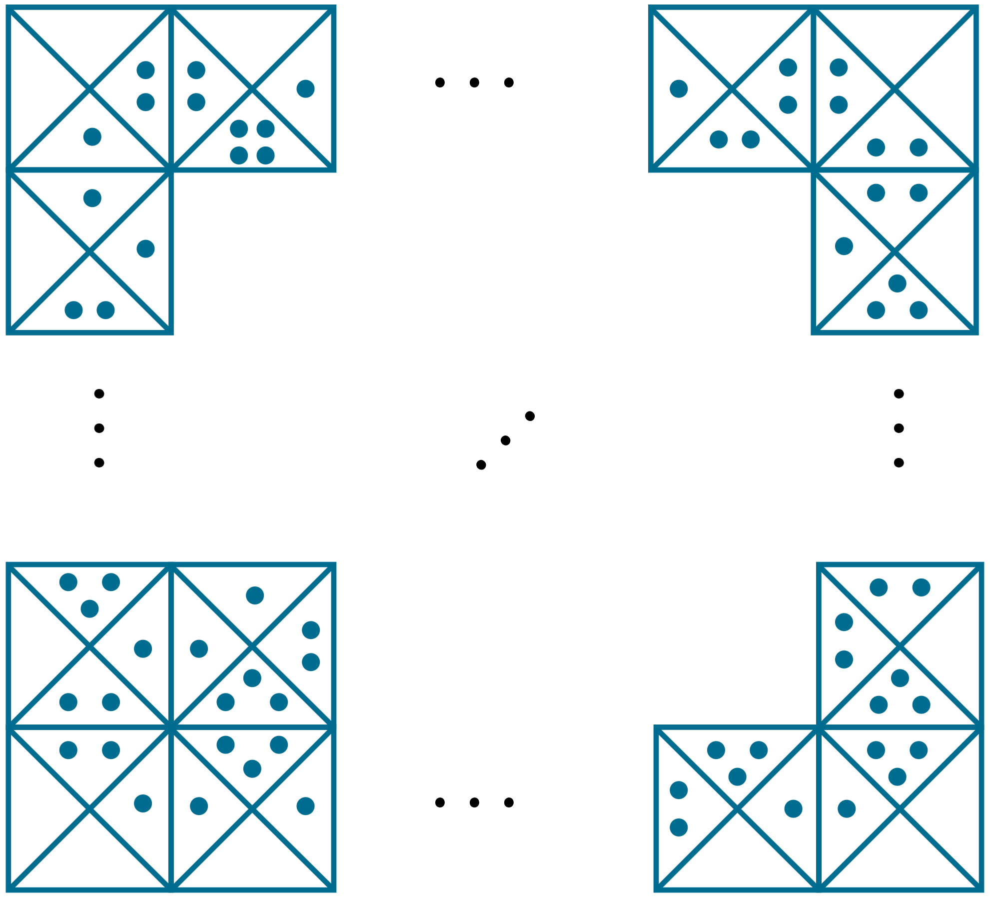
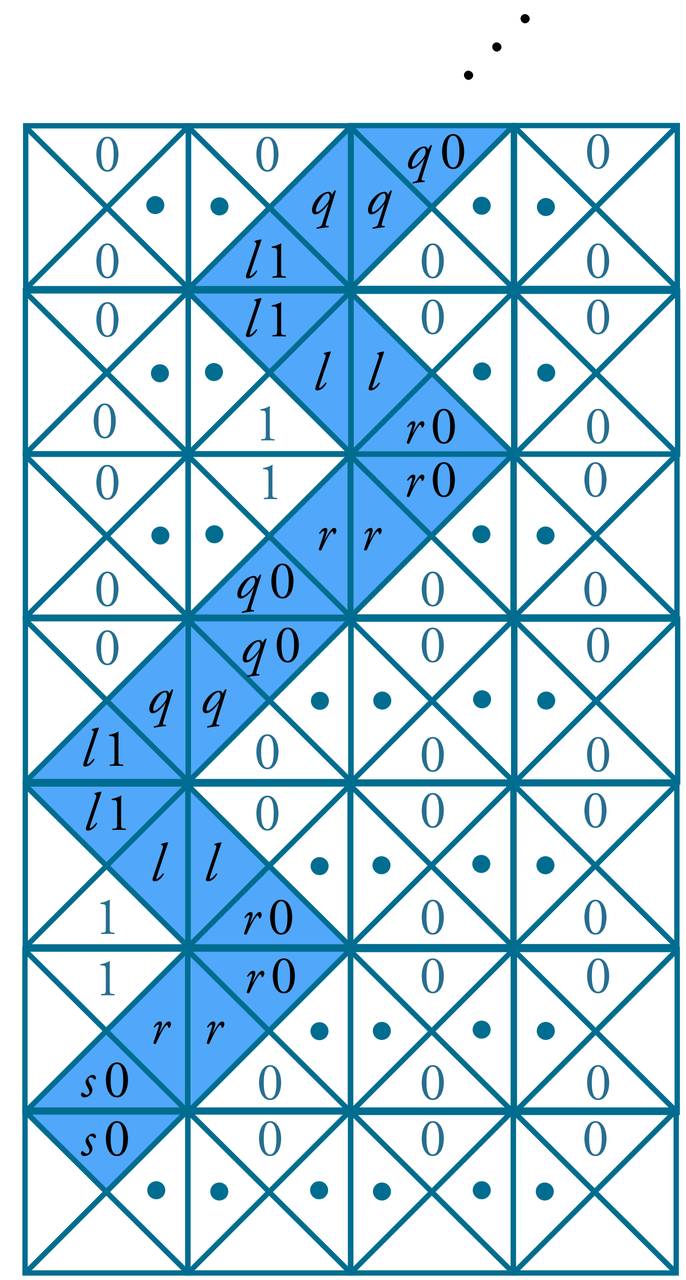


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

The (undecidable) Domino problem

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:



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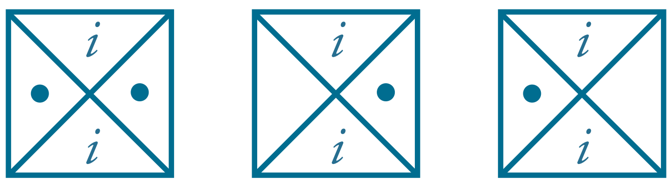

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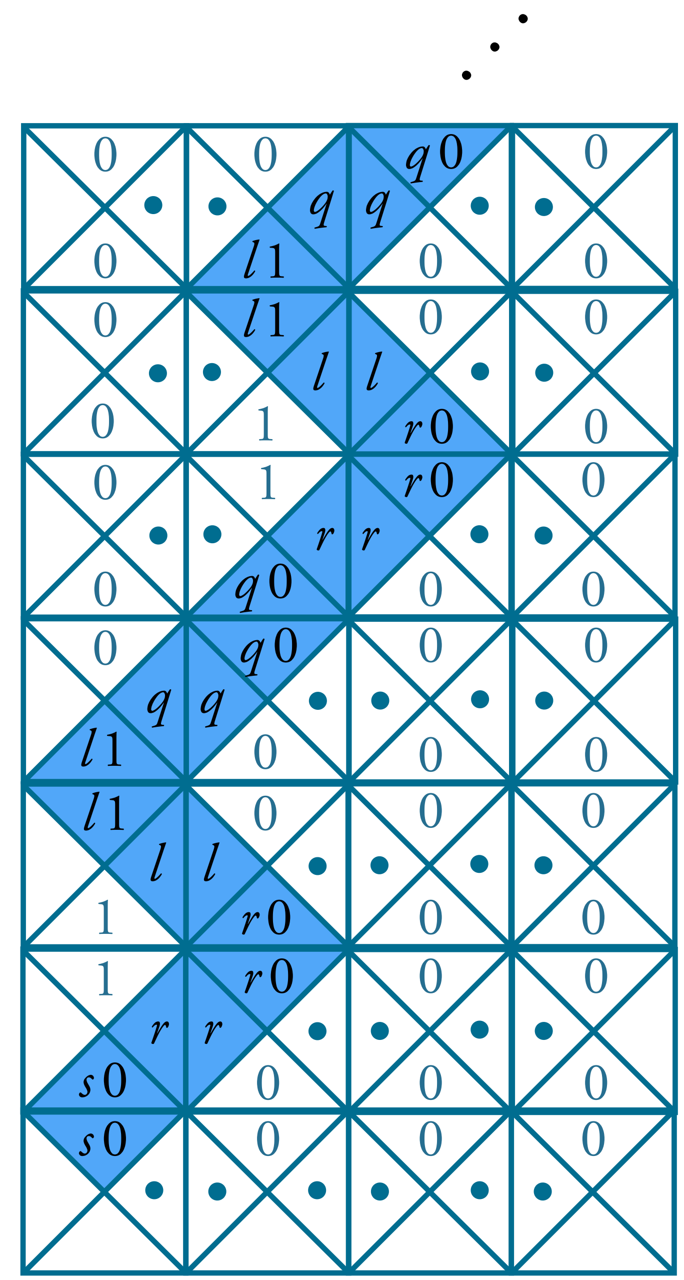

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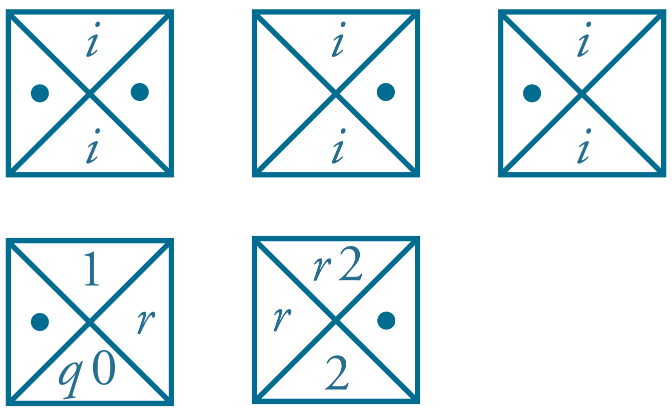

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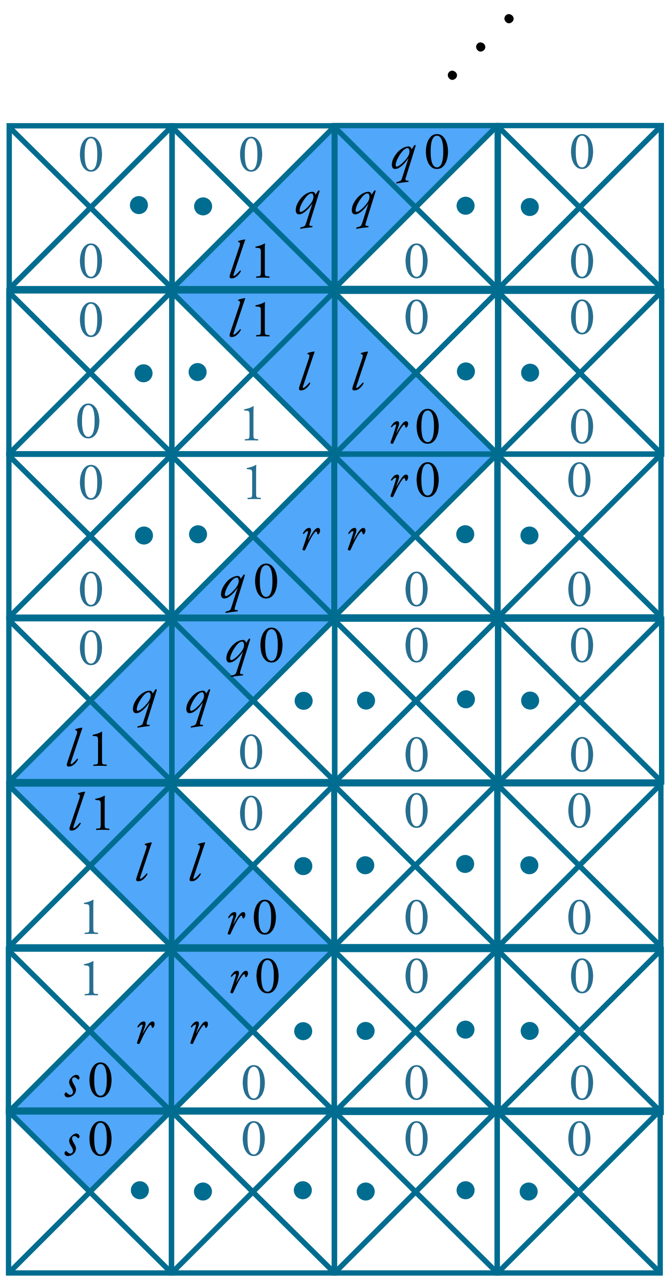

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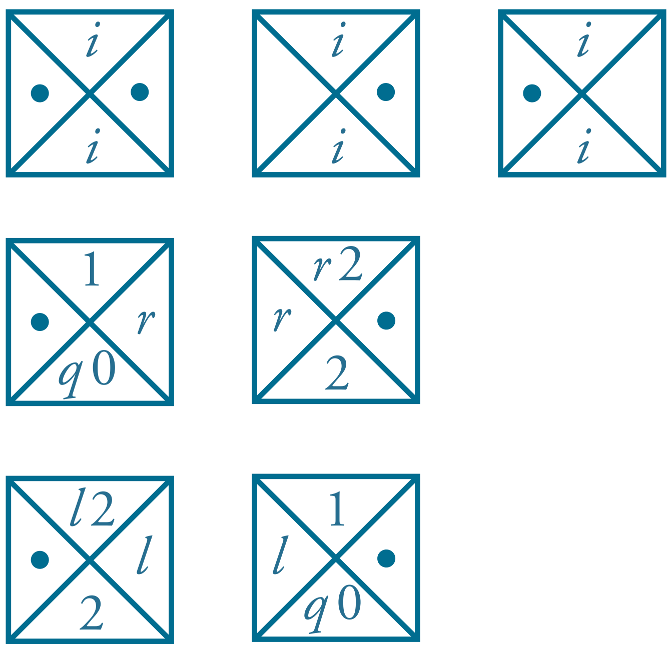

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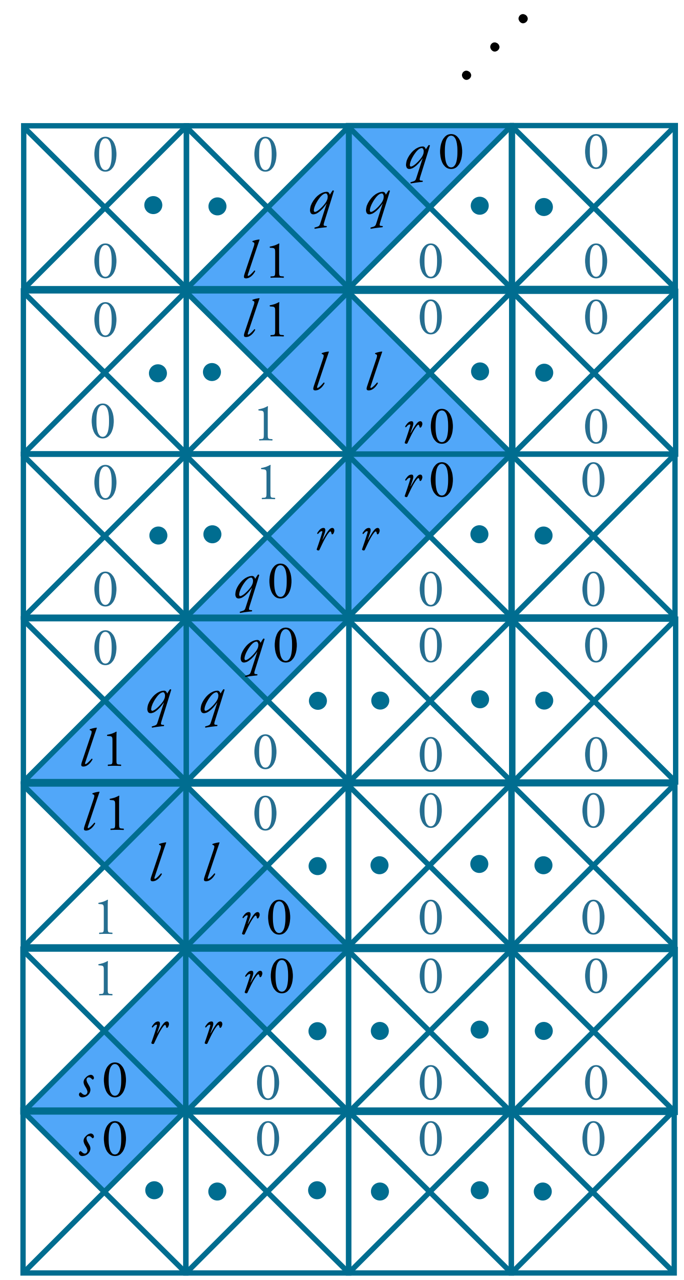

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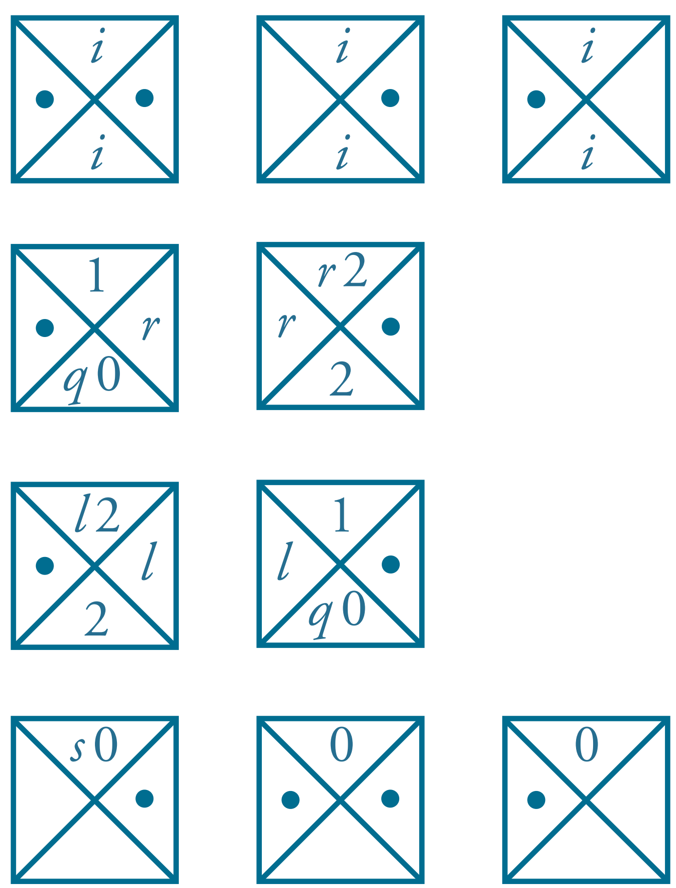

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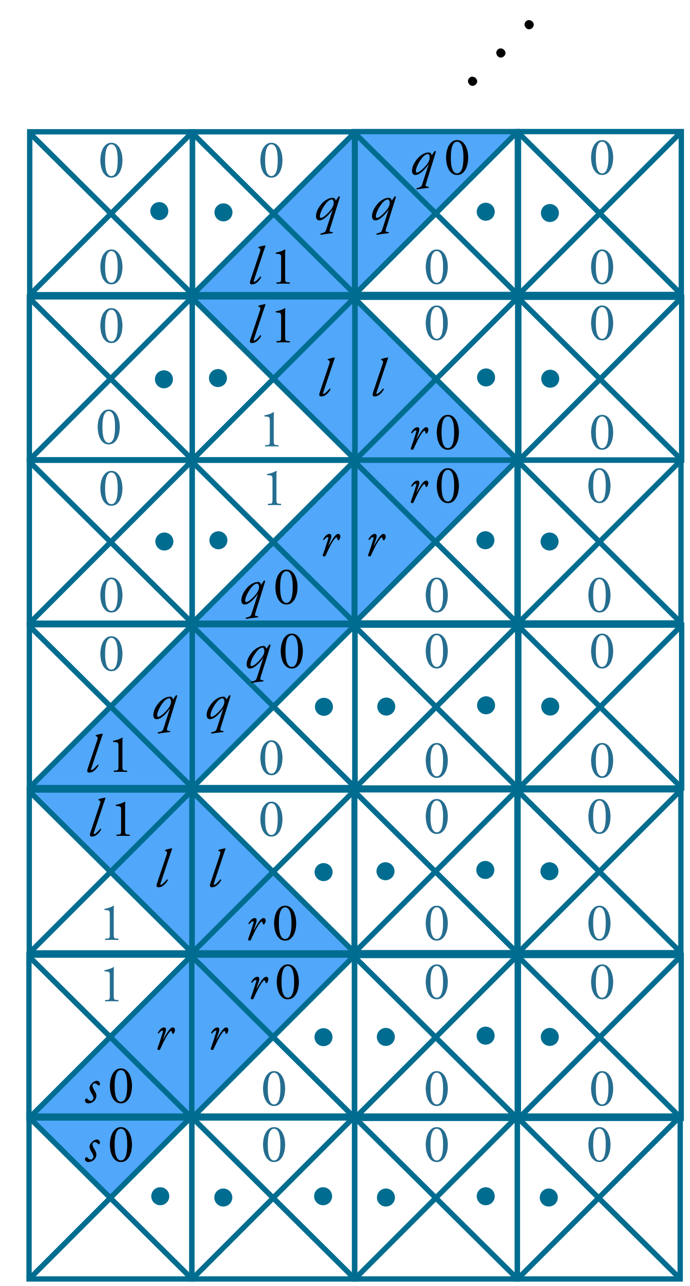

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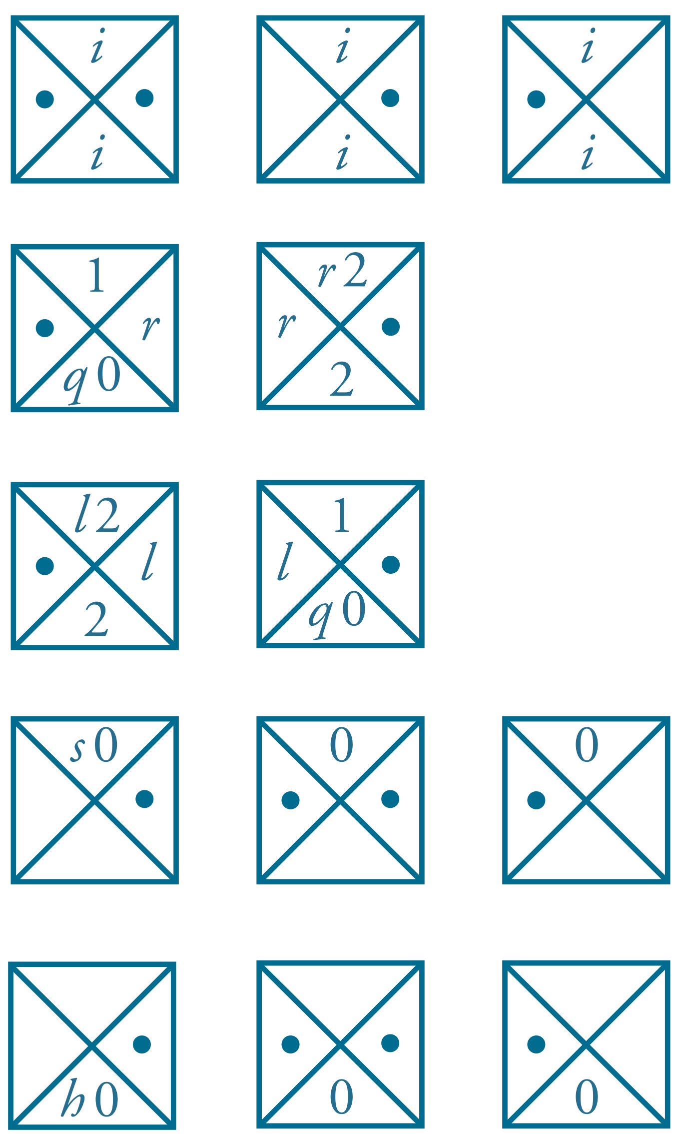

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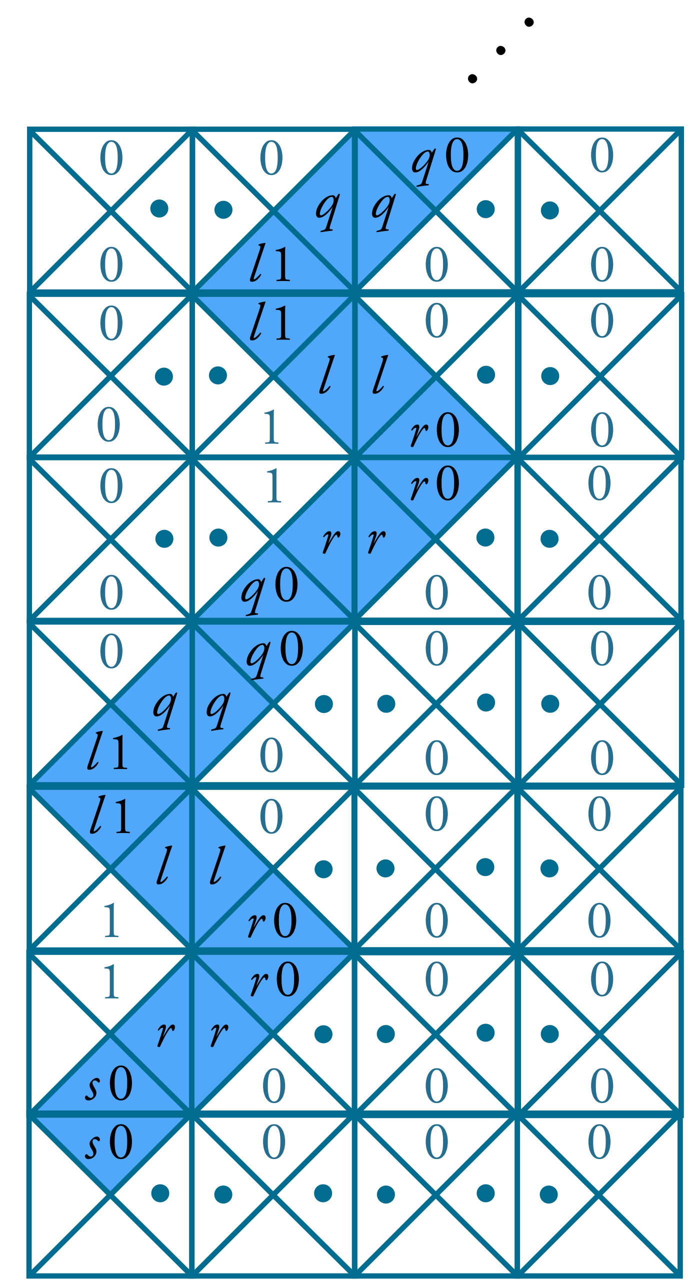

Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…

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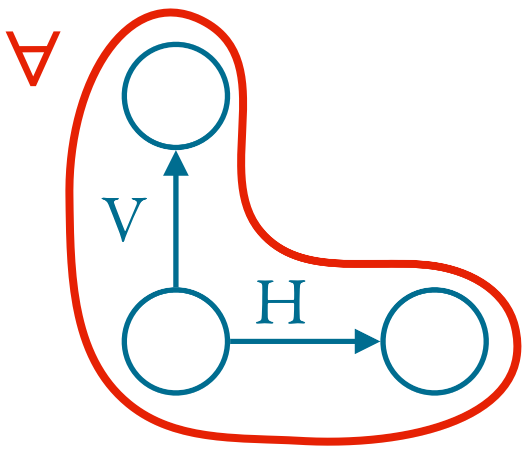

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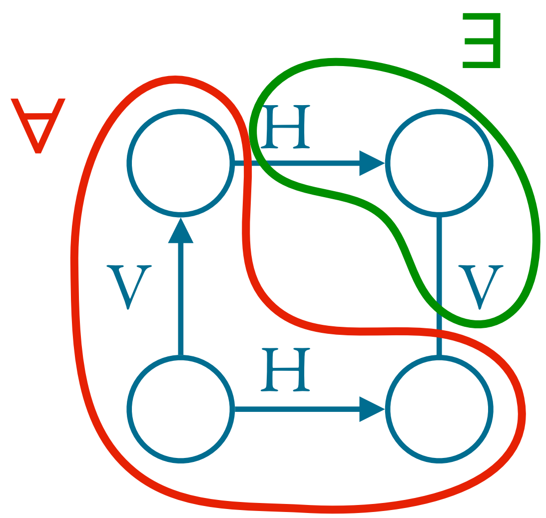

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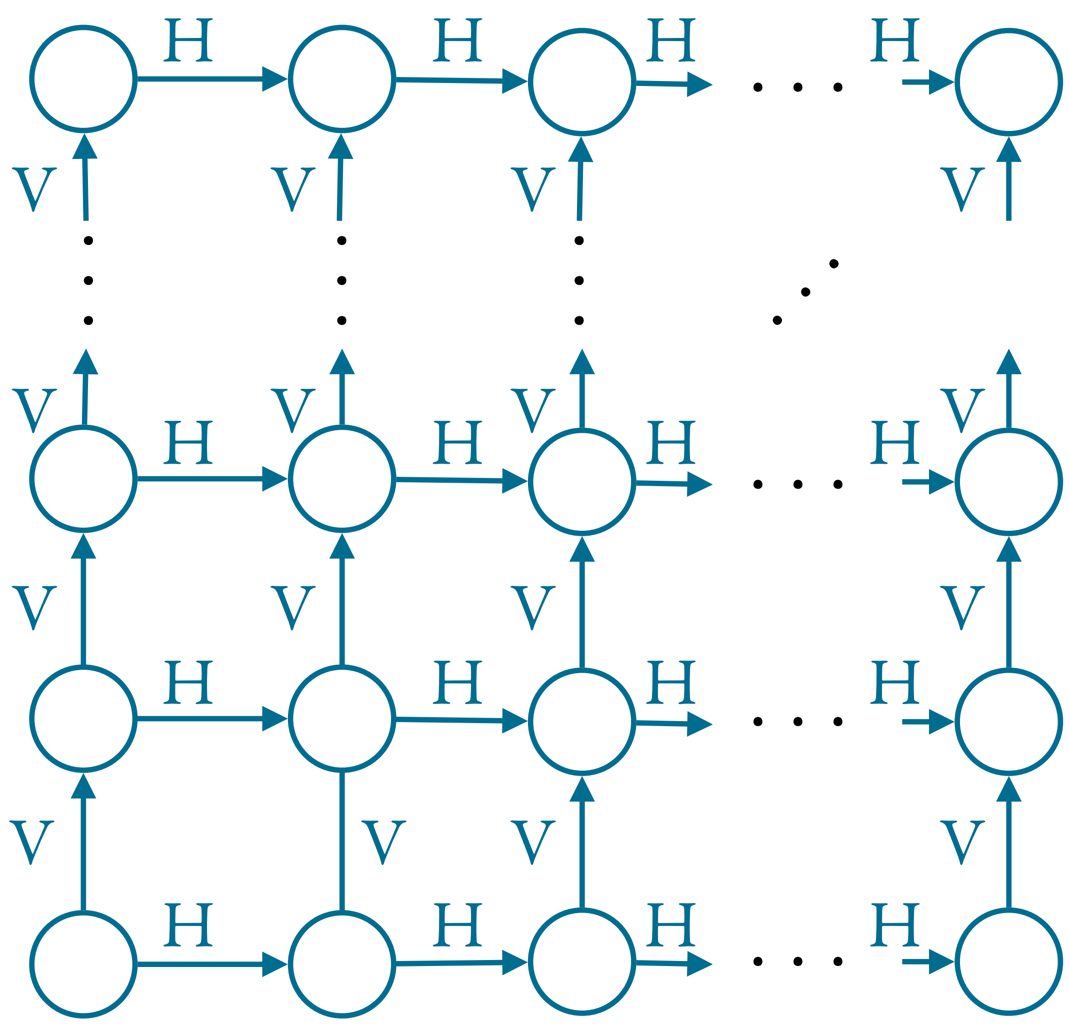

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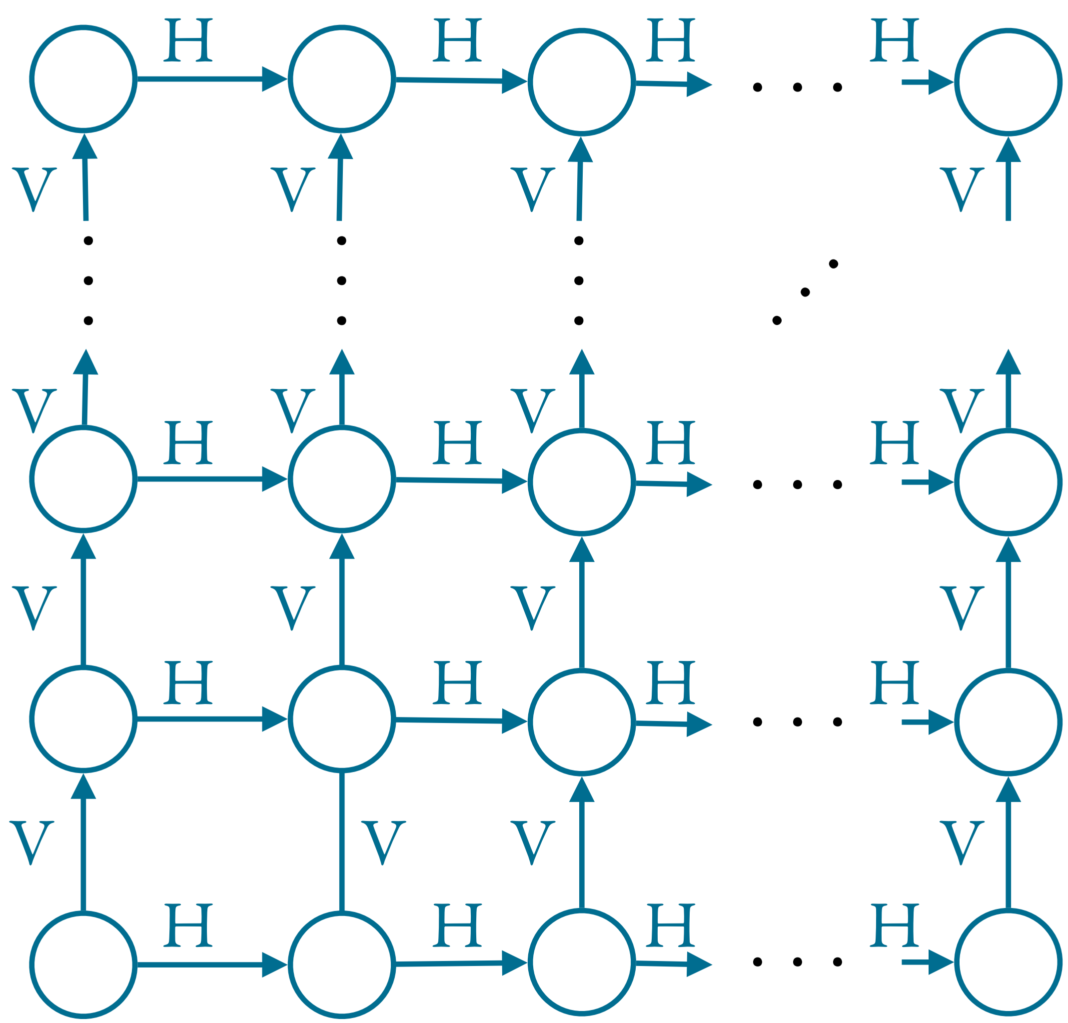

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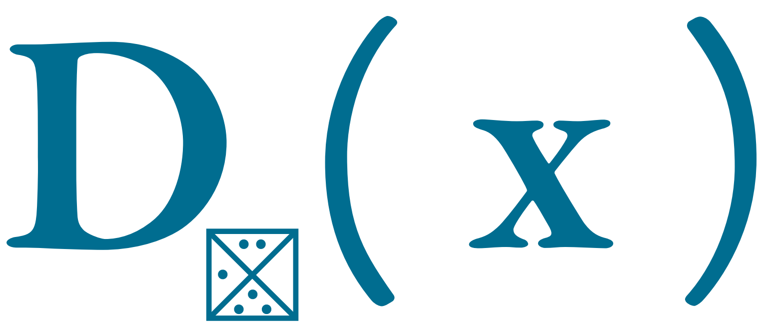
a unary relation



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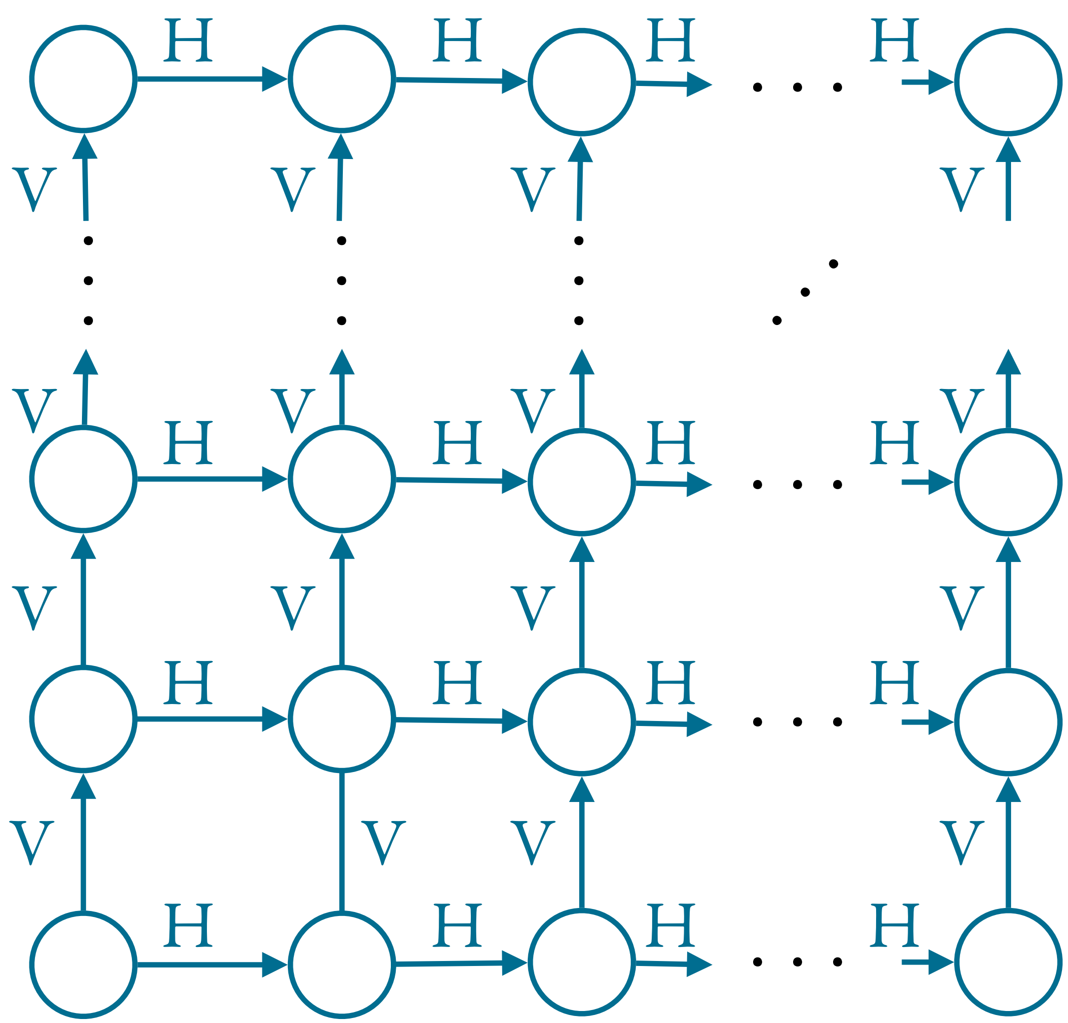

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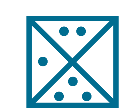
for each domino 

3. **Match the sides** $\forall x,y$

if $H(x,y)$, then $D_a(x) \wedge D_b(y)$

for some dominos **a,b** that 'match' horizontally (Idem vertically)

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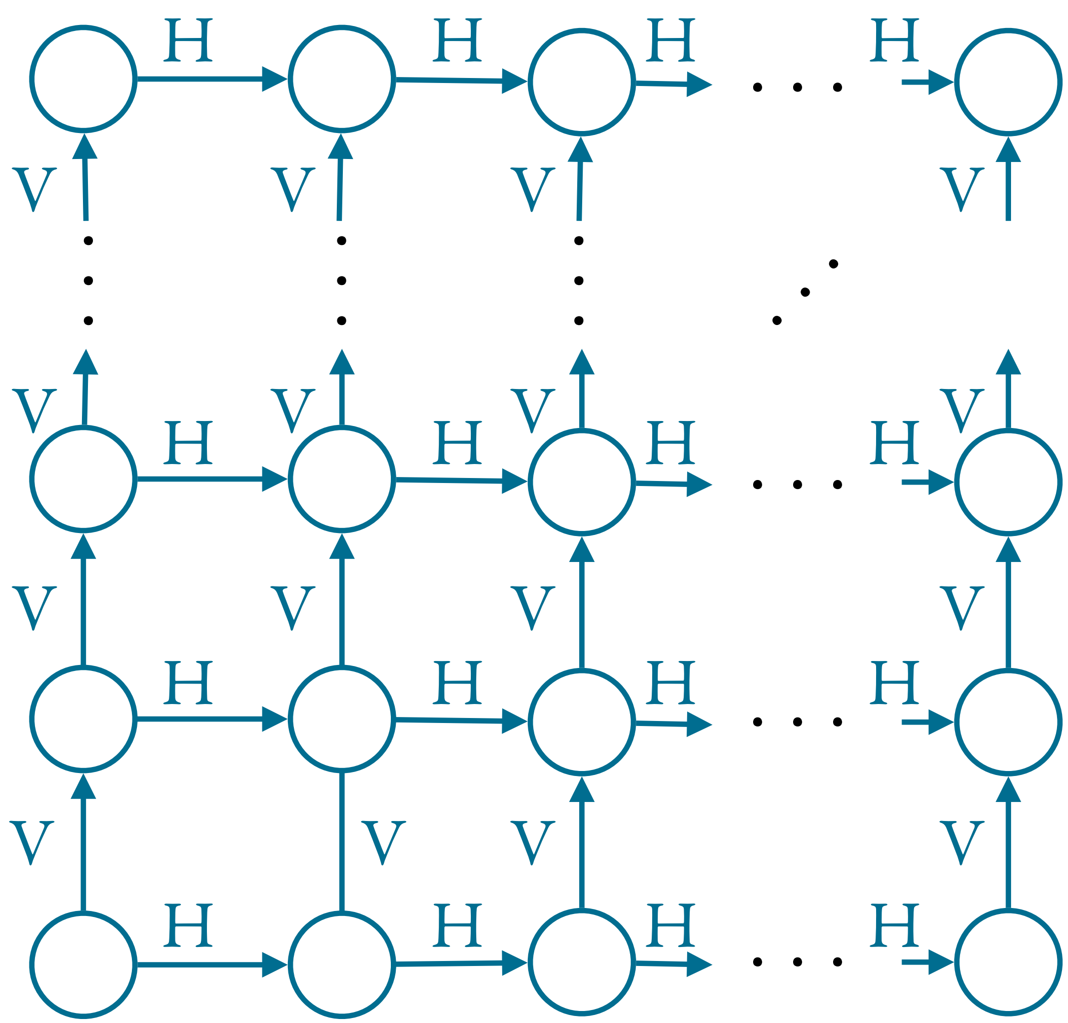

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4. **Borders are white.**

Evaluation problem for FO

Input:

$$\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$$

Output: $G \models_\alpha \phi$?

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Encoding of $G = (V, E)$

- each node is coded with a bit string of size $\log(|V|)$,
- edge set is encoded by its tuples, e.g. (100,101), (010, 010), ...

Cost of coding: $||G|| = |E| \cdot 2 \cdot \log(|V|) \approx |V|$ (mod a polynomial)

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Encoding of $\alpha = \{x_1,\dots,x_n\} \longrightarrow V$

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Cost of coding: $||\alpha|| = n \cdot \log(|V|)$

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- If $\phi(x_1,\dots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

- If $\phi(x_1,\dots,x_n) = \psi(x_1,\dots,x_n) \wedge \psi'(x_1,\dots,x_n)$:
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- If $\phi(x_1,\dots,x_n) = \neg\psi(x_1,\dots,x_n)$:
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- If $\phi(x_1,\dots,x_n) = \exists y \,.\, \psi(x_1,\dots,x_n,y)$:
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Question:
How much space
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use 4 pointers ⇝ LOGSPACE

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  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  ⇝ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))

- If $\phi(x_1,\dots,x_n) = \neg\psi(x_1,\dots,x_n)$:
  answer NO iff $G \models_\alpha \psi$

- If $\phi(x_1,\dots,x_n) = \exists y \,.\, \psi(x_1,\dots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{cases}$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\dots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers $\rightsquigarrow$ LOGSPACE

- If $\phi(x_1,\dots,x_n) = \psi(x_1,\dots,x_n) \wedge \psi'(x_1,\dots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$

  $\rightsquigarrow$ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))

- If $\phi(x_1,\dots,x_n) = \neg\psi(x_1,\dots,x_n)$:
  answer NO iff $G \models_\alpha \psi$

  $\rightsquigarrow$ SPACE($G \models_\alpha \psi$))

- If $\phi(x_1,\dots,x_n) = \exists y \,.\, \psi(x_1,\dots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$

Output: $G \vDash_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers $\rightsquigarrow$ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \vDash_\alpha \psi$ and $G \vDash_\alpha \psi'$

  $\rightsquigarrow$ MAX(SPACE($G \vDash_\alpha \psi$)), SPACE($G \vDash_\alpha \psi'$)))

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \vDash_\alpha \psi$

  $\rightsquigarrow$ SPACE($G \vDash_\alpha \psi$))

- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \vDash_{\alpha'} \psi$.

  $\rightsquigarrow$ $2{\cdot}\log(|G|)$ + SPACE($G \vDash_{\alpha'} \psi$)

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\left(\begin{array}{l} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{array} \right.$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\dots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$ — use 4 pointers ⇝ LOGSPACE
- If $\phi(x_1,\dots,x_n) = \psi(x_1,\dots,x_n) \wedge \psi'(x_1,\dots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$ — ⇝ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))
- If $\phi(x_1,\dots,x_n) = \neg\psi(x_1,\dots,x_n)$:
  answer NO iff $G \models_\alpha \psi$ — ⇝ SPACE($G \models_\alpha \psi$))
- If $\phi(x_1,\dots,x_n) = \exists y \,.\, \psi(x_1,\dots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$ we have $G \models_{\alpha'} \psi$. — ⇝ $2 \cdot \log(|G|)$ + SPACE($G \models_{\alpha'} \psi$)

Question:
How much space does it take?

$\underbrace{2 \cdot \log(|G|) + \cdots + 2 \cdot \log(|G|)}_{\leq |\phi| \text{ times}} + k \cdot \log(|\alpha|+|G|)$ space

Evaluation problem for FO in PSPACE

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
  use 4 pointers ⇝ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  ⇝ $\mathrm{MAX}(\,\mathrm{SPACE}(G \models_\alpha \psi)), \mathrm{SPACE}(G \models_\alpha \psi'))\,)$

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
  ⇝ $\mathrm{SPACE}(G \models_\alpha \psi))$

- If $\phi(x_1,\ldots,x_n) = \exists y\,.\,\psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$ we have $G \models_{\alpha'} \psi$.
  ⇝ $2 \cdot \log(|G|) + \mathrm{SPACE}(G \models_{\alpha'} \psi\,)$

Question:
How much space does it take?

$$\underbrace{2 \cdot \log(|G|) + \cdots + 2 \cdot \log(|G|)}_{\leq |\phi| \text{ times}} + k \cdot \log(|\alpha|+|G|) \text{ space}$$

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input:

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input: • **query** +

Combined, Query, and Data comp

Problem: Usual scen

Input: • **query** +

database

Combined, Query, and Data comp

Problem: Usual scen

Input: • **query** +

database

But we don't distinguish this in the analysis: $\mathrm{TIME}(2^{|\text{query}|} + |\text{data}|) = \mathrm{TIME}(|\text{query}| + 2^{|\text{data}|})$

Combined, Query, and Data complexities

[Vardi, 1982]

Query and data play very **different roles.**

Separation of concerns: How the resources grow with respect to

- the size of the data
- the query size

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed)**:** input size is |query|

Data complexity (|query| fixed)**:** input size is |data|

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed)**:** input size is |query|

Data complexity (|query| fixed)**:** input size is |data|

$O(2^{|query|} + |data|)$ is
- exponential in **combined** complexity
- exponential in **query** complexity
- linear in **data** complexity

$O(|query| + 2^{|data|})$ is
- exponential in **combined** complexity
- linear in **query** complexity
- exponential in **data** complexity

Question

What is the data, query and combined complexity for the evaluation problem for FO?

Remember: **data** complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

Question

What is the data, query and combined complexity
for the evaluation problem for FO?

Remember: **data** complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k{\cdot}\log(|\alpha|+|G|)$ space

O(log(|data|)·|query|) space

PSPACE combined and query complexity

LOGSPACE data complexity

Bounded Degree

$\mathcal{D}_k$—the class of structures $\mathbb{A}$ in which every element has at most k neighbours in $G\mathbb{A}$.

$O(f(k) \cdot n)$

Theorem (Seese)

For every sentence φ of FO and every k there is a linear time algorithm which, given a structure $\mathbb{A} \in \mathcal{D}_k$ determines whether $\mathbb{A} \models \varphi$.

Note: this is not true for MSO unless P = NP.

The proof is based on *locality* of first-order logic. Specifically, *Hanf's theorem*.

Hanf Types

For an element a in a structure $\mathbb{A}$, define

> $N_r^{\mathbb{A}}(a)$—*the substructure of* $\mathbb{A}$ *generated by the elements whose distance from* a *(in* $G\mathbb{A}$*) is at most* r*.*

We say $\mathbb{A}$ and $\mathbb{B}$ are *Hanf equivalent* with radius r and threshold q ($\mathbb{A} \simeq_{r,q} \mathbb{B}$) if, for every $a \in A$ the two sets

$$\{a' \in A \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{A}}(a')\} \quad \text{and} \quad \{b \in B \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{B}}(b)\}$$

either have the same size or both have size greater than q;
and, similarly for every $b \in B$.

Hanf Locality Theorem

Theorem (Hanf)
For every vocabulary σ and every p there qre r and q such that for any σ-structures $\mathbb{A}$ and $\mathbb{B}$: if $\mathbb{A} \simeq_{r,q} \mathbb{B}$ then $\mathbb{A} \equiv_p \mathbb{B}$.

For $\mathbb{A} \in \mathcal{D}_k$:

$N_r^{\mathbb{A}}(a)$ has at most $k^r + 1$ elements

each $\simeq_{r,q}$ has finite index.

Each $\simeq_{r,q}$-class t can be characterised by a finite table, I_t, giving isomorphism types of neighbourhoods and numbers of their occurrences up to threshold q.

Satisfaction on $\mathcal{D}_k$

For a sentence φ of FO, we can compute a set of tables $\{I_1, \ldots, I_s\}$ describing $\simeq_{r,q}$-classes consistent with it.
This computation is independent of any structure $\mathbb{A}$.

Given a structure $\mathbb{A} \in \mathcal{D}_k$,

for each a, determine the isomorphism type of $N_r^{\mathbb{A}}(a)$

construct the table describing the $\simeq_{r,q}$-class of $\mathbb{A}$.

compare against $\{I_1, \ldots, I_s\}$ to determine whether $\mathbb{A} \models \varphi$.

For fixed k, r, q, this requires time *linear* in the size of $\mathbb{A}$.

Note: evaluation for FO is in $O(f(l, k)n)$.