Many-valued Horn Logic is Hard

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1 Introduction

Fuzzy Description Logics (FDLs) have been introduced to reason about vague or imprecise knowledge in application domains. In recent years, reasoning in many FDLs based on infinitely many values has been proved to be undecidable [3,15] and systematic studies have been undertaken on this topic [8]. On the other hand, every finite-valued FDL that has been studied in the recent literature has not only been proved to be decidable, but even to belong to the same complexity class as the corresponding crisp DL [6,7,11,12]. A question that naturally arises is whether the finite-valued fuzzy framework is not more complex (w.r.t. computational complexity) than the crisp-valued formalism in general. A common opinion is that everything that can be expressed in finite-valued FDLs can be reduced to the corresponding crisp DLs without any serious loss of efficiency. Indeed, although some known translations of finite-valued FDLs into crisp DLs are exponential [5], more efficient reasoning can be achieved through direct algorithms.

The fact that a significant difference in computational complexity between the crisp and the finite-valued case has not yet been found is mainly due to the high expressivity of the languages studied so far. Indeed, these languages already contain significant sources of nondeterminism in the crisp case. Our idea is that the proof of a possible difference in the complexity between both formalisms has to be searched in languages that allow for a lower expressivity. In such languages, the sources of nondeterminism inherent in the classical framework have not yet shown up, while the same languages may be already affected by the inherent nondeterminism of the basic logical connectives in the finite-valued framework. For this purpose, we want to take advantage of the "revival" that simple DL

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languages have experienced in recent times. Due to practical reasons, in the last years there have been different works on the complexity behavior of simple languages such as those of the $\mathcal{EL}$ and *DL-Lite* families [2,14]. In [13] it is proven that the subsumption problem with respect to GCIs in the classical language $\mathcal{EL}$ can be solved in polynomial time, and in [2] further extensions of $\mathcal{EL}$ are investigated.

The question about the computational complexity of $\mathcal{EL}$ under a fuzzy semantics has been already considered in [9,10,20]. In [9], the subsumption problem of $\mathcal{EL}$ under infinite Łukasiewicz semantics has been proven coNP-hard. The proof employs a reduction of the *vertex cover* problem to fuzzy $\mathcal{EL}$ subsumption. Unfortunately, this proof cannot be applied when the semantics is based on a finite Łukasiewicz chain. The reason is that for every fixed finite Łukasiewicz chain, only finitely many instances of the vertex cover problem can be encoded. On the other hand, the polynomial algorithm used in [13] to solve the same problem under crisp semantics cannot be straightforwardly applied to fuzzy $\mathcal{EL}$ under a finite Łukasiewicz chain. This is due to the fact that under this semantics, an $\mathcal{EL}$ TBox cannot always be transformed into an equivalent TBox in normal form. All these facts make the subsumption problem of $\mathcal{EL}$ under finite Łukasiewicz semantics a suitable candidate for finding a problem that is computationally different in the crisp and in the finite-valued case.

In this work we are not facing the problem directly. Rather, we consider propositional Horn clauses, which can be seen as a restricted form of $\mathcal{EL}_\bot$-axioms. Reasoning in both formalisms is polynomial under classical two-valued semantics [2,18]. We show that for these clauses a finite-valued conjunction operator (in particular under finite Łukasiewicz semantics) can induce additional nondeterminism. In order to prove this, we reduce the problem of deciding classical satisfiability of propositional formulas to the satisfiability problem for Horn clauses with finite-valued constraints.

The consequences of our result are not restricted to the framework of Fuzzy Description Logics, but they obviously go beyond that. Indeed, our result contributes to achieving a deeper insight into the complexity of reasoning in fragments of finite-valued propositional Łukasiewicz logic. To the best of our knowledge, although there are numerous works on the computational complexity of finite-valued propositional logics [16,19], this kind of problem has not yet been dealt with in the literature. Perhaps the closest to our approach is the study of fuzzy answer set programming. In this context, it has been shown that satisfiability of a set of Horn clauses in restricted form can be decided in polynomial time [1,4] for the infinite-valued Łukasiewicz t-norm. In a nutshell, the main difference between the Horn theories considered is that we allow also conjunctions in the head of a clause while in [1,4] the head can only contain one atom; the latter is a restriction when using fuzzy semantics.

2 Preliminaries

We now introduce the syntax and semantics of the logical formalism Ł_n-Horn. The former is given in terms of implications between conjunctions of literals. In contrast to classical propositional logic, we need to allow conjunctions also in the head of an implication due to our many-valued semantics.

This semantics is defined using the operators of finite fuzzy Łukasiewicz chains. For any natural number $n \geq 2$, we consider the set of n truth values $\text{Ł}_n := \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ together with the following operators:

- The *finite Łukasiewicz t-norm* $*\colon \text{Ł}_n \times \text{Ł}_n \longrightarrow \text{Ł}_n$ defined by

$$x * y := \max\{0, x + y - 1\}.$$

- The *residuum* of the finite Łukasiewicz t-norm $\Rightarrow\colon \text{Ł}_n \times \text{Ł}_n \longrightarrow \text{Ł}_n$ computed as

$$x \Rightarrow y := \min\{1, 1 - x + y\}.$$

As usual in fuzzy logic, these two operators satisfy the property that $x * y \leq z$ iff $y \leq x \Rightarrow z$ for all $x, y, z \in \text{Ł}_n$. The algebra $\langle \text{Ł}_n, *, \Rightarrow, 0, 1\rangle$ is called the *finite Łukasiewicz chain* of length n.

Definition 1 (syntax). *We consider a set $\mathcal{V}$ of* propositional variables*. A* fuzzy Horn clause *is of the form*

$$\langle x_1 \;\&\; \ldots \;\&\; x_k \to y_1 \;\&\; \ldots \;\&\; y_m \geq p\rangle \quad \text{or} \tag{1}$$

$$\langle x_1 \;\&\; \ldots \;\&\; x_k \to \overline{0} \geq p\rangle, \tag{2}$$

where $k \geq 0$, $m \geq 1$, $x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathcal{V}$, and $p \in \text{Ł}_n$. A fuzzy Horn theory *is a finite set of fuzzy Horn clauses.*

Notice that the conjunction on the left-hand side of a Horn clause might be empty; that is, a Horn clause could have the form $\langle \to y_1 \;\&\; \ldots \;\&\; y_m \geq p\rangle$.

The semantics of this logic is given in terms of valuations mapping each propositional variable to one of the possible truth degrees.

Definition 2 (semantics). *A* valuation *of the propositional variables is a function $v\colon \mathcal{V} \longrightarrow \text{Ł}_n$. A fuzzy Horn clause of the form* (1) *or* (2) *is* satisfied *by v if*

$$\big(v(x_1) * \cdots * v(x_k)\big) \Rightarrow \big(v(y_1) * \cdots * v(y_m)\big) \geq p \text{ or}$$
$$\big(v(x_1) * \cdots * v(x_k)\big) \Rightarrow 0 \geq p, \text{ respectively,}$$

where the operation of the left-hand side of the implication is evaluated to 1 *whenever $k = 0$. A fuzzy Horn theory $\mathcal{H}$ is* satisfiable *if there is a valuation that satisfies all fuzzy Horn clauses in $\mathcal{H}$.*

We show that satisfiability of fuzzy Horn theories in Ł_n-Horn is NP-complete, in contrast to the classical case, where it can be checked in linear time. The upper bound follows from the fact that one can simply guess a valuation in polynomial time ($\mathcal{O}(n \cdot |\mathcal{V}|)$) and then check satisfaction of all clauses.

3 Hardness

In this section we reduce a variant of the satisfiability problem for (classical) propositional formulae to satisfiability of Horn theories in Ł_n-Horn, for any $n \geq 4$. This shows that the problem is NP-hard already for Łukasiewicz chains of length 4; i.e., for chains containing four membership degrees.

Given a set $\mathcal{V}$ of variables, the set of *literals* is $\mathcal{V} \cup \{\neg x \mid x \in \mathcal{V}\}$; that is, a literal is either a variable or a negated variable. For a natural number m, an *m-clause* is a disjuction of m literals $\bigvee_{i=1}^{m} \ell_i$. A propositional formula ϕ is in *m-conjunctive normal form* (m-CNF) if it is a conjunction of m-clauses; that is, if ϕ is of the form $\bigwedge_{i=1}^{k} C_i$ for some $k \geq 0$, where each $C_i, 1 \leq i \leq k$ is an m-clause. It is well-known that deciding the satisfiability of m-CNF formulae is NP-hard, for any $m \geq 3$ [17].

Let now $n \geq 4$ and define $m := n - 1$. Given an m-CNF formula ϕ, we construct a fuzzy Horn theory $\mathcal{H}_\phi$ such that ϕ is satisfiable in Ł_n-Horn if and only if $\mathcal{H}_\phi$ is satisfiable. For the rest of this section, let $\phi = \bigwedge_{i=1}^{k} C_i$ be an arbitrary but fixed formula in m-CNF, and $\mathsf{var}(\phi) \subseteq \mathcal{V}$ be the set of all propositional variables appearing in ϕ. For each $x \in \mathsf{var}(\phi)$, we employ a fresh variable $x' \in \mathcal{V}$ to simulate the literal $\neg x$ in the fuzzy Horn theory $\mathcal{H}_\phi$.

As a first step, we define the Horn theory

$$\begin{aligned}\mathcal{H}_\ell := \{&\langle x \mathbin{\&} x' \to \overline{0} \geq \tfrac{1}{m}\rangle \mid x \in \mathsf{var}(\phi)\} \cup \\ &\{\langle\ \to x \mathbin{\&} x' \geq \tfrac{m-1}{m}\rangle \mid x \in \mathsf{var}(\phi)\}.\end{aligned}$$

It is easy to see that any valuation $v\colon \mathcal{V} \longrightarrow \text{Ł}_n$ that satisfies the Horn clause $\langle\ \to x \mathbin{\&} x' \geq \frac{m-1}{m}\rangle$ must be such that $v(x) \geq \frac{m-1}{m}$ and $v(x') \geq \frac{m-1}{m}$, but $\max\{v(x), v(x')\} = 1$. Moreover, if v also satisfies $\langle x \mathbin{\&} x' \to \overline{0} \geq \frac{1}{m}\rangle$, then it must be the case that $\min\{v(x), v(x')\} = \frac{m-1}{m}$. Overall, this means that for every $x \in \mathsf{var}(\phi)$ and for all valuations v satisfying $\mathcal{H}_\ell$, exactly one of x, x' is evaluated by v to 1, while the other is evaluated to $\frac{m-1}{m}$. The intuition of this construction is that we will read $v(x) = 1$ as evaluating the variable x to true, and $v(x) \neq 1$ (and hence $v(x') = 1$) as evaluating x to false.

Consider now the translation ρ that maps literals to variables, defined by

$$\rho(\ell) := \begin{cases} x & \text{if } \ell = x \in \mathcal{V} \\ x' & \text{if } \ell = \neg x, x \in \mathcal{V}. \end{cases}$$

We extend this mapping to m-clauses by setting

$$\rho\Big(\bigvee_{i=1}^{m} \ell_i\Big) := \mathop{\&}_{i=1}^{m} \rho(\ell_i).$$

Observe that a valuation satisfies the Horn clause $\langle\ \to \&_{i=1}^{m} \rho(\ell_i) \geq \frac{1}{m}\rangle$ if and only if at least one of the conjuncts $\rho(\ell_i)$ is evaluated to 1; this will correspond to the literal satisfying the clause $\bigvee_{i=1}^{m} \ell_i$. We thus define the fuzzy Horn theory

$$\mathcal{H}_\phi := \mathcal{H}_\ell \cup \{\langle\ \to \rho(C_i) \geq \tfrac{1}{m}\rangle \mid 1 \leq i \leq k\}.$$

Theorem 3. *The m-CNF formula ϕ is satisfiable iff the fuzzy Horn theory $\mathcal{H}_\phi$ is satisfiable in* Ł_n*-Horn.*

Proof. If ϕ is satisfiable, then there is a propositional valuation V satisfying ϕ. We use V to define a fuzzy valuation $v\colon \mathcal{V} \longrightarrow \text{Ł}_n$ by setting for every $x \in \mathsf{var}(\phi)$

$$v(x) := \begin{cases} 1 & \text{if } V(x) = \mathsf{true} \\ \frac{m-1}{m} & \text{otherwise,} \end{cases}$$
$$v(x') := \frac{2m-1}{m} - v(x).$$

By construction, this valuation satisfies all the Horn clauses in $\mathcal{H}_\ell$. Consider now the Horn clause $\langle \rightarrow \rho(C) \geq \frac{1}{m} \rangle$ for some m-clause $C = \bigvee_{i=1}^{m} \ell_i$ of ϕ. Since V is a model of ϕ, there exists an $i, 1 \leq i \leq m$ such that ℓ_i is evaluated to true by V. By construction, $v(\rho(\ell_i)) = 1$, and hence v satisfies the Horn clause.

Conversely, let v be a model of $\mathcal{H}_\phi$, and consider the valuation V that maps every variable x to true if $v(x) = 1$ and to false if $v(x) = \frac{m-1}{m}$. For any m-clause in ϕ, we know that $v(\rho(C)) \geq \frac{1}{m}$. Thus, there is at least one literal ℓ appearing in C such that $v(\rho(\ell)) = 1$. If ℓ is a propositional variable x, then $v(x) = 1$ and V evaluates x to true; hence V satisfies the clause C. Otherwise, we have that $\ell = \neg x$ for some propositional variable x. In this case, $v(x) = \frac{m-1}{m}$ and V evaluates x to false; i.e., V evaluates $\neg x$ to true, and hence satisfies C. □

Recall that satisfiability of m-CNF formulae is NP-hard for any $m \geq 3$. Using our construction, since $n = m + 1$, we obtain that satisfiability of fuzzy Horn theories in Ł_n-Horn is also NP-hard for any $n \geq 4$, as desired.

Corollary 4. *Satisfiability of fuzzy Horn theories in* Ł_n*-Horn is* NP*-complete for any $n \geq 4$.*

For $n = 4$, we obtain the chain $\text{Ł}_4 = \{0, 1/3, 2/3, 1\}$ containing four membership degrees; thus, our results prove hardness for four-or-more-valued Horn logics. For two-valued Horn logics, which correspond to classical Horn logic, it is well-known that satisfiability is decidable in linear time [18]. Unfortunately, the case of three-valued semantics is not covered by our result.

4 Conclusions

In this paper we have shown that increasing the set of membership degrees over which propositional variables are interpreted may have negative effects on the complexity of reasoning. Specifically, for Horn theories under finite-valued Łukasiewicz semantics, the complexity of deciding satisfiability increases from linear time—for the classical two-valued case—to NP-complete for the four-valued (or higher) case. To the best of our knowledge, the precise complexity of satisfiability in three-valued Horn theories is still unknown, but we conjecture that it is NP-complete in this case as well.

The main motivation for our work is to understand the complexity of reasoning in fuzzy extensions of tractable description logics, such as $\mathcal{EL}$. Horn clauses of the form (1) are expressible in (fuzzy) $\mathcal{EL}$ using fuzzy general concept inclusions like $\langle A_1 \sqcap \cdots \sqcap A_k \sqsubseteq B_1 \sqcap \cdots \sqcap B_m \geq p\rangle$, where A_i, B_j are concept names. Unfortunately, to express the constant $\overline{0}$ in clauses of the form (2), we need the additional constructor $\bot$. Thus, while our hardness results do not transfer directly to the case of Ł_n-$\mathcal{EL}$, our result shows that reasoning in Ł_n-$\mathcal{EL}_\bot$, which is also polynomial in the two-valued case [2], is NP-hard whenever $n \geq 4$.

As future work we plan to continue our task to determine the complexity of reasoning in fuzzy extensions of description logics. Our first goal is to find tight complexity bounds for extensions of $\mathcal{EL}$ with different fuzzy semantics. We are also interested in covering the missing case of Horn theories in Ł_3-Horn.

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