Gödel Description Logics with General Models*

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Abstract. In the last few years, the complexity of reasoning in fuzzy description logics has been studied in depth. Surprisingly, despite being arguably the simplest form of fuzzy semantics, not much is known about the complexity of reasoning in fuzzy description logics using the Gödel t-norm. It was recently shown that in the logic G-$\mathfrak{I}\mathcal{ALC}$ under witnessed model semantics, all standard reasoning problems can be solved in exponential time, matching the complexity of reasoning in classical $\mathcal{ALC}$. We show that this also holds under general model semantics.

1 Introduction

Fuzzy Description Logics (DLs) have been studied as a means of representing vague or imprecise knowledge in a formal and well-understood manner. In contrast to classical DLs, the semantics of fuzzy DLs are based on *fuzzy sets*. Fuzzy sets associate every element of the domain with a number from the interval $[0, 1]$, which represents the *degree* to which the element belongs to the fuzzy set.

When defining a fuzzy DL, one must decide how to interpret the logical constructors to handle the truth degrees. The simplest approach is to use the *minimum* operator for conjunctions to generalize intersection to fuzzy sets. Thus, the degree of membership of a conjunction is interpreted as the minimum of the membership degrees of the conjuncts. This operation, called the *Gödel t-norm*, can be used to interpret all other logical constructors in a formally justified manner [19,22]. The quantifiers $\forall$ and $\exists$ are interpreted as infima and suprema of truth values, respectively. To avoid problems with infinitely many truth values, reasoning in fuzzy DLs is often restricted to so-called *witnessed models* [21].

The study of fuzzy DLs underwent a large change in recent years, after some relatively inexpressive fuzzy DLs were shown to be undecidable when reasoning w.r.t. general ontologies [3,4,16]. Since then, the limits of decidability have been explored, yielding very expressive decidable logics on the one hand [10], and inexpressive undecidable logics on the other [1,13]. All existing approaches for reasoning in fuzzy DLs depend on limiting models to finitely many truth degrees. For these approaches to work, one must either (i) restrict the semantics to a finite set of truth degrees [6,7,8,9,14,15,28]; (ii) prove that reasoning can be restricted

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to a finite set of degrees [5,10,27]; or (iii) prove that models can be built from a finite pattern [26,29]. In all three cases, the proofs of correctness of these algorithms imply the *finitely-valued model property*: an ontology has a model iff it has a model using only finitely many truth values. Conversely, the proofs of undecidability [3,4,13,16] are based on the construction of a model that uses infinitely many truth degrees. Thus, the finitely-valued model property seems to be a good indicator of the decidability of a fuzzy DL.

Despite being widely regarded as the simplest t-norm, surprisingly little is known about fuzzy DLs based on Gödel semantics. It was generally believed that these logics are decidable, but no proof existed to support this claim. The only results for similar logics restrict reasoning *a priori* to a finite subset of $[0,1]$; in this case, a reduction to classical reasoning yields decidability [6,7]. The fuzzy DL $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ does not have the finitely-valued model property; neither under witnessed model semantics [12] nor w.r.t. general models [20]. Despite this, all standard reasoning problems in this logic have recently been shown to be decidable (ExpTime-complete) when considering only witnessed models [12].

In this paper, we extend the analysis of reasoning in $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ to the case of general models, adapt the automata-based algorithm from [12] to deal with this slightly more difficult semantics, and show that all reasoning problems remain ExpTime-complete. The main idea is that under Gödel semantics, one only needs to know an ordering between the relevant truth degrees, rather than the precise values they take. This idea has already been used for deciding validity of formulae in propositional Gödel logic [18]. In [23], a tableaux algorithm using a similar approach has been used to reason in a fuzzy DL under Zadeh semantics that can additionally express order relations between arbitrary concepts. The main difference of the algorithm in this paper to that of [12] lies in the treatment of existential and value restrictions in Definition 5 and Propositions 6 and 7.

2 Preliminaries

We briefly introduce the basic notions of $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ and order structures. The two basic operators of Gödel fuzzy logic are conjunction and implication, interpreted by the *Gödel t-norm* and its *residuum*. The Gödel t-norm is the binary function $\min(x,y)$ on [0,1]; its residuum $\Rightarrow$ is uniquely defined by the equivalence $\min(x,y) \leq z$ iff $y \leq (x \Rightarrow z)$ for all $x,y,z \in [0,1]$, and is computed as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

For a deeper introduction to t-norm-based fuzzy logics, see [17,19,22].

A *total preorder* over a set S is a transitive and total binary relation $\lesssim_*$ on S. For $x,y \in S$, we write $x \equiv_* y$ if $x \lesssim_* y$ and $y \lesssim_* x$. Notice that $\equiv_*$ is an equivalence relation on S. Similarly, we write $x <_* y$ if $x \lesssim_* y$, but not $y \lesssim_* x$. We write $\bowtie$ for an arbitrary element of $\{=,\geq,>,\leq,<\}$, and $\bowtie_*$ for the corresponding relation induced by $\lesssim_*$, i.e. $\equiv_*$, $\gtrsim_*$, $>_*$, $\lesssim_*$, or $<_*$. Subscripts are used to distinguish these relations for different total preorders.

Table 1. Semantics of $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$

Constructor	Syntax	Semantics
top concept	$\top$	1
involutive negation	$\neg C$	$1 - C^{\mathcal{I}}(x)$
conjunction	$C \sqcap D$	$\min(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$
implication	$C \to D$	$C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
existential restriction	$\exists r.C$	$\sup_{y\in\Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y))$
value restriction	$\forall r.C$	$\inf_{y\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$

An *order structure* S is a finite set containing the numbers 0, 0.5, and 1, together with an involutive unary operation $\mathrm{inv}\colon S \to S$ such that $\mathrm{inv}(x) = 1-x$ for all $x \in S \cap [0,1]$. For an order structure S, $\mathrm{order}(S)$ denotes the set of all total preorders $\lesssim_*$ over S that have 0 and 1 as least and greatest element, respectively, preserve the order of real numbers on $S \cap [0,1]$, and satisfy $x \lesssim_* y$ iff $\mathrm{inv}(y) \lesssim_* \mathrm{inv}(x)$ for all $x, y \in S$. Given $\lesssim_* \in \mathrm{order}(S)$, the following functions on S are well-defined since $\lesssim_*$ is total:

$$\min{}_*(x,y) := \begin{cases} x & \text{if } x \lesssim_* y \\ y & \text{otherwise} \end{cases} \qquad \mathrm{res}_*(x,y) := \begin{cases} 1 & \text{if } x \lesssim_* y \\ y & \text{otherwise} \end{cases}$$

It is easy to see that these operators agree with min and $\Rightarrow$ on the set $S \cap [0,1]$.

Let $\mathsf{N_I}$, $\mathsf{N_R}$, and $\mathsf{N_C}$ be sets of *individual*, *role*, and *concept names*, respectively. $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ *concepts* are built as follows, where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$:

$$C ::= A \mid \top \mid \neg C \mid C \sqcap C \mid C \to C \mid \exists r.C \mid \forall r.C.$$

We call concepts of the form $\exists r.C$ or $\forall r.C$ *quantified concepts*. An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and $\cdot^{\mathcal{I}}$ maps every $a \in \mathsf{N_I}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, every $A \in \mathsf{N_C}$ to a fuzzy set $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$, and every $r \in \mathsf{N_R}$ to a fuzzy binary relation $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. The interpretation of complex concepts is shown in Table 1. In $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$, one can simulate the additional constructors bottom, residual negation, and disjunction, by using $\top$, $\neg$, $\sqcap$, and $\to$. The knowledge of a domain is represented using axioms that restrict the class of interpretations that are relevant for the different reasoning tasks.

Definition 1 (axioms). *A* crisp assertion *is either a* concept assertion $a{:}C$ *or a* role assertion $(a,b){:}r$ *for a concept C, $r \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. An* (order) assertion *is of the form $\langle \alpha \bowtie \beta \rangle$, where α is a crisp assertion and β is either a crisp assertion or a value from $[0,1]$. An interpretation $\mathcal{I}$* satisfies *an order assertion $\langle \alpha \bowtie \beta \rangle$ if $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$, where $(a{:}C)^{\mathcal{I}} := C^{\mathcal{I}}(a^{\mathcal{I}})$, $((a,b){:}r)^{\mathcal{I}} := r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $q^{\mathcal{I}} := q$ for all $q \in [0,1]$. An* ordered ABox $\mathcal{A}$ *is a finite set of order assertions. An interpretation is a* model *of $\mathcal{A}$ if it satisfies all order assertions in $\mathcal{A}$.*

A general concept inclusion (GCI) *is an expression of the form $\langle C \sqsubseteq D \geq q \rangle$ for concepts C, D, and $q \in [0,1]$. An interpretation $\mathcal{I}$ satisfies this GCI if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq q$ holds for all $x \in \Delta^{\mathcal{I}}$. A* TBox *is a finite set of GCIs.*

An ontology *is a pair* $\mathcal{O} = (\mathcal{A}, \mathcal{T})$*, where* $\mathcal{A}$ *is an ordered ABox and* $\mathcal{T}$ *is a TBox. An interpretation is a* model *of a TBox* $\mathcal{T}$ *if it satisfies all GCIs in* $\mathcal{T}$*, and it is a* model *of an ontology* $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ *if it is a model of both* $\mathcal{A}$ *and* $\mathcal{T}$*.*

Ordered ABoxes are more expressive than ABoxes usually considered in fuzzy DLs [27] since they allow to state order relations between concepts and roles. This more general kind of ABox is better suited for our algorithms.

We denote by $\mathsf{sub}(\mathcal{O})$ the closure under negation of the set of all subconcepts appearing in an ontology $\mathcal{O}$. The concepts $\neg\neg C$ and C are equivalent, and we regard them here as equal; thus, $\mathsf{sub}(\mathcal{O})$ is finite. By $\mathcal{V}_\mathcal{O}$ we denote the closure under the operator $x \mapsto 1 - x$ of the set of all truth degrees appearing in $\mathcal{O}$, together with 0, 0.5, and 1. Since this operator is involutive, $\mathcal{V}_\mathcal{O}$ is also finite. We denote the elements of $\mathcal{V}_\mathcal{O} \subseteq [0, 1]$ as $0 = q_0 < q_1 < \cdots < q_k = 1$.

As with classical DLs, the most basic reasoning task in $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ is to decide ontology consistency. One may also be interested in computing the degree to which an entailment holds.

Definition 2 (reasoning). *An ontology* $\mathcal{O}$ *is* consistent *if it has a model. Given* $p \in [0, 1]$*, a concept* C *is* p-satisfiable *w.r.t.* $\mathcal{O}$ *if there is a model* $\mathcal{I}$ *of* $\mathcal{O}$ *and an* $x \in \Delta^\mathcal{I}$ *with* $C^\mathcal{I}(x) \geq p$*. The* best satisfiability degree *of* C *w.r.t.* $\mathcal{O}$ *is the supremum over all* p *such that* C *is* p*-satisfiable w.r.t.* $\mathcal{O}$*.* C *is* p-subsumed *by a concept* D *w.r.t.* $\mathcal{O}$ *if all models of* $\mathcal{O}$ *satisfy the GCI* $\langle C \sqsubseteq D \geq p\rangle$*. The* best subsumption degree *of* C *and* D *w.r.t.* $\mathcal{O}$ *is the supremum over all* p *such that* C *is* p*-subsumed by* D *w.r.t.* $\mathcal{O}$*.*

In this paper, we consider only the consistency problem for ontologies $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ where $\mathcal{A}$ is a *local* ordered ABox, i.e. it contains no role assertions and uses only a single individual name a. For all other reasoning problems, one can use exactly the same reductions to local consistency as in [12].

3 Deciding Local Consistency

Let $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ be an ontology with a local ordered ABox using the individual name a. Our algorithm is based on the observation that the axioms and the semantics of the constructors only introduce restrictions on the *order* of the values that models can assign to concepts, not on the values themselves. For example, an interpretation $\mathcal{I}$ satisfies $\langle a\!: (A \to B) = p\rangle$ with $p < 1$ iff $A^\mathcal{I}(a^\mathcal{I}) > B^\mathcal{I}(a^\mathcal{I})$ and $B^\mathcal{I}(a^\mathcal{I}) = p$. Thus, rather than building a model directly, we first create an abstract representation of a model that encodes only the order between concepts. This will be achieved through an order structure on $\mathcal{V}_\mathcal{O} \cup \mathsf{sub}(\mathcal{O})$.

As in classical $\mathcal{ALC}$, it suffices to consider *tree-shaped* models. Since the values of concepts in a node of this tree are also restricted by the values of concepts at the parent node, we additionally introduce expressions of the form $C_\uparrow$ to refer to the value of C at the parent node. We additionally use a new element λ to represent the degree of the role connection from the parent node.

Definition 3 (order structure $\mathcal{U}$). *We define* $\mathsf{sub}_{\uparrow}(\mathcal{O}) := \{C_{\uparrow} \mid C \in \mathsf{sub}(\mathcal{O})\}$ *and the order structure* $\mathcal{U} := \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ *with* $\mathrm{inv}(\lambda) := \neg\lambda$, $\mathrm{inv}(C) := \neg C$, *and* $\mathrm{inv}(C_{\uparrow}) := (\neg C)_{\uparrow}$ *for all* $C \in \mathsf{sub}(\mathcal{O})$.

For convenience, we extend the notation of $\mathsf{sub}_{\uparrow}(\mathcal{O})$ by setting $q_{\uparrow} := q$ for $q \in \mathcal{V}_{\mathcal{O}}$.

Using total preorders from $\mathrm{order}(\mathcal{U})$, we can describe the relationships between all the subconcepts from $\mathcal{O}$ and the truth degrees from $\mathcal{V}_{\mathcal{O}}$ at given domain elements. Such a preorder can be seen as the *type* of a domain element, from which a tree-shaped interpretation, represented by a *Hintikka tree*, can be built.

In the following, let n be the number of quantified concepts in $\mathsf{sub}(\mathcal{O})$ and ϕ a fixed bijection between the set of all quantified concepts in $\mathsf{sub}(\mathcal{O})$ and $\{1, \ldots, n\}$ that specifies which quantified concept is satisfied by which successor in the Hintikka tree. For a given role $r \in \mathsf{N_R}$, we denote by Φ_r the set of all indices $\phi(E)$ where $E \in \mathsf{sub}(\mathcal{O})$ is a quantified concept of the form $\exists r.C$ or $\forall r.C$.

Definition 4 (Hintikka ordering). *A* Hintikka ordering *is a total preorder* $\lesssim_H \in \mathrm{order}(\mathcal{U})$ *that satisfies the following conditions for every* $C \in \mathsf{sub}(\mathcal{O})$*:*

- $C = \top$ *implies* $C \equiv_H 1$,
- *if* $C = D_1 \sqcap D_2$, *then* $C \equiv_H \min_H(D_1, D_2)$,
- *if* $C = D_1 \to D_2$, *then* $C \equiv_H \mathrm{res}_H(D_1, D_2)$.

This preorder is compatible *with the TBox* $\mathcal{T}$ *if for every GCI* $\langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}$ *we have* $\mathrm{res}_H(C, D) \gtrsim_H q$. *It is* compatible *with* $\mathcal{A}$ *if for every order assertion* $\langle a{:}C \bowtie q\rangle$ *or* $\langle a{:}C \bowtie a{:}D\rangle$ *in* $\mathcal{A}$, *we have* $C \bowtie_H q$ *or* $C \bowtie_H D$, *respectively.*

These conditions ensure that the semantics of all *propositional* constructors is preserved (the order structure $\mathcal{U}$ already takes care of the involutive negation). The following definition deals with the *quantified* concepts.

Definition 5 (Hintikka condition). *A tuple* $(\lesssim_0, \lesssim_1, \ldots, \lesssim_n)$ *of Hintikka orderings satisfies the* Hintikka condition *if:*

- *for every* $1 \leq i \leq n$ *and all* $\alpha, \beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$, *we have* $\alpha \lesssim_0 \beta$ *iff* $\alpha_{\uparrow} \lesssim_i \beta_{\uparrow}$;
- *for every* $\exists r.C \in \mathsf{sub}(\mathcal{O})$, *we have*
  - $(\exists r.C)_{\uparrow} \gtrsim_i \min_i(\lambda, C)$ *for all* $i \in \Phi_r$, *and*
  - *for* $i = \phi(\exists r.C)$ *and every* $\alpha \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ *with* $(\exists r.C)_{\uparrow} >_i \alpha_{\uparrow}$ *we have* $\min_i(\lambda, C) >_i \alpha_{\uparrow}$; *and*
- *for every* $\forall r.C \in \mathsf{sub}(\mathcal{O})$, *we have*
  - $(\forall r.C)_{\uparrow} \lesssim_i \mathrm{res}_i(\lambda, C)$ *for all* $i \in \Phi_r$, *and*
  - *for* $i = \phi(\forall r.C)$ *and every* $\alpha \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ *with* $(\forall r.C)_{\uparrow} <_i \alpha_{\uparrow}$, *we have* $\mathrm{res}_i(\lambda, C) <_i \alpha_{\uparrow}$.

Here lies the main difference to [12], where the second condition for $\exists r.C$ is replaced by $(\exists r.C)_{\uparrow} \equiv_i \min_i(\lambda, C)$ to obtain a witness, and similarly for value restrictions. Since we consider general models, we instead have to ensure that the value of $\min_i(\lambda, C)$ can be moved arbitrarily close to that of $(\exists r.C)_{\uparrow}$ to satisfy the semantics of $\exists r.C$, which is based on a supremum. This is only possible if no other value from the parent node enforces their separation (for details, see the proof of Proposition 6).

A *Hintikka tree* for $\mathcal{O}$ is an infinite n-ary tree,[3] where every node u is associated with a Hintikka ordering $\lesssim_u$ compatible with $\mathcal{T}$, such that:

- every tuple $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$ satisfies the Hintikka condition, and
- $\lesssim_\varepsilon$ is compatible with $\mathcal{A}$.

Proposition 6. *If there is a Hintikka tree for $\mathcal{O}$, then $\mathcal{O}$ has a model.*

Proof. Given a Hintikka tree, we construct a model in two steps. In the first step, we recursively define a function $v\colon \mathcal{U} \times (\{1,\dots,n\} \times \mathbb{N})^* \to [0,1]$. For a word $u = (i_0, m_0)\dots(i_\ell, m_\ell) \in (\{1,\dots,n\} \times \mathbb{N})^*$, let $\pi_1(u) := i_0 \dots i_\ell \in \{1,\dots,n\}^*$ denote its projection to the first component. The mapping v will satisfy the following conditions for all $\alpha, \beta \in \mathcal{U}$ and all $u \in (\{1,\dots,n\} \times \mathbb{N})^*$:

(P1) for all values $q \in \mathcal{V}_\mathcal{O}$ we have $v(q,u) = q$,
(P2) $v(\alpha,u) \leq v(\beta,u)$ iff $\alpha \lesssim_{\pi_1(u)} \beta$,
(P3) $v(\mathrm{inv}(\alpha),u) = 1 - v(\alpha,u)$,
(P4) for all $\exists r.C \in \mathsf{sub}(\mathcal{O})$, we have

$$v(\exists r.C, u) = \sup_{i\in\Phi_r} \sup_{m\in\mathbb{N}} \min(v(\lambda, u\cdot(i,m)), v(C, u\cdot(i,m))),$$

(P5) for all $\forall r.C \in \mathsf{sub}(\mathcal{O})$, we have

$$v(\forall r.C, u) = \inf_{i\in\Phi_r} \inf_{m\in\mathbb{N}} v(\lambda, u\cdot(i,m)) \Rightarrow v(C, u\cdot(i,m)).$$

In the second step, we construct, with the help of this function v, an interpretation $\mathcal{I}_v = ((\{1,\dots,n\} \times \mathbb{N})^*, \cdot^{\mathcal{I}_v})$ satisfying $C^{\mathcal{I}_v}(u) = v(C,u)$ for all concepts C and all $u \in (\{1,\dots,n\} \times \mathbb{N})^*$, and show that $\mathcal{I}_v$ is indeed a model of $\mathcal{O}$.

Step 1. The function v is defined recursively, starting from the root node ε. Let $\mathcal{U}/{\equiv_\varepsilon}$ be the set of all equivalence classes of $\equiv_\varepsilon$. Then $\lesssim_\varepsilon$ yields a total order on $\mathcal{U}/{\equiv_\varepsilon}$. In particular, $[0]_\varepsilon <_\varepsilon [q_1]_\varepsilon <_\varepsilon [q_2]_\varepsilon <_\varepsilon \cdots <_\varepsilon [q_{k-1}]_\varepsilon <_\varepsilon [1]_\varepsilon$ holds if we extend $<_\varepsilon$ to $\mathcal{U}/{\equiv_\varepsilon}$ in the obvious way. For an equivalence class $[\alpha]_\varepsilon$, we set $\mathrm{inv}([\alpha]_\varepsilon) := [\mathrm{inv}(\alpha)]_\varepsilon$, which is well-defined since $\lesssim_\varepsilon$ is an element of $\mathrm{order}(\mathcal{U})$.

We first define an auxiliary function $\tilde{v}_\varepsilon\colon \mathcal{U}/{\equiv_\varepsilon} \to [0,1]$. For all $q \in \mathcal{V}_\mathcal{O}$ we set $\tilde{v}_\varepsilon([q]_\varepsilon) := q$. It remains to define a value for all equivalence classes that do not contain a value from $\mathcal{V}_\mathcal{O}$. Notice that due to the minimality of $[0]_\varepsilon$ and maximality of $[1]_\varepsilon$ every such class must be strictly between $[q_i]_\varepsilon$ and $[q_{i+1}]_\varepsilon$ for two adjacent truth degrees q_i, q_{i+1}. For every $i \in \{0,\dots,k-1\}$, let ν_i be the number of equivalence classes that are strictly between $[q_i]_\varepsilon$ and $[q_{i+1}]_\varepsilon$. We assume that these classes are denoted by E^i_j such that $[q_i]_\varepsilon <_\varepsilon E^i_1 <_\varepsilon E^i_2 <_\varepsilon \cdots <_\varepsilon E^i_{\nu_i} <_\varepsilon [q_{i+1}]_\varepsilon$. We then define values $q_i < s^i_1 < s^i_2 < \cdots < s^i_{\nu_i} < q_{i+1}$ as

$$s^i_j := q_i + \tfrac{j}{\nu_i+1}(q_{i+1} - q_i) \tag{1}$$

[3] We use words from $\{1,\dots,n\}^*$ to denote the nodes of such a tree, as usual.

and set $\tilde{v}_\varepsilon(E^i_j) := s^i_j$ for every j, $1 \leq j \leq \nu_i$. Finally, we define $v(\alpha, \varepsilon) := \tilde{v}_\varepsilon([\alpha]_\varepsilon)$ for all $\alpha \in \mathcal{U}$. This construction ensures that (P1) and (P2) hold at the node ε. To see that (P3) is also satisfied, note that $1 - q_{i+1}$ and $1 - q_i$ are also adjacent in $\mathcal{V}_\mathcal{O}$ and have exactly the inverses $\mathrm{inv}(E^i_j)$ between them in reversed order.

For the recursion step, assume that we have already defined v for a node $u \in (\{1, \dots, n\} \times \mathbb{N})^*$ such that (P1)–(P3) are satisfied at u, let $u_1 := \pi_1(u)$, and consider the next pair $(i, m) \in \{1, \dots, n\} \times \mathbb{N}$. We initialize the auxiliary function $\tilde{v}_{u,i} \colon \mathcal{U}/{\equiv_{u_1 i}} \to [0, 1]$ by setting $\tilde{v}_{u,i}([q]_{u_1 i}) := q$ for all $q \in \mathcal{V}_\mathcal{O}$ and $\tilde{v}_{u,i}([C_\uparrow]_{u_1 i}) := v(C, u)$ for all $C \in \mathsf{sub}(\mathcal{O})$. To see that this is well-defined, consider the case that $[C_\uparrow]_{u_1 i} = [D_\uparrow]_{u_1 i}$, i.e. $C_\uparrow \equiv_{u_1 i} D_\uparrow$. From the Hintikka condition, we get $C \equiv_{\pi_1(u)} D$, and from (P2) at u we obtain $v(C, u) = v(D, u)$. Similarly, one can show that $[q]_{u_1 i} = [C_\uparrow]_{u_1 i}$ implies $v(q, u) = v(C, u)$. For the remaining equivalence classes, we use a construction similar to (1) by considering all neighboring equivalence classes that contain an element of $\mathcal{V}_\mathcal{O} \cup \mathsf{sub}_\uparrow(\mathcal{O})$ (whose values are already fixed) and evenly distributing the values between them.

To ensure that the semantics of the role restrictions are respected at the node u, we consider first the case that $i = \phi(\exists r.C)$ for $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and define $v(\alpha, u \cdot (i, m))$ for $\alpha \in \mathcal{U}$ as follows:

$$\begin{cases} \frac{2^m-1}{2^m} f((\exists r.C)_\uparrow) + \frac{1}{2^m} f(\alpha) & \text{if } \min_{u_1 i}(\lambda, C) \lesssim_{u_1 i} \alpha <_{u_1 i} (\exists r.C)_\uparrow, \\ \frac{2^m-1}{2^m} f((\neg\exists r.C)_\uparrow) + \frac{1}{2^m} f(\alpha) & \text{if } (\neg\exists r.C)_\uparrow <_{u_1 i} \alpha \lesssim_{u_1 i} \mathrm{inv}(\min_{u_1 i}(\lambda, C)), \\ f(\alpha) & \text{otherwise,} \end{cases}$$

where $f(\alpha) := \tilde{v}_{u,i}([\alpha]_{u_1 i})$. That is, we let $\min(v(\lambda, u \cdot (i, m)), v(C, u \cdot (i, m)))$ approach $v(\exists r.C, u)$ with increasing m, and do the same for all values in between. This is well-defined since, if α lies in both intervals, then the Hintikka condition yields $\min_{u_1 i}(\lambda, C) \leq_{u_1 i} \mathrm{inv}(\min_{u_1 i}(\lambda, C)) <_{u_1 i} (\exists r.C)_\uparrow$. But this is impossible since $\mathcal{U}$ is an order structure, and thus $\min_{u_1 i}(\lambda, C) \leq_{u_1 i} 0.5 <_{u_1 i} (\exists r.C)_\uparrow$, which contradicts the Hintikka condition. If i corresponds to a value restriction, we use a similar definition where $(\exists r.C)_\uparrow$ is replaced by $(\forall r.C)_\uparrow$, $\min_{u_1 i}(\lambda, C)$ is replaced by $\mathrm{res}_{u_1 i}(\lambda, C)$, and the order is inverted.

We now verify that (P1)–(P5) hold for this v at all $u \cdot (i, m)$. (P1) is satisfied by the definition of $\tilde{v}_{u,i}$ and the Hintikka condition. For (P2), observe that the Hintikka ordering $\lesssim_{u_1 i}$ is preserved by the definition of $\tilde{v}_{u,i}$ and the definition of v at $u \cdot (i, m)$ only compresses the distances between neighboring equivalence classes, but does not affect their ordering. (P3) also holds because it is valid at $u \cdot (i, 0)$ and all shifts towards $(\exists r.C)_\uparrow$ for increasing m are mirrored for $(\neg\exists r.C)_\uparrow$, and similarly for value restrictions. We now verify (P4); (P5) follows from dual arguments. For this, consider any $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and $i := \phi(\exists r.C)$. From the construction and the Hintikka condition, we know that

$$\sup_{m \in \mathbb{N}} \min(v(\lambda, u \cdot (i, m)), v(C, u \cdot (i, m))) = v((\exists r.C)_\uparrow, u \cdot (i, m)) = v(\exists r.C, u).$$

Furthermore, for every other $j \in \Phi_r$ and all $m \in \mathbb{N}$, we obtain

$$\min(v(\lambda, u \cdot (j, m)), v(C, u \cdot (j, m))) \leq v((\exists r.C)_\uparrow, u \cdot (j, m)) = v(\exists r.C, u)$$

from the Hintikka condition and (P2).

Step 2. We define the interpretation $\mathcal{I}_v$ over the domain $\Delta^{\mathcal{I}} := (\{1, \dots, n\} \times \mathbb{N})^*$ as follows. For every concept name $A \in \mathsf{N_C}$ and all domain elements u, we set

$$A^{\mathcal{I}_v}(u) := \begin{cases} v(A, u) & \text{if } A \in \mathsf{sub}(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

For every role name $r \in \mathsf{N_R}$ and all domain elements u, we likewise define

$$r^{\mathcal{I}_v}(u, w) := \begin{cases} v(\lambda, w) & \text{if } w = u \cdot (i, m) \text{ with } i \in \Phi_r, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define $a^{\mathcal{I}_v} := \varepsilon$ for the individual name a. It can be shown by induction on the structure of C, using similar arguments as in [11], that

$$C^{\mathcal{I}_v}(u) = v(C, u) \text{ for all } C \in \mathsf{sub}(\mathcal{O}) \text{ and } u \in \Delta^{\mathcal{I}} \tag{2}$$

holds. In this proof by induction

- the base case follows trivially from the definition of $\mathcal{I}_v$,
- the cases $\top$, $C \sqcap D$, and $C \to D$ follow from (P1), (P2), and Definition 4,
- the case $\neg C$ follows from (P3), and
- (P4) and (P5) entail the cases $\exists r.C$ and $\forall r.C$, respectively.

It remains to show that $\mathcal{I}_v$ is indeed a model of $\mathcal{O}$. For every $\langle a : C \bowtie q \rangle \in \mathcal{A}$, the Hintikka tree satisfies $C \bowtie_\varepsilon q$, and thus we obtain from (2), (P1), and (P2):

$$C^{\mathcal{I}_v}(a^{\mathcal{I}_v}) = v(C, \varepsilon) \bowtie v(q, \varepsilon) = q,$$

and similarly for assertions of the form $\langle a : C \bowtie a : D \rangle$.

Consider now $u \in \Delta^{\mathcal{I}}$ and $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$. Since $q \in \mathcal{V}_{\mathcal{O}}$ and $\lesssim_{\pi_1(u)}$ is compatible with $\mathcal{T}$, it must hold that

$$q \lesssim_{\pi_1(u)} \mathrm{res}_{\pi_1(u)}(C, D) = \left\{ \begin{matrix} 1 \text{ if } C \lesssim_{\pi_1(u)} D \\ D \text{ if } D <_{\pi_1(u)} C \end{matrix} \right\} = \left\{ \begin{matrix} 1 \text{ if } v(C, u) \leq v(D, u) \\ D \text{ if } v(D, u) < v(C, u), \end{matrix} \right.$$

where the second equality is due to (P2). Thus, we obtain

$$q = v(q, u) \leq \left\{ \begin{matrix} v(1, u) \text{ if } v(C, u) \leq v(D, u) \\ v(D, u) \text{ if } v(D, u) < v(C, u) \end{matrix} \right\} = C^{\mathcal{I}_v}(u) \Rightarrow D^{\mathcal{I}_v}(u).$$

from (2), (P1), and (P2). □

Conversely, every model can be *unraveled* into an infinite tree, and then we can abstract from the specific values by just considering the ordering between the elements of $\mathcal{U}$, which yields a Hintikka tree.

Proposition 7. *If $\mathcal{O}$ has a model, then there is a Hintikka tree for $\mathcal{O}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}$. We use this model to construct a Hintikka tree for $\mathcal{O}$ and recursively generate a mapping $g\colon \{1,\ldots,n\}^* \to \Delta^{\mathcal{I}}$ specifying which domain elements correspond to the nodes in the tree. This mapping satisfies the following condition for all $\alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ and all $u \in \{1,\ldots,n\}^*$:

(P6) $\alpha \lesssim_u \beta$ iff $\alpha^{\mathcal{I}}(g(u)) \leq \beta^{\mathcal{I}}(g(u))$,

where we define $q^{\mathcal{I}}(x) := q$ for all $q \in \mathcal{V}_{\mathcal{O}}$ and $x \in \Delta^{\mathcal{I}}$.

We first consider the root node ε of the tree. Recall that the ontology contains a local ordered ABox that uses only the individual name a. We define $g(\varepsilon) := a^{\mathcal{I}}$ and the Hintikka ordering $\lesssim_\varepsilon$ as follows for all $\alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$:

$$\alpha \lesssim_\varepsilon \beta \text{ iff } \alpha^{\mathcal{I}}(a^{\mathcal{I}}) \leq \beta^{\mathcal{I}}(a^{\mathcal{I}}).$$

We extend this order to the elements in $\mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ arbitrarily, such that for all $\alpha,\beta \in \mathcal{U}$ we have $\alpha \lesssim_\varepsilon \beta$ iff $\mathrm{inv}(\beta) \lesssim_\varepsilon \mathrm{inv}(\alpha)$. It is easy to show that $\lesssim_\varepsilon$ is an element of $\mathrm{order}(\mathcal{U})$ satisfying (P6) at ε, and that $\lesssim_\varepsilon$ is a Hintikka ordering that is compatible with $\mathcal{T}$ (cf. [11]).

Assume now that we have already defined $g(u)$ and $\lesssim_u$ for $u \in \{1,\ldots,n\}^*$ such that (P6) is satisfied. For all $i \in \{1,\ldots,n\}$, we construct $\lesssim_{ui}$ such that $(\lesssim_u, \lesssim_{u1}, \ldots, \lesssim_{un})$ satisfies the Hintikka condition. For brevity, we consider only the case $i = \phi(\exists r.C)$; value restrictions can be handled using similar arguments.

If there is a $y_i \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(g(u)) = \min(r^{\mathcal{I}}(g(u),y_i), C^{\mathcal{I}}(y_i))$, then we define $g(ui) := y_i$, and $\lesssim_{ui}$ for all $\alpha,\beta \in \mathcal{U}$ by

$$\alpha \lesssim_{ui} \beta \text{ iff } \alpha^{\mathcal{I}}(g(ui)) \leq \beta^{\mathcal{I}}(g(ui)), \tag{3}$$

where we abbreviate $\lambda^{\mathcal{I}}(g(ui)) := r^{\mathcal{I}}(g(u), g(ui))$ and $(D_{\uparrow})^{\mathcal{I}}(g(ui)) := D^{\mathcal{I}}(g(u))$ for all concepts $D \in \mathsf{sub}(\mathcal{O})$. It is clear that $\lesssim_{ui}$ behaves on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}_{\uparrow}(\mathcal{O})$ exactly as $\lesssim_u$ does on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$. As for the root node, it is easy to show that $\lesssim_{ui}$ is actually a Hintikka ordering compatible with $\mathcal{T}$.

If there is no such element y_i, then the set $\{\min(r^{\mathcal{I}}(g(u),y), C^{\mathcal{I}}(y)) \mid y \in \Delta^{\mathcal{I}}\}$ must contain an infinite increasing chain whose supremum is $(\exists r.C)^{\mathcal{I}}(g(u))$. Let $(y^j)_{j\in\mathbb{N}}$ be the domain elements corresponding to these increasing values, and define $\lesssim_{y^j}$ for each $j \in \mathbb{N}$ as follows for all $\alpha,\beta \in \mathcal{U}$:

$$\alpha \lesssim_{y^j} \beta \text{ iff } \alpha^{\mathcal{I}}(y^j) \leq \beta^{\mathcal{I}}(y^j). \tag{4}$$

As before, this defines Hintikka orderings compatible with $\mathcal{T}$ that behave on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}_{\uparrow}(\mathcal{O})$ exactly as $\lesssim_u$ on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$. Since there are only finitely many such orderings, we can find a Hintikka ordering $\lesssim_{ui}$ and an infinite subsequence $(y^{j_\ell})_{\ell\in\mathbb{N}}$ such that $\lesssim_{y^{j_\ell}} = \lesssim_{ui}$ for all $\ell \in \mathbb{N}$. We now define $g(ui) := y^{j_0}$ and $\lesssim_{ui}$ as above, which obviously satisfies (P6).

We show the Hintikka condition for $(\lesssim_u, \lesssim_{u1}, \ldots, \lesssim_{un})$, again considering only the existential restrictions $\exists r.C \in \mathsf{sub}(\mathcal{O})$. For all $i \in \Phi_r$, we have

$$\begin{aligned}(\exists r.C)^{\mathcal{I}}(g(u)) &= \sup_{y\in\Delta^{\mathcal{I}}} \min\left(r^{\mathcal{I}}(g(u),y), C^{\mathcal{I}}(y)\right) \\ &\geq \min\left(r^{\mathcal{I}}(g(u),g(ui)), C^{\mathcal{I}}(g(ui))\right),\end{aligned}$$

which shows that $(\exists r.C)_{\uparrow} \gtrsim_{ui} \min_{ui}(\lambda, C)$. For $i = \phi(\exists r.C)$, assume that there is an $\alpha \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ with $(\exists r.C)_{\uparrow} >_{ui} \alpha_{\uparrow}$, and thus we have $\exists r.C >_u \alpha$. By (P6), we obtain $(\exists r.C)^{\mathcal{I}}(g(u)) > \alpha^{\mathcal{I}}(g(u))$. If we have defined $\lesssim_{ui}$ directly via (3), then $\min(\lambda^{\mathcal{I}}(g(ui)), C^{\mathcal{I}}(g(ui))) = (\exists r.C)^{\mathcal{I}}(g(u)) > \alpha^{\mathcal{I}}(g(u))$, and thus $\min_{ui}(\lambda, C) >_{ui} \alpha_{\uparrow}$, as required. If we have chosen $\lesssim_{ui}$ as one of the Hintikka orderings defined by (4), assume that $\min_{ui}(\lambda, C) \lesssim_{ui} \alpha_{\uparrow}$. Then, for all $\ell \in \mathbb{N}$, $\min(r^{\mathcal{I}}(g(u), y^{j_\ell}), C^{\mathcal{I}}(y^{j_\ell})) \leq \alpha^{\mathcal{I}}(g(u)) < (\exists r.C)^{\mathcal{I}}(g(u))$; thus the supremum of these values is not equal to $(\exists r.C)^{\mathcal{I}}(g(u))$, contradicting our construction.

Finally, for every $\langle a\!:\!C \bowtie q\rangle \in \mathcal{A}$, we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie q$, and thus $C \bowtie_{\varepsilon} q$ by the definition of $\lesssim_{\varepsilon}$, and similarly for assertions of the form $\langle a\!:\!C \bowtie a\!:\!D\rangle$. Hence, the tree defined by $\lesssim_u$, for $u \in \{1, \dots, n\}^*$, is a Hintikka tree for $\mathcal{O}$. □

These propositions show that Hintikka trees characterize consistency of ontologies with a local ordered ABox. That is, deciding the existence of a Hintikka tree for $\mathcal{O}$ suffices for deciding consistency of $\mathcal{O}$. We now show that the former problem can be solved in exponential time in the size of $\mathcal{O}$. For this, we construct a *looping tree automaton* whose runs correspond exactly to such Hintikka trees. This automaton accepts a non-empty language iff the ontology $\mathcal{O}$ is consistent.

A *looping automaton* over n-ary (infinite) trees is a tuple $\mathbf{A} = (Q, I, \Delta)$, consisting of a non-empty set Q of *states*, a subset $I \subseteq Q$ of *initial states*, and a *transition relation* $\Delta \subseteq Q^{n+1}$. A *run* of this automaton is a mapping $\rho\colon \{1, \dots, n\}^* \to Q$ such that (i) $\rho(\varepsilon) \in I$, and (ii) for all $u \in \{1, \dots, n\}^*$, we have $\big(\rho(u), \rho(u1), \dots, \rho(un)\big) \in \Delta$. $\mathbf{A}$ is *non-empty* iff it has a run.

Definition 8. *The* Hintikka automaton *for an ontology $\mathcal{O}$ is the looping tree automaton $\mathbf{A}_{\mathcal{O}} := (Q_{\mathcal{O}}, I_{\mathcal{O}}, \Delta_{\mathcal{O}})$, where*

- *$Q_{\mathcal{O}}$ is the set of all Hintikka orderings compatible with $\mathcal{T}$,*
- *$I_{\mathcal{O}} := \{\lesssim_H \in Q_{\mathcal{O}} \mid \lesssim_H$ is compatible with $\mathcal{A}\}$, and*
- *$\Delta_{\mathcal{O}}$ contains all tuples from $Q_{\mathcal{O}}^{n+1}$ that satisfy the Hintikka condition.*

It is easy to see that the runs of $\mathbf{A}_{\mathcal{O}}$ are exactly the Hintikka trees for $\mathcal{O}$. The cardinality of $\mathcal{U} = \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ is linear in the size of $\mathcal{O}$ and the number of Hintikka orderings for $\mathcal{O}$ is bounded by $2^{|\mathcal{U}|^2}$. Likewise, the arity n of $\mathbf{A}_{\mathcal{O}}$ is bounded by $|\mathsf{sub}(\mathcal{O})|$, which is linear in the size of $\mathcal{O}$. Thus, the size of the Hintikka automaton $\mathbf{A}_{\mathcal{O}}$ is exponential in the size of $\mathcal{O}$. Since emptiness of looping tree automata can be decided in polynomial time [30], we obtain an ExpTime-decision procedure for consistency of ontologies with local ordered ABoxes in G-$\mathfrak{I}\mathcal{ALC}$. The complexity of classical $\mathcal{ALC}$ [25] yields a matching lower bound.

Theorem 9. *Consistency in* G-$\mathfrak{I}\mathcal{ALC}$ *w.r.t. local ordered ABoxes and general models is* ExpTime*-complete.*

It is easy to adapt the decision procedures for ontology consistency where the ABox need not be local, and for concept satisfiability and subsumption from [12] to general model semantics. In fact, the *pre-completion* used to reduce consistency to local consistency is not concerned about witnesses at all, but only about the values of the concepts at the named domain elements and the role connections between them. The task of finding witnesses for quantified concepts is delegated to polynomially many local consistency tests.

The algorithm for concept satisfiability is based on the observation that the ABox is irrelevant for this inference since $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ does not include nominals and we can check consistency of the ontology beforehand. Thus, C is p-satisfiable w.r.t. $\mathcal{O} = (\emptyset, \mathcal{T})$ iff $(\{\langle a{:}C \geq p\rangle\}, \mathcal{T})$ is (locally) consistent, where a is an arbitrary individual name. To obtain the *best* satisfiability degree, only polynomially many local consistency tests of the above kind, one for each value in $\mathcal{V}_\mathcal{O}$, are needed. The reason for this is that, given $p, p' \in (q_i, q_{i+1})$ for two consecutive values $q_i, q_{i+1} \in \mathcal{V}_\mathcal{O}$, the Hintikka trees for $(\{\langle a{:}C \geq p\rangle\}, \mathcal{T})$ stand in a natural bijection to those for $(\{\langle a{:}C \geq p'\rangle\}, \mathcal{T})$, and thus C is p-satisfiable iff it is p'-satisfiable. Similar arguments hold for deciding subsumption and computing best subsumption degrees between concepts.

4 Conclusions

We have studied the standard reasoning problems for the fuzzy DL $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ w.r.t. general model semantics. We showed that all standard reasoning problems can be solved in exponential time. To achieve this, we developed an automaton that decides the existence of a Hintikka tree, which is an abstract representation of a model of a given ontology. The main insight needed for this approach is that we can abstract from the precise truth degrees assigned by an interpretation, and focus only on their ordering.

Our results complement those recently developed in [12], by showing that the exponential time reasoning is preserved in this Gödel DL, even if general models are considered. Recall that with this semantics, a consistent ontology may have only models where every domain element has infinitely many role successors with positive degree [20]. Thus, finding a finite abstract representation of these models is fundamental for effective reasoning.

As an added benefit, in our formalism we can express order assertions like $\langle \mathsf{ana}{:}\mathsf{Tall} > \mathsf{bob}{:}\mathsf{Tall}\rangle$, intuitively stating that Ana is taller than Bob, without needing to specify the precise degrees to which ana and bob belong to the concept Tall. Such assertions provide useful expressivity for the representation of domain knowledge. This is similar to [23,24], where values can even be compared at unnamed domain elements.

As we have developed an automata-based algorithm, it is natural to ask whether previous automata-based approaches [2,14] can be adapted to this setting in order to handle the expressivity up to $\mathsf{G}\text{-}\mathfrak{I}\mathcal{SCHI}$, or provide better upper-bounds for reasoning w.r.t. *acyclic* TBoxes. We will study these problems in future work. We also plan to adapt the presented ideas into a tableau-based algorithm which is more suitable for implementation.

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