

COMPLEXITY THEORY

Lecture 26: Interactive Proof Systems

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Knowledge-Based Systems

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https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en

Believing without proof?

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Is this a sensible statement?

In other words: Could a rational mathematician be convinced of a formal claim without having the slightest idea of how to prove it?

Motivation: Provers and Verifiers

Recall: languages in NP admit short, easy-to-check membership certificates

NP membership checking as an interaction of two parties:

- The **Prover** produces a certificate (proof of membership) that it claims to be valid
- The **Verifier** validates the certificate to decide upon acceptance

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Can we generalise this idea?

- A (untrusted) Prover tries to convince the Verifier of membership
- Verifier sceptically checks the Prover's arguments before making a decision
- The interaction might involve several rounds of communication
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For which languages can such a polytime Verifier ensure that it can be convinced of membership exactly for the words that really are in the language?

Example: Graph Isomorphism

We consider (undirected) graphs over a set of numbered vertices $1, 2, \dots, n$.

Two graphs are **isomorphic** if one can be obtained from the other by a bijective renaming (permutation) of vertices.

GRAPH ISOMORPHISM

Input: Two graphs G_1 and G_2 .

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However, **GRAPH ISOMORPHISM** is not known (or believed) to be NP-hard

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Protocol: given non-isomorphic graphs G_1 and G_2

- Verifier: randomly select $i \in \{1, 2\}$; randomly permute vertices of G_i to obtain a new graph H ; send H to the Prover
- Prover: determine which G_j ($j \in \{1, 2\}$) the graph H is isomorphic to; send j
- Verifier: accept if $i = j$, else reject

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Analysis: The Prover can ensure acceptance for non-isomorphic graphs, but for isomorphic graphs it can only achieve acceptance with probability 0.5 (which can be reduced further by repeating the interaction several times) □

Zero-Knowledge Proofs

Running the previous protocol is interestingly uninformative:

- The Verifier can be convinced that $\langle G_1, G_2 \rangle \in \mathbf{GRAPH\ NON-ISOMORPHISM}$
- But the Verifier learns nothing about the reasons
- In particular, the Verifier would not be able to prove this to anybody else

This is called a **zero-knowledge proof**.

Note: The mathematical property that characterises such proofs formally is that the Verifier could have produced the whole interaction all by itself, without the assistance of a Prover. This would not convince the Verifier, of course, but would not be distinguishable otherwise.

Making interactive proofs formal (1)

The interaction can be viewed as a sequence of messages m_1, m_2, \dots, m_k , followed by the Verifier declaring “accept” or “reject”.

The **Verifier** may consider the following:

- The **input string** w
- A string r of **random bits** (certificate-style view of random computation)
- A (partial) **message history** $m_1\#m_2\#\dots\#m_i$ of messages exchanged so far (odd-index messages are sent by Verifier, even-index messages by Prover)

\rightsquigarrow Verifier can be described by a function $V : \Sigma^* \times \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \cup \{\text{accept, reject}\}$

The **Prover** may consider the following:

- The **input string** w
- A (partial) **message history** $m_1\#m_2\#\dots\#m_i$ of messages exchanged so far

\rightsquigarrow Prover can be described by a function $P : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$

Making interactive proofs formal (2)

Definition 26.1: A word w is **accepted by V and P with random string r** if there is a sequence of messages $m_1\#m_2\#\dots\#m_k$ such that

- for all even $i \geq 0$ we have $m_{i+1} = V(w, r, m_1\#\dots\#m_i)$
- for all odd $i \geq 0$ we have $m_{i+1} = P(w, m_1\#\dots\#m_i)$
- $m_k = \text{accept}$ (and in particular k is odd)

In this case, we write $(V \leftrightarrow P)(w, r) = \text{accept}$.

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Definition 26.2: A **polynomial verifier** V with bound p is a verifier function that ensures that, for all inputs w , random strings r , and provers P , at most $p(|w|)$ computation steps are performed overall (across all interactions).

Note: Polynomial verifiers could, for example, use messages to store the number of available steps that remain, and reject when this is used up.

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Note: Polynomial verifiers could, for example, use messages to store the number of available steps that remain, and reject when this is used up.

Definition 26.3: A polynomial verifier V with bound p and a prover P **accept a word w with probability** $\Pr[V \leftrightarrow P \text{ accepts } w] = \Pr_{r \in \{0,1\}^{p(|w|)}}[(V \leftrightarrow P)(w, r) = \text{accept}]$.

The class IP

We can now formally define a class of languages that are accepted by polytime Verifiers using interactive proofs:

Definition 26.4: A language \mathbf{L} is in **IP** if there is a polynomial verifier V such that, for every word w :

- (1) if $w \in \mathbf{L}$ then there is a prover P with $\Pr[V \leftrightarrow P \text{ accepts } w] \geq \frac{2}{3}$,
- (2) if $w \notin \mathbf{L}$ then for all provers \tilde{P} we have $\Pr[V \leftrightarrow \tilde{P} \text{ accepts } w] \leq \frac{1}{3}$.

In words:

- there is a “good” prover P that can convince V to accept $w \in \mathbf{L}$ with high probability
(note that the existence of one good prover for each $w \in \mathbf{L}$ implies that there is one globally good prover)
- not even a “bad” prover \tilde{P} can convince V to accept words $w \notin \mathbf{L}$ with more than a low probability

Probabilistic interactions

The definition of IP uses probabilistic computations

- The Verifier is a polynomially time-bounded probabilistic TM
- The Prover does not use randomness (and including it would not change IP)
- As discussed for BPP, we can amplify probabilities; in particular, the bounds $\frac{2}{3}$ and $\frac{1}{3}$ are not essential to the definition

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The use of randomness in the Verifier is important for expressive power:

Theorem 26.5: Let IP_d be the restriction of IP that is obtained when requiring V to be deterministic (ignoring the random bits). Then $IP_d = NP$.

The proof is not hard (exercise; or see Arora & Barak, Lemma 8.4)

Obvious sub-classes of IP

Some observations are straightforward:

Theorem 26.6: $NP \subseteq IP$.

Proof: Use definition of NP via polynomial-time verifiers. □

Theorem 26.7: $BPP \subseteq IP$.

Proof: Verifier can solve BPP problems without talking to Prover. □

A superclass of IP

Interestingly, we can use another well-known class to capture IP from above:

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Proof: Consider $L \in IP$ with polynomial verifier V . For any word w , let

$$\Pr[V \text{ accepts } w] = \max_P \Pr[V \leftrightarrow P \text{ accepts } w].$$

Then $\Pr[V \text{ accepts } w] \geq \frac{2}{3}$ if $w \in L$ and $\Pr[V \text{ accepts } w] \leq \frac{1}{3}$ otherwise.

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Notation:

- Let M_j abbreviate a message sequence $m_1 \# m_2 \# \dots \# m_j$
- $(V \leftrightarrow P)(w, r, M_j) = \text{accept}$ if $(V \leftrightarrow P)(w, r) = \text{accept}$ for a message sequence $m_1 \# m_2 \# \dots \# m_k$ that extends M_j (in particular: M_j is possible with r , V and P)
- $\Pr[(V \leftrightarrow P) \text{ accepts } w \text{ starting from } M_j] = \Pr_{r \in \{0,1\}^{p(|w|)}}[(V \leftrightarrow P)(w, r, M_j) = \text{accept}]$
- $\Pr[V \text{ accepts } w \text{ starting from } M_j] = \max_P \Pr[(V \leftrightarrow P) \text{ accepts } w \text{ starting from } M_j]$

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Theorem 26.8: $IP \subseteq PSpace$.

Proof (cont.): What we seek is $\Pr [V \text{ accepts } w] = \Pr [V \text{ accepts } w \text{ starting from } M_0]$, where M_0 is the empty message sequence.

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We define numbers $N[M_j]$ recursively, with the longest possible sequences as base case:

1. If M_j is cannot be produced by V for any r (and P), then $N[M_j] = 0$.
2. Else, if j is odd and $M_j = m_1 \# \dots \# m_j$, then
 - 2.1 If $m_j = \text{accept}$ then $N[M_j] = 1$
 - 2.2 If $m_j = \text{reject}$ then $N[M_j] = 0$
 - 2.3 If $m_j \notin \{\text{accept, reject}\}$ then $N[M_j] = \max_{m_{j+1}} N[M_j \# m_{j+1}]$
3. Else, if j is even, then $N[M_j] = \text{wt-avg}_{m_{j+1}} N[M_j \# m_{j+1}]$
where $\text{wt-avg}_{m_{j+1}} N[M_j \# m_{j+1}] = \sum_{m_{j+1}} \Pr_{r \in \{0,1\}^{p(|w|)}} [V(w, r, M_j) = m_{j+1}] \cdot N[M_j \# m_{j+1}]$

In all cases, m_{j+1} ranges over (a superset of) the messages possible at this step (which can be assumed to be of polynomial length, and are therefore bounded).

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Note 1: Case 2.3 corresponds to best possible answer of any Prover

Note 2: Case 3 corresponds to probability-weighted average for given Verifier

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Proof (cont.): We need to show two claims:

- **Claim 1:** $N[M_j] = \Pr[V \text{ accepts } w \text{ starting from } M_j]$
- **Claim 2:** $N[M_j]$ can be computed in polynomial space

Together, this would show that $N[M_0] = \Pr[V \text{ accepts } w]$ can be computed in polynomial space.

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Claim 2 is not hard to see:

- The recursive computation of $N[M_j]$ is of polynomially bounded depth (longer message sequences are never consistent with a polynomial verifier V)
- Checking consistency with some $r \in \{0, 1\}^{p(|w|)}$ can be done by iterating over these r
- Computing $\max_{m_{j+1}} N[M_j \# m_{j+1}]$ is similar by iterating over all m_{j+1}
- Computing $\text{wt-avg}_{m_{j+1}} N[M_j \# m_{j+1}]$ is also similar, using two iterations (over m_{j+1} and over all r)

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Proof (cont.): Claim 1 can be shown by induction.

The **base cases** are when M_j is not consistent (impossible message sequence), or already ends in accept or reject. The claim is clear for these cases.

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For the **induction step**, assume the claim holds for all $N[M_{j+1}]$ (ind. hypothesis, IH)

- For the case $N[M_j] = \text{wt-avg}_{m_{j+1}} N[M_j \# m_{j+1}]$, we compute:

$$\begin{aligned} N[M_j] &\stackrel{\text{def}}{=} \sum_{m_{j+1}} \Pr_{r \in \{0,1\}^{p(|w|)}} [V(w, r, M_j) = m_{j+1}] \cdot N[M_j \# m_{j+1}] \\ &\stackrel{\text{IH}}{=} \sum_{m_{j+1}} \Pr_{r \in \{0,1\}^{p(|w|)}} [V(w, r, M_j) = m_{j+1}] \cdot \Pr [V \text{ accepts } w \text{ starting from } M_j \# m_{j+1}] \\ &= \Pr [V \text{ accepts } w \text{ starting from } M_j] \quad (\text{def. of acceptance probability for steps of } V) \end{aligned}$$

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The second equality follows since this probability can be achieved by a Prover that sends the message m_{j+1} that maximises $N[M + j]$, and no higher probability can be achieved by any other message.

This finishes the proof of the theorem.

□

The power of the prover

Our definition of IP allows Prover to have unlimited computational power (possibly even uncomputable behaviour).

However, our proof of $IP \subseteq PSpace$ showed that the optimal Prover output for any given Verifier can be computed in polynomial space, so we get:

Corollary 26.9: The class IP remains the same if the Prover is required to compute its responses in polynomial space.

The power of IP

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As we will see, IP can do much more. We start with the following problem:

#SAT

Input: A propositional logic formula φ .

Problem: The number of satisfying assignments of φ

Note:

- **#SAT** is not a decision problem. Let $\mathbf{\#SAT}_D = \{\langle \varphi, k \rangle \mid k \text{ is the solution of } \mathbf{\#SAT} \text{ on } \varphi\}$ be the corresponding decision problem
- Computing **#SAT** solves propositional satisfiability as well as unsatisfiability.
- Indeed, it is complete for the powerful class #P

Solving **#SAT**_D in IP

Theorem 26.10: **#SAT**_D ∈ IP

We consider a formula φ of size n and with m propositional variables x_1, \dots, x_m .

For $1 \leq i \leq m$, let $f_i : \{0, 1\}^i \rightarrow \mathbb{N}$ be the function that maps $\langle a_1, \dots, a_m \rangle$ to the number of satisfying assignments of φ with $x_1 = a_1, \dots, x_i = a_i$.

- Then $f_0()$ is the solution to **#SAT**
- We find $f_i(a_1, \dots, a_i) = f_{i+1}(a_1, \dots, a_i, 0) + f_{i+1}(a_1, \dots, a_i, 1)$

#SAT_D ∈ IP: first attempt

Protocol: to check if $\langle \varphi, k \rangle \in \text{\#SAT}_D$

- P : send $f_0()$ to V
- V : check if $f_0() = k$ and reject if this fails
- For $i = 1, \dots, m$:
 - P : send $f_i(a_1, \dots, a_i)$ to V for all $\langle a_1, \dots, a_i \rangle \in \{0, 1\}^i$
 - V : check, for all $\vec{a} \in \{0, 1\}^{i-1}$, if $f_{i-1}(\vec{a}) = f_i(\vec{a}, 0) + f_i(\vec{a}, 1)$, reject if not
- V : check if, for all $\langle a_1, \dots, a_m \rangle \in \{0, 1\}^m$, $f_m(a_1, \dots, a_m) = 1$ if and only if $\{x_1 \mapsto a_1, \dots, x_m \mapsto a_m\}$ is a satisfying assignment for φ ; accept iff

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This protocol does not show $\text{\#SAT}_D \in \text{IP}$:

- it requires exponential time to perform exponentially many checks.

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This protocol does not show $\#SAT_D \in IP$:

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However, the protocol is otherwise correct:

- if k is the correct result, a truthful Prover can convince the Verifier
- if k is not correct, not even a mischievous Prover can convince the verifier (exercise: why?)

Arithmetisation

To reduce the number of messages and checks, we use [arithmetisation](#).

φ is transformed into an arithmetic expression Φ by replacing subexpressions:

- $\alpha \wedge \beta$ becomes $\alpha\beta$
- $\neg\alpha$ becomes $(1 - \alpha)$
- $\alpha \vee \beta$ becomes $\alpha * \beta = 1 - (1 - \alpha)(1 - \beta)$

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Some observations:

- Φ is a multivariate polynomial function over variables x_1, \dots, x_m
- The degree of Φ is bounded by the size n of φ
- The value of Φ for inputs $x_i \in \{0, 1\}$ is also in $\{0, 1\}$, and corresponds to the valuation of φ on the corresponding truth values
- We can evaluate Φ over an arbitrary field

Example 26.11: For a prime number p , the algebra of natural numbers $\{0, 1, \dots, p - 1\}$ and where $+$ and \cdot are addition and multiplication modulo p is a finite field. This field is denoted $\text{GF}(p)$.

#SAT_D ∈ IP

Theorem 26.10: #SAT_D ∈ IP

Proof: By our prior observation, k is a solution to #SAT exactly if

$$k = \sum_{a_1 \in \{0,1\}} \dots \sum_{a_m \in \{0,1\}} \Phi(a_1, \dots, a_m) \quad (1)$$

The Prover tries to convince the Verifier of this.

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Initialisation:

The Prover sends a prime number p with $2^n < p \leq 2^{2n}$ (n : size of φ). All calculations will be performed in $\text{GF}(p)$.

Note: The right side of (1) is at most $2^m \leq 2^n$, so the value is unaffected by this restriction.

The Verifier checks that p is really prime (primality is known to be in P)

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Proof (cont.): Given a multi-variate polynomial $g(x_1, \dots, x_\ell)$, let $h(x_1)$ denote the (univariate) polynomial $\sum_{a_2 \in \{0,1\}} \dots \sum_{a_m \in \{0,1\}} g(x_1, a_2, \dots, a_\ell)$.

Protocol: to check $K = \sum_{a_1 \in \{0,1\}} \dots \sum_{a_m \in \{0,1\}} g(a_1, \dots, a_m) \pmod p$ for multi-variate polynomial g that has a polynomial-size representation and polynomial degree

- V : if $m = 1$, verify $g(0) + g(1) = K$ and reject or accept accordingly;
if $m \geq 2$, ask P to send a polynomial-size representation of $h(x_1)$
- P : send a polynomial $\tilde{h}(x_1)$ (if P is truthful, it sends $\tilde{h} = h$)
- V : check if \tilde{h} is polynomially sized and of degree $\leq n$;
check if $K = \tilde{h}(0) + \tilde{h}(1)$; reject if any of these fail;
pick a random $b \in \text{GF}(p)$ and send b to P
- Recursively use the same protocol to verify
 $\tilde{h}(b) = \sum_{a_2 \in \{0,1\}} \dots \sum_{a_m \in \{0,1\}} g(b, a_2, \dots, a_m) \pmod p$

#SAT_D ∈ IP

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Proof (cont.): It is not hard to verify that the protocol can be implemented by a polynomial verifier:

- All polynomials are given by polynomial representations and have polynomial degree
- They can therefore be evaluated in polynomial time (using binary encoding of numbers)
- The random number $b \leq p \leq 2^{2n}$ consists of $2n$ random bits
- There are $\leq m$ recursive applications of the protocol

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Proof (cont.): If the claim is true, a truthful Prover can ensure that V accepts.

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Proof (cont.): If the claim is true, a truthful Prover can ensure that V accepts.

If the claim is false, the probability that V accepts is very small:

- For $m = 1$, the probability is 0 (V will just check directly)
- For $m > 1$, P must send some $\tilde{h} \neq h$ in order to pass the check $K = \tilde{h}(0) + \tilde{h}(1)$
If V selects b such that $\tilde{h}(b) = \sum_{a_2 \in \{0,1\}} \cdots \sum_{a_m \in \{0,1\}} g(b, a_2, \dots, a_m) \pmod p$, then P can continue to play truthfully and V will eventually accept
- Overall, there are $m - 1$ opportunities for P to be lucky in this sense.
- But if $\tilde{h} \neq h$, then the chance of a random $b \in \{0, \dots, p\} \supseteq \{0, \dots, 2^m\}$ to be such that $\tilde{h}(b) - h(b) = 0$ is $\leq d/2^n$, where d is the degree of $\tilde{h} - h$ (Schwartz-Zippel Lemma).
- The degree of h and any reasonable \tilde{h} is bounded by the size n of φ (linear), while 2^n is exponential, hence the success rate is small for sufficiently large φ .
- The overall chance of P tricking V to accept a wrong claim is $\leq 1 - (1 - n/2^n)^{m-1}$, which is $\leq 1/n$ for $n \geq 10$. □

Main result

The main insight about IP is as follows:

Theorem 26.12: $IP = PSpace$

Proof: We have already shown $IP \subseteq PSpace$. For the converse, we adopt our proof of $\#SAT_D \in IP$ to show that **TRUEQBF** $\in IP$. This suffices (why?).

Main result

The main insight about IP is as follows:

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Consider a QBF of the form $\psi = \forall x_1. \exists x_2. \forall x_3. \dots \exists x_m. \varphi[x_1, \dots, x_n]$ (this is w.l.o.g. – why?).

Using the arithmetisation Ψ of φ , we find $\psi \in \mathbf{TRUEQBF}$ iff

$$\sum_{a_1 \in \{0,1\}}^* \prod_{a_2 \in \{0,1\}} \sum_{a_3 \in \{0,1\}}^* \dots \sum_{a_m \in \{0,1\}}^* \Phi(a_1, \dots, a_m) = 1 \quad (2)$$

where $\sum_{a \in \{0,1\}}^* P(a) = P(0) * P(1) = 1 - (1 - P(0))(1 - P(1))$.

We would like to verify (2) using similar ideas as for $\#SAT_D \in IP$.

Showing $IP = PSpace$

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Problem: The degree of polynomials such as

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\leadsto no polynomial-size description, no polytime evaluation

Solution: Reduce degrees of all relevant polynomials in a way that preserves truth values

- Idea: if $x \in \{0, 1\}$, then $x^d = x$ and $P(x) = xP(1) + (1 - x)P(0)$
- We define an operator R with $Rx.P(x) = xP(1) + (1 - x)P(0)$
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We redefine the polynomial we want evaluate as follows:

$$\exists x_1.Rx_1 \forall x_2.Rx_1.Rx_2. \exists x_3.Rx_1.Rx_2.Rx_3. \cdots \exists x_m.Rx_1. \cdots Rx_m. \Phi(x_1, \dots, x_m)$$

where $\exists x.P(x) = P(0) * P(1)$ and $\forall x.P(x) = P(0) \cdot P(1)$.

Note: This expression is of quadratic size compared to ψ .

Showing $IP = PSpace$

We write $\exists x_1.Rx_1 \forall x_2.Rx_1.Rx_2.\exists x_3.Rx_1.Rx_2.Rx_3.\dots \exists x_m.Rx_1.\dots Rx_m.\Phi(x_1, \dots, x_m)$ as $O_1y_1.O_2y_2.\dots.O_ky_k.\Phi(x_1, \dots, x_m)$, where $O_i \in \{\exists, \forall, R\}$ and $y_i \in \{x_1, \dots, x_m\}$.

Verifier picks a prime $p > n^4$ (for n the size of ψ); we calculate in $GF(p)$.

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Verifier picks a prime $p > n^4$ (for n the size of ψ); we calculate in $\text{GF}(p)$.

Protocol: to check $K = O_1y_1.O_2y_2.\dots.O_ky_k.g(b_1, \dots, b_\ell) \pmod p$ where $O_1y_1.O_2y_2.\dots.O_ky_k.g$ is a polynomial in ℓ variables that has a polynomial-size representation and polynomial degree

- V : if $k = 0$, verify $g(b_1, \dots, b_\ell) = K$ and reject or accept accordingly; else, ask P for a representation of $O_2y_2.\dots.O_ky_k.g(b_1, \dots, b_\ell)[y_1 \mapsto \text{undef}]$
- P : send a polynomial $\tilde{h}(y_1)$
- V : check if \tilde{h} is polynomially sized and of degree $\leq m$; check if $K = O_1y_1.\tilde{h}(y_1)$; reject if any of these fail; pick a random $b \in \text{GF}(p)$ and send b to P
- Recursively use the same protocol to verify $\tilde{h}(b) = O_2y_2.\dots.O_ky_k.g(b_1, \dots, b_\ell)[y_1 \mapsto b] \pmod p$

Explanations

The following notes may help to understand the protocol.

- The function $O_1 y_1 \cdots .O_k y_k .g$ is a function on variables x_1, \dots, x_ℓ
 - Variables x_i ($i > \ell$) are bound by \exists or \forall , hence eliminated
 - Variables x_i ($i \leq \ell$) may still occur in R operators, but they do not remove them
- $O_2 y_2 \cdots .O_k y_k .g$ is a function on variables $x_1, \dots, x_\ell, x_{\ell+1}$ if $O_1 \in \{\exists, \forall\}$
- $O_2 y_2 \cdots .O_k y_k .g$ is a function on variables x_1, \dots, x_ℓ if $O_1 = R$
- $O_2 y_2 \cdots .O_k y_k .g(b_1, \dots, b_\ell)[y_1 \mapsto \text{undef}]$ denotes the function over y_1 obtained by ignoring the binding b_i for $y_1 = x_i$ (only relevant if $O_1 = R$)
- $O_2 y_2 \cdots .O_k y_k .g(b_1, \dots, b_\ell)[y_1 \mapsto b]$ denotes the function over y_1 obtained by redefining the binding b_i for $y_1 = x_i$ to be b (only relevant if $O_1 = R$)
- The check $K = O_1 y_1 .\tilde{h}(y_1)$ is evaluated as required for O_1

Finishing the proof

Theorem 26.12: $IP = PSpace$

Proof: Summary of approach:

- The problem is arithmetised and extended with degree-reduction operators
- A prime $p > n^4$ is chosen to define a field $GF(p)$ for calculations
- A protocol is followed to verify the arithmetisation yields 1 =

As in the case of $\#SAT_D$, the Prover's chances of fooling the Verifier are small:

- Wrong claims require to send wrong polynomials $\tilde{h}(y_1)$
- It is unlikely that V picks a random value b on which $\tilde{h}(p)$ agrees with the correct function's value ($p > n^4$ suffices here since the degree of the functions are small)

This finishes the proof. □

Summary and Outlook

Interactive proofs enable probabilistic machines to solve problems beyond NP

GRAPH NON-ISOMORPHISM has an interesting interactive zero-knowledge proof protocol

$IP = PSpace$

What's next?

- Summary & consultation
- Examinations