TECHNISCHE UNIVERSITÄT DRESDEN

DEDUCTION SYSTEMS

Optimizations for Tableau Procedures

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Chair for Knowledge-Based Systems

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DRESDEN concept Exzellenz aus Wissenschaft und Kultur

Agenda

- Recap Tableau Calculus
- Optimizations
  - Unfolding
  - Absorption
  - Dependency-Directed Backtracking
  - Further Optimizations
- Classification
- Summary

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Tableau Algorithm for $\mathcal{ALC}$ Concepts and TBoxes

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- C is satisfiable iff there is a successful tableau construction

Treatment of Knowledge Bases

we condense the TBox into one concept:
for $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$, $C_\mathcal{T} = \mathsf{NNF}(\bigsqcap_{1 \leq i \leq n} \neg C_i \sqcup D_i)$

we extend the rules of the $\mathcal{ALC}$ tableau algorithm:

$\mathcal{T}$-rule: for an arbitrary $v \in V$ with $C_\mathcal{T} \notin L(v)$,
let $L(v) := L(v) \cup \{C_\mathcal{T}\}$.

in order to take an ABox $\mathcal{A}$ into account, initialize G such that

- V contains a node v_a for every individual a in $\mathcal{A}$
- $L(v_a) = \{C \mid C(a) \in \mathcal{A}\}$
- $\langle v_a, v_b \rangle \in E$ iff $r(a, b) \in \mathcal{A}$

Extensions of the Logic

- plus inverses ($\mathcal{ALCI}$): inverse roles in edge labels, definition and use of r-neighbors instead of r-successors in tableau rules
- plus functional roles ($\mathcal{ALCIF}$): merging of nodes to account for functionality

blocking guarantees termination:

- $\mathcal{ALC}$ subset-blocking
- plus inverses ($\mathcal{ALCI}$): equality blocking
- plus functional roles ($\mathcal{ALCIF}$): pairwise blocking

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  - $\mathcal{T}$-rule adds one disjunction per axiom to the corresponding node
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Unfolding

- $\mathcal{T}$-rule is not necessary if $\mathcal{T}$ is unfoldable, i.e., every axiom is:
  - definitorial: form $A \sqsubseteq C$ or $A \equiv C$ for A a concept name ($A \equiv C$ corresponds to $A \sqsubseteq C$ and $C \sqsubseteq A$)
  - acyclic: C uses A neither directly nor indirectly
  - unique: only one such axiom exists for every concept name A

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  - acyclic: C uses A neither directly nor indirectly
  - unique: only one such axiom exists for every concept name A
- If $\mathcal{T}$ is unfoldable, the TBox can be (unfolded) into a concept

Unfolding Example

- We check satisfiability of A w.r.t. the TBox $\mathcal{T}$

$\mathcal{T}$:

$$A \sqsubseteq B \sqcap \exists r.C$$
$$B \equiv C \sqcup D$$
$$C \sqsubseteq \exists r.D$$

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$$\begin{aligned} & A \\ \rightsquigarrow\ & A \sqcap B \sqcap \exists r.C \\ \rightsquigarrow\ & A \sqcap (C \sqcup D) \sqcap \exists r.C \end{aligned}$$

$\mathcal{T}$:

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$$\begin{aligned} & A \\ \rightsquigarrow\ & A \sqcap B \sqcap \exists r.C \\ \rightsquigarrow\ & A \sqcap (C \sqcup D) \sqcap \exists r.C \\ \rightsquigarrow\ & A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D) \end{aligned}$$

Unfolding Example

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$\mathcal{T}$:

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\begin{aligned}
A &\sqsubseteq B \sqcap \exists r.C \\
B &\equiv C \sqcup D \\
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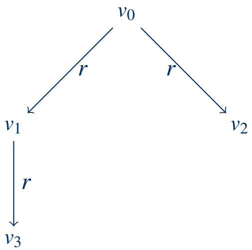
- A is satisfiable w.r.t. $\mathcal{T}$ iff

$$A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D)$$

  is satisfiable w.r.t. the empty TBox

Tableau Algorithm Example with Unfolding

We obtain the following contradiction-free tableau for the satisfiability of
$U = A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D)$:



$$
\begin{aligned}
L(v_0) &= \{U, A, (C \sqcap \exists r.D) \sqcup D, \\
&\quad\ \exists r.(C \sqcap \exists r.D), C \sqcap \exists r.D, \\
&\quad\ C, \exists r.D\} \\
L(v_1) &= \{C \sqcap \exists r.D, C, \exists r.D\} \\
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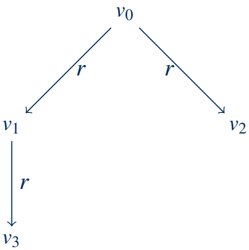

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Only one disjunctive decision left!

Lazy Unfolding

- computation of NNF together with unfolding may decrease performance, e.g.:
  - satisfiability of $C \sqcap \neg C$ w.r.t. $\mathcal{T} = \{C \sqsubseteq A \sqcap B\}$
  - unfolding: $C \sqcap A \sqcap B \sqcap \neg(C \sqcap A \sqcap B)$
  - NNF + unfolding: $C \sqcap A \sqcap B \sqcap (\neg C \sqcup \neg A \sqcup \neg B)$

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  - NNF + unfolding: $C \sqcap A \sqcap B \sqcap (\neg C \sqcup \neg A \sqcup \neg B)$
- better: apply NNF and unfolding if needed, via corresponding tableau rules:
  - $A \equiv C \rightsquigarrow A \sqsubseteq C$ and $A \sqsupseteq C$

$\sqsubseteq$-rule: For $v \in V$ such that $A \sqsubseteq C \in \mathcal{T}$, $A \in L(v)$ and $C \notin L(v)$
let $L(v) := L(v) \cup C$.

$\sqsupseteq$-rule: For $v \in V$ such that $A \sqsupseteq C \in \mathcal{T}$, $\neg A \in L(v)$ and $\neg C \notin L(v)$
let $L(v) := L(v) \cup \{\neg C\}$.

$\neg$-rule: For $v \in V$ such that $\neg C \in L(v)$ and NNF$(\neg C) \notin L(v)$,
let $L(v) := L(v) \cup \{$NNF$(\neg C)\}$.

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- What if $\mathcal{T}$ is not unfoldable?
  - Separate $\mathcal{T}$ into $\mathcal{T}_u$ (unfoldable part) and $\mathcal{T}_g$ (GCIs, not unfoldable)
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- absorption decreases $\mathcal{T}_g$ and increases $\mathcal{T}_u$
  1. take an axiom from $\mathcal{T}_g$, e.g., $A \sqcap B \sqsubseteq C$
  2. transform the axiom: $A \sqsubseteq C \sqcup \neg B$
  3. if $\mathcal{T}_u$ contains an axiom of the form $A \equiv D$ $\quad$ ($A \sqsubseteq D$ and $D \sqsupseteq A$), then $A \sqsubseteq C \sqcup \neg B$ cannot be absorbed;
     $A \sqsubseteq C \sqcup \neg B$ remains in $\mathcal{T}_g$
  4. otherwise, if $\mathcal{T}_u$ contains an axiom of the form $A \sqsubseteq D$,
     then absorb $A \sqsubseteq C \sqcup \neg B$ resulting in $A \sqsubseteq D \sqcap (C \sqcup \neg B)$
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- nondeterministic: $B \sqsubseteq C \sqcup \neg A$ also possible

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Dependency-Directed Backtracking

- despite those optimizations, search space often too big
- let $v \in V$ with $(C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n) \sqcap \exists r.\neg A \sqcap \forall r.A \in L(v)$

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Dependency-Directed Backtracking

- despite those optimizations, search space often too big
- let $v \in V$ with $(C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n) \sqcap \exists r.\neg A \sqcap \forall r.A \in L(v)$

$\sqcap$-rule	$L(v)$	$:=$	$L(v) \cup \{(C_1 \sqcup D_1), \ldots, (C_n \sqcup D_n), \exists r.\neg A, \forall r.A\}$	
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- exponentially big search space is traversed

Dependency-Directed Backtracking

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  - concepts in the node label are tagged with a set of integers (dependency set) allowing to identify the concept's "origin"
  - initially, all concepts are tagged with $\emptyset$
  - tableau rules combine and extend these tags
  - $\sqcup$-rule adds the tag $\{d\}$ to the existing tag, where d is the $\sqcup$-depth (number of $\sqcup$-rules applied by now)
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- irrelevant part of the search space is not considered

Dependency-Directed Backtracking Example

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- Output false (unsatisfiable)

Agenda

- Recap Tableau Calculus
- Optimizations
  - Unfolding
  - Absorption
  - Dependency-Directed Backtracking
  - Further Optimizations
- Classification
- Summary

Further Optimizations

- Simplification and Normalization
  - quick recognition of trivial contradictions
  - normalization, e.g., $A \sqcap (B \sqcap C) \equiv \sqcap\{A, B, C\}$, $\forall r.C \equiv \neg\exists r.\neg C$
  - simplification, e.g., $\sqcap\{A, \ldots, \neg A, \ldots\} \equiv \bot$, $\exists r.\bot \equiv \bot$, $\forall r.\top \equiv \top$

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  - prevents the repeated construction of equal subtrees
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One of the most wide-spread tasks for automated reasoning is classification

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- check for $\mathcal{T} \models C \sqsubseteq D$ can be reduced to checking satisfiability of $\mathcal{T}$ together with the ABox $(C \sqcap \neg D)(a)$ (or, equivalenty: $C(a), (\neg D)(a)$)
  - $\leadsto$ if $\top$ is satisfiable: subsumption does not hold (as we have constructed a counter-model)
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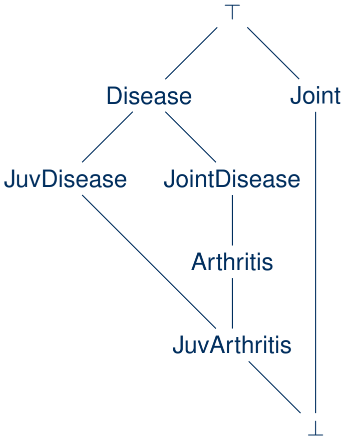
  - $\rightsquigarrow$ if $\top$ is satisfiable: subsumption does not hold (as we have constructed a counter-model)
  - $\rightsquigarrow$ if $\top$ is unsatisfiable: subsumption holds (no counter-model exists)
- naïve approach needs n^2 subsumption checks for n concept names
- normally cached in the concept hierarchy graph

Concept Hierarchy Graph



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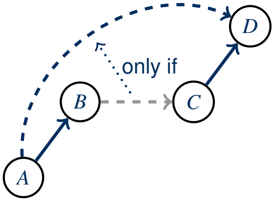
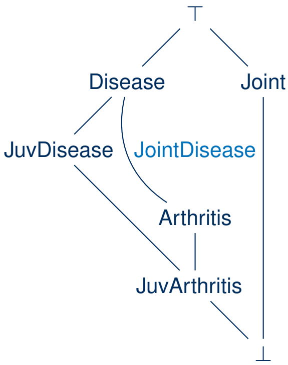
- hierarchy is created incrementally by introducing concept after concept
- top-down phase: recognize direct superconcepts
- bottom-up phase: recognize direct subconcepts
- transitivity of $\sqsubseteq$ used to save checks



- If $A \sqsubseteq B$ and $C \sqsubseteq D$ hold,
- then $B \sqsubseteq C \longrightarrow A \sqsubseteq D$
- and $A \not\sqsubseteq D \longrightarrow B \not\sqsubseteq C$

Enhanced Traversal Example

already created hierarchy:



Goal: insertion of JointDisease

Top-Down Phase:

Bottom-Up Phase:

Enhanced Traversal Example

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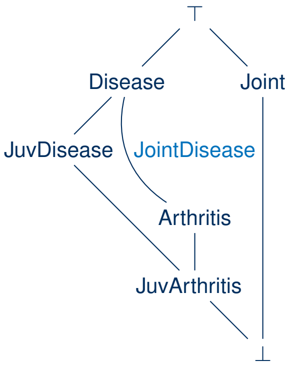

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Enhanced Traversal Example

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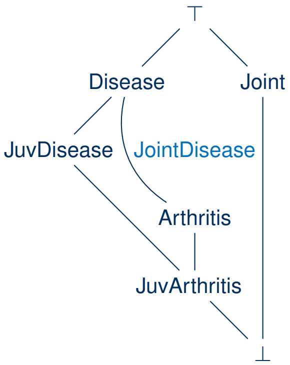

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- JointDisease $\sqsubseteq^?$ JuvDisease

Bottom-Up Phase:

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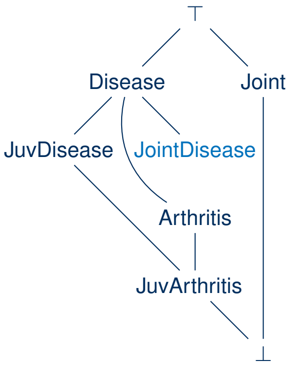

Goal: insertion of JointDisease

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- JointDisease $\sqsubseteq^?$ Arthritis

Bottom-Up Phase:

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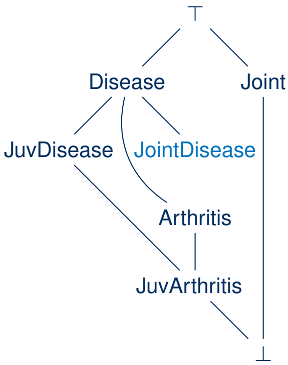

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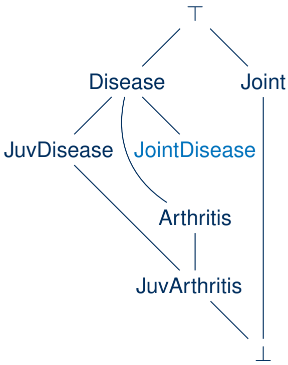

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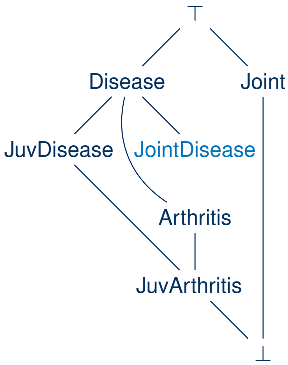

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Bottom-Up Phase:

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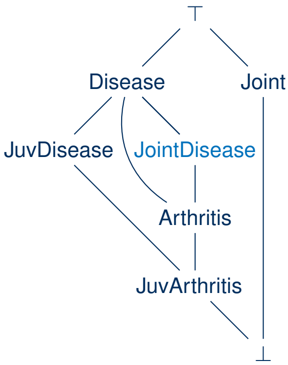

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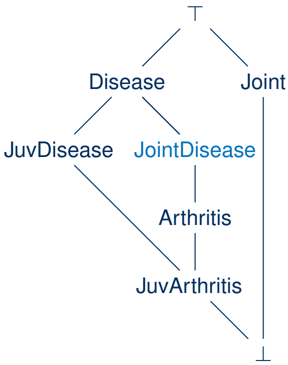

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Summary

- we have a tableau algorithm for $\mathcal{ALCIF}$ knowledge bases
  - ABox treated like for $\mathcal{ALC}$
  - number restrictions are treated similar to functionality and existential quantifiers
- termination via cycle detection
  - becomes harder as the logic becomes more expressive
- naive tableau algorithm not sufficiently performant
- diverse optimizations improve average case
- specific methods for classification
  - enhanced traversal
- tableaux algorithms or variants modifications thereof are the basis of many OWL reasoners