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Complexity Theory

Exercise 6: Diagonalisation and Alternation

Exercise 6.1. Show that Cook-reducibility is transitive. In other words, show that if **A** is Cook-reducible to **B** and **B** is Cook-reducible to **C**, then **A** is Cook-reducible to **C**.

Exercise 6.2. Show that there exists an oracle **C** such that $\mathrm{NP}^{\mathbf{C}} \neq \mathrm{coNP}^{\mathbf{C}}$.

Hint:
What kind of Turing machines exist for languages in coNP? Use the answer to adapt the proof of the Baker-Gill-Solovay Theorem for coNP instead of P.

Exercise 6.3. Describe a polynomial-time ATM solving **EXACT INDEPENDENT SET**:

Input:	Given a graph G and some number k.
Question:	Does there exists a maximal independent set in G of size exactly k?

Exercise 6.4. Consider the Japanese game *go-moku* that is played by two players X and O on a 19x19 board. Players alternately place markers on the board, and the first one to have five of its markers in a row, column, or diagonal wins.

Consider the generalised version of go-moku on an $n \times n$ board. Say that a *position* of go-moku is a placement of markers on such a board as it could occur during the game. Define

$$\mathbf{GM} = \{\langle B \rangle \mid B \text{ is a position of go-moku where X has a winning strategy}\}.$$

Describe a polynomial-time ATM solving **GM**.

Exercise 6.5. Show that $\text{AExpTime} = \text{ExpSpace}$.

∗ **Exercise 6.6.** Show that Σ_2QBF is complete for Σ_2P. Generalise your argument to show that Σ_iQBF is complete for Σ_iP for all $i \geq 1$.

Exercise 6.7. Show that if P = NP, then P = PH.

Exercise 6.8. Describe a polynomial-time ATM solving **EXACT INDEPENDENT SET**.

$$\textbf{ExactIS} = \{(G,k) \mid \ |S| = k \text{ for some independent set } S \text{ in } G \text{ and } |S'| \leq k \text{ for every independent set } S' \text{ in } G\}$$

Find a level of the polynomial hierarchy where this problem is contained in.

Exercise 6.9. Consider the Japanese game *go-moku* that is played by two players X and O on a 19x19 board. Players alternately place markers on the board, and the first one to have five of its markers in a row, column, or diagonal wins.

Consider the generalised version of go-moku on an $n \times n$ board. Say that a *position* of go-moku is a placement of markers on such a board as it could occur during the game. Define

$$\mathbf{GM} = \{\langle B \rangle \mid B \text{ is a position of go-moku where X has a winning strategy}\}.$$

Describe a polynomial-time ATM solving **GM** and informally argue why this problem is not in any level of the polynomial hierarchy.

Exercise 6.10. Show $\mathsf{NP}^{\mathsf{SAT}} \subseteq \Sigma_2\mathsf{P}$.

Exercise 6.11. Show the following result: *If there is any k such that $\Sigma_k^{\mathrm{P}} = \Sigma_{k+1}^{\mathrm{P}}$ then $\Sigma_j^{\mathrm{P}} = \Pi_j^{\mathrm{P}} = \Sigma_k^{\mathrm{P}}$ for all $j > k$, and therefore $\mathrm{PH} = \Sigma_k^{\mathrm{P}}$.*

Exercise 6.12. Show that $\mathrm{PH} \subseteq \mathrm{PS{\scriptsize PACE}}$.

Exercise 6.13. Let $\mathbf{A}$ be a language and let $\mathbf{F}$ be a finite set with $\mathbf{A} \cap \mathbf{F} = \emptyset$.
Show that $\mathrm{P}^{\mathbf{A}} = \mathrm{P}^{\mathbf{A} \cup \mathbf{F}}$ and $\mathrm{NP}^{\mathbf{A}} = \mathrm{NP}^{\mathbf{A} \cup \mathbf{F}}$.