On Logics and Homorphism Closure

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29.06.2021

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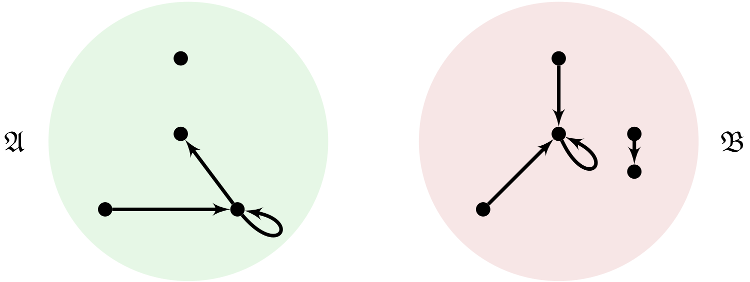

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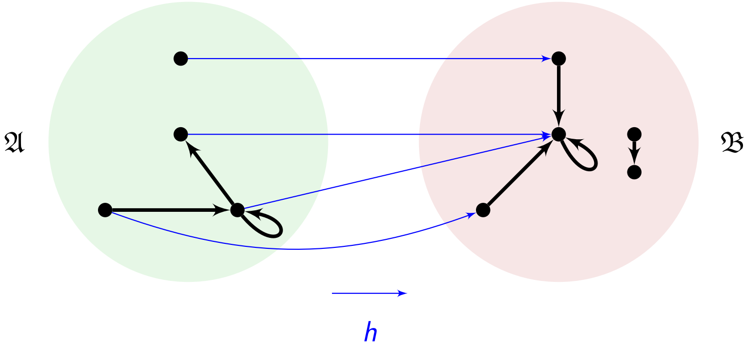

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- special structures: $\mathfrak{I}_\tau$ and $\mathfrak{F}_\tau$

- $\mathfrak{I}_\tau$ hom-maps into every structure / all structures hom-map onto $\mathfrak{F}_\tau$

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Theorem (Homomorphism Preservation Theorem, (Lyndon and Łos, Tarski / Rossman))

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Note: Also some of our results hold in the finite and infinite. For brevity there will be no explicit mention of the finite case.

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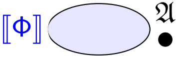

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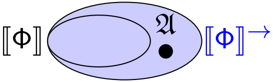

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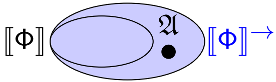

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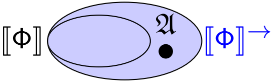

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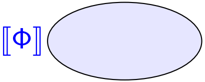

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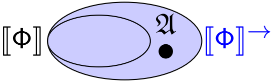

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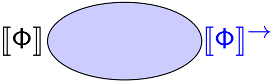

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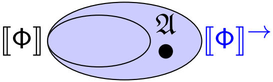

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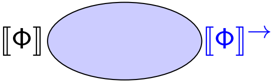

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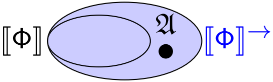

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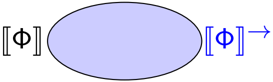

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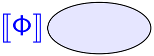

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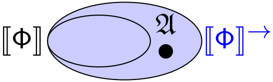

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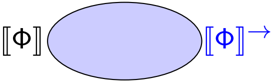

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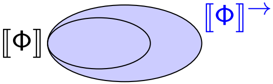

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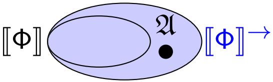

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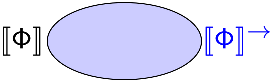

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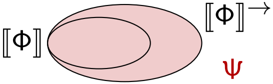

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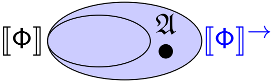

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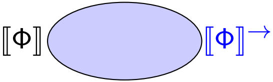

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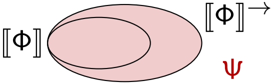

4. **Homclosed normal forms.** For which logics exists a "homclosed normal form" (i.e. a syntactic fragment representing all and only the homclosed formulae)?

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- Second order logic ($\mathbb{SO}$) and existential second order logic ($\exists\mathbb{SO}$)
- Tuple-Generating Dependencies ($\mathbb{TGD}$) and their disjunctive ($\mathbb{DTGD}$) and mildly disjunctive ($\mathbb{MDTGD}$) variants

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Problem: $\mathrm{InHomCl}$
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- $\mathrm{N}2\textsc{ExpTime}$ *for* $\mathbb{TGF}$,
- $\mathrm{N}\textsc{ExpTime}$ *for* $\mathbb{FO}^2_=$, $\exists^*\forall\forall\exists^*\mathbb{FO}$, $\exists^*\forall^*\mathbb{FO}_=$, $\forall^*\mathbb{FO}_=$,
- NP *for* $\exists^*\mathbb{FO}_=$ *and* $\exists^*\mathbb{FO}^{\pm}_=$.

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But: undecidable for $\mathbb{TGD}$ (and hence $\mathbb{MDTGD}$ and $\mathbb{DTGD}$)

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- **aim:** show that attention can be focused on specific kinds of spoilers

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Question 2: Homclosedness, Tools

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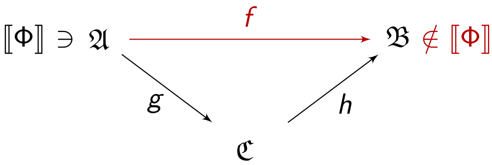

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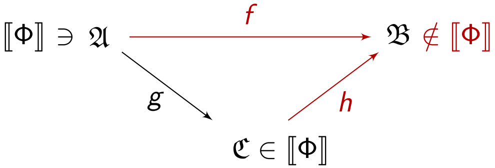

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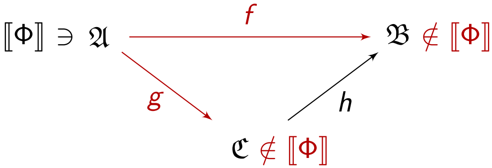

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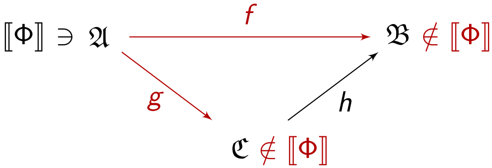

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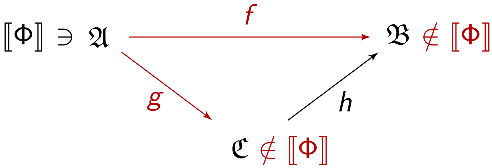

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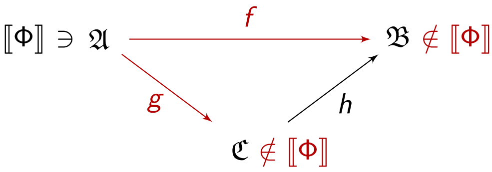

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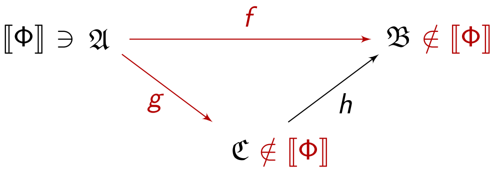

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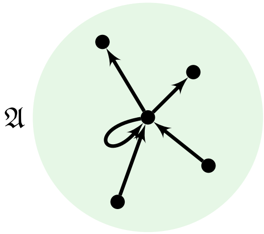

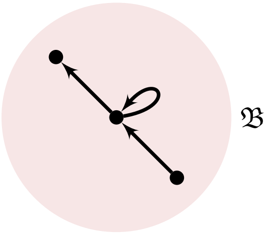

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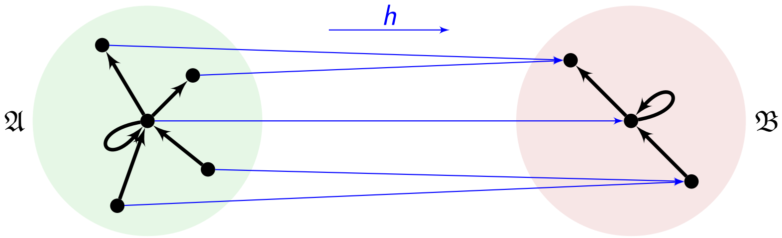

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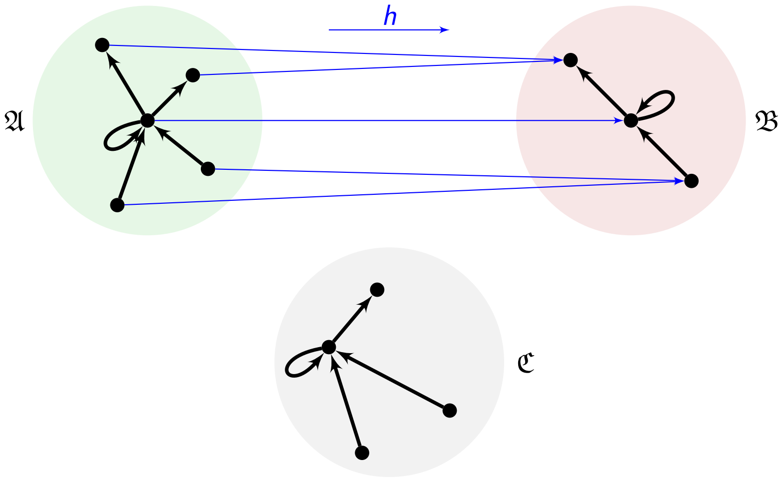

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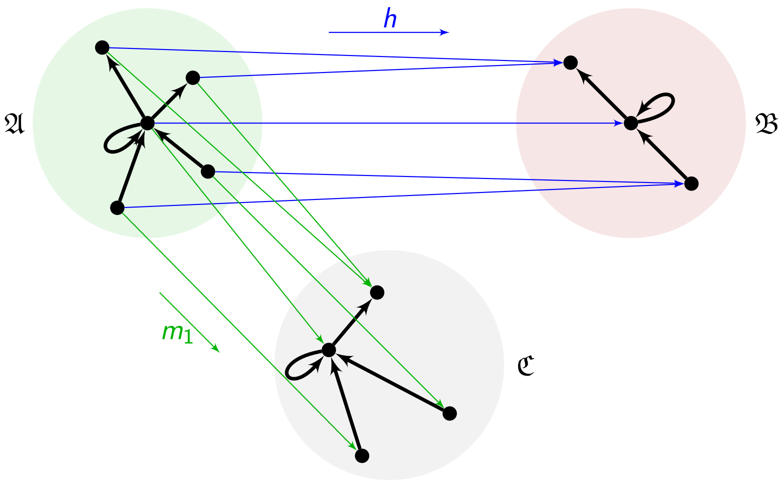

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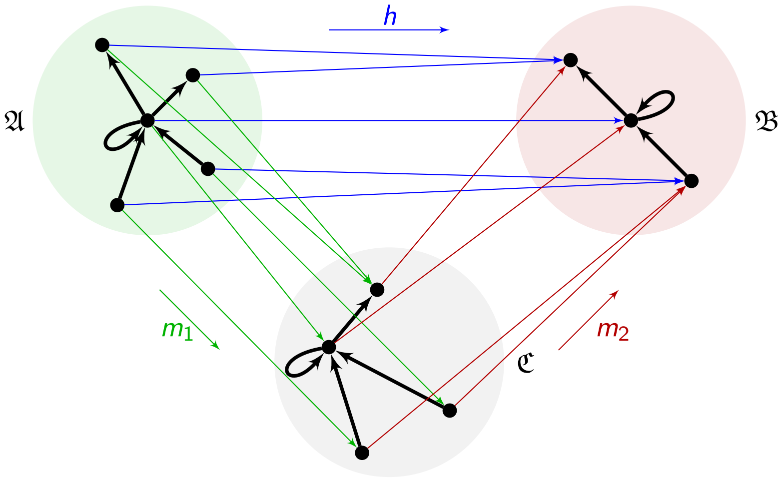

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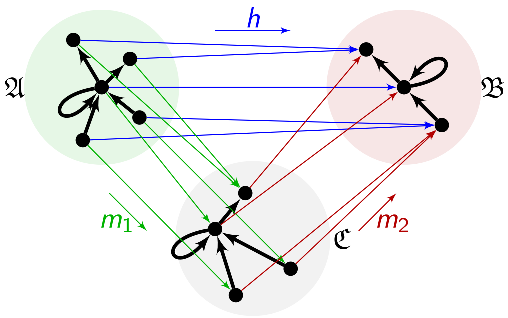

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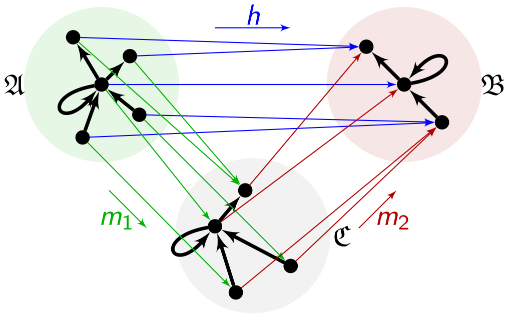

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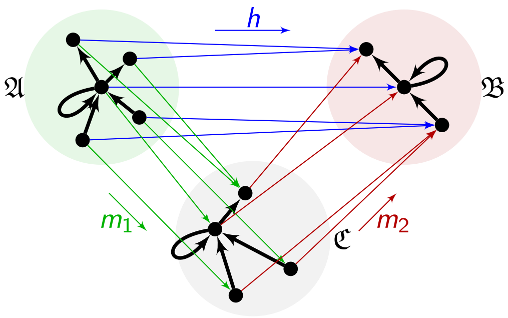

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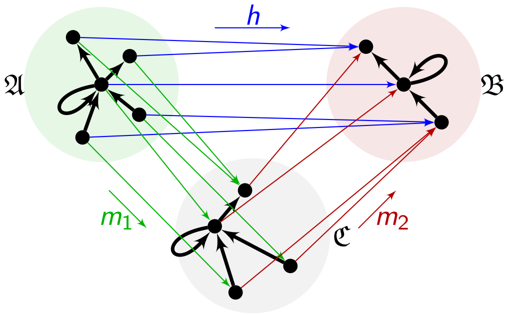

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Question 2: Homclosedness, Results

Theorem

HomClosed *for*

- $\mathbb{GNFO}_=$ *is* 2ExpTime*-complete,*
- $\mathbb{TGF}$ *is* coN2ExpTime*-complete,*
- *any of* $\mathbb{FO}^2_=$, $\forall^*\mathbb{FO}_=$, $\exists^*\mathbb{FO}_=$, $\forall\forall\exists\exists\mathbb{FO}$, *and* $\exists\exists\forall\forall\mathbb{FO}$ *is* coNExpTime*-complete.*

- for $\mathbb{TGD}$ NP-complete
- **but:** undecidable for $\mathbb{MDTGD}$ (and thus $\mathbb{DTGD}$)

Question 3: Characterizability

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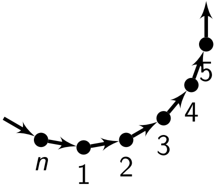

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- characterization in $\mathbb{FO}_{=}$ fails
- hence looking for characterizing logics more expressive than $\mathbb{FO}_{=}$

Question 3: Characterizability, Results 1

- employing standard type-based approaches

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$$\exists \mathtt{Lv}.\big(\exists x.\big(\mathtt{Lv}(x)\big) \wedge \forall x.\big(\mathtt{Lv}(x) \Rightarrow \exists y.(\mathtt{P}(x,y) \wedge \mathtt{Lv}(y))\big)\big)$$

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- $\mathbb{GFO}_=$ and $\mathbb{GNFO}_=$ on the other hand...

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Caveat: Normal form sentence might be non-elementary in the size of the given one!
(Rossman, 2008)

Question 4: Normal Forms for Homclosure, A Normal Form for $\mathbb{SO}$

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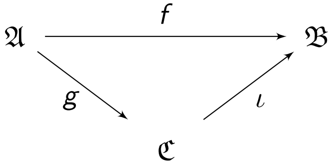

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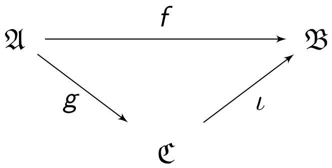

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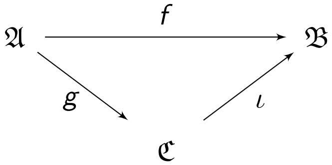

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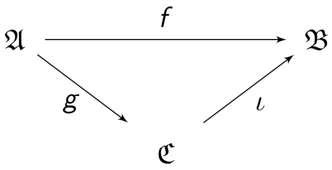

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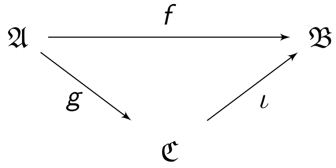

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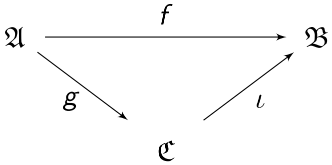

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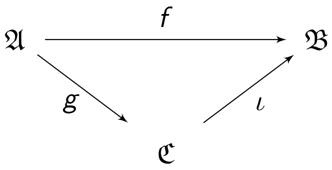

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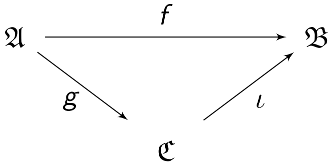

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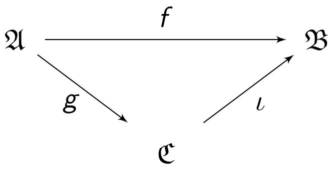

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- $(\Phi^{\text{shom}})^{\text{sup}}$ is always homclosed
- $\Phi \equiv (\Phi^{\text{shom}})^{\text{sup}}$ if and only if Φ homclosed
- yields syntactic fragment, $\mathbb{HSO}$; transformations are polytime-computable!

Overview and Summary

logic name	SAT fin/arb	finite model property (size)	closure $\neg$	closure $\wedge$	InHomCl comb.	InHomCl data	HomClosed fin/arb	homclosure characterizable in logic	normal form fragment
$\mathbb{FO}_=$	und.	no	yes	yes	und.	und.	und.	none	$\exists^*\mathbb{FO}^+_=$
$\mathbb{DTGD}$	trivial	yes (1)	no	yes	und.	und.	und.	none	$\mathbb{UCQ}$
$\mathbb{MDTGD}$	trivial	yes (1)	no	no	und.	und.	und.	none	$\mathbb{CQ}\vee\mathbb{CQ}$
$\mathbb{TGD}$	trivial	yes (1)	no	yes	und.	und.	NP	none	$\mathbb{CQ}$
$\mathbb{TGF}$	N2Exp	yes (2Exp)	yes	yes	N2Exp	NP	coN2Exp	$\exists\mathbb{SO}(\mathbb{TGF})$	$\mathbb{HTGF}$
$\mathbb{FO}^2_=$	NExp	yes (Exp)	yes	yes	NExp	NP	coNExp	$\exists\mathbb{SO}(\mathbb{FO}^2_=)$	$\mathbb{HFO}^2_=$
$\mathbb{GNFO}_=$	2Exp	yes (2Exp)	yes	yes	2Exp	P	2Exp	$\mathbb{FO}^{\mathsf{lfp}}_=$ / $\exists\mathbb{SO}(\mathbb{GFO}_=)$	$\exists^*\mathbb{FO}^+_=$
$\mathbb{GFO}_=$	2Exp	yes (2Exp)	yes	yes	2Exp	P	2Exp	$\mathbb{FO}^{\mathsf{lfp}}_=$ / $\exists\mathbb{SO}(\mathbb{GFO}_=)$	$\exists^*\mathbb{FO}^+_=$
$\forall\forall\forall\exists\mathbb{FO}$	und.	no	no	no	und.	und.	und.	none	?
$\exists^*\forall\forall\exists^*\mathbb{FO}$	NExp	yes (2Exp)	no	yes	NExp	NP	und.	$\exists\mathbb{SO}(\mathbb{TGF})$	$\exists^*\mathbb{FO}^+_=$
$\forall\forall\exists\exists\mathbb{FO}$	NExp	yes (2Exp)	no	no	NExp	NP	coNExp	$\exists\mathbb{SO}(\mathbb{TGF})$	$\mathbb{H}\forall\forall\exists\exists\mathbb{FO}$
$\exists^*\forall^*\mathbb{FO}_=$	NExp	yes (C+Ex)	no	yes	NExp	AC^0	und.	$\exists^*\mathbb{FO}^+_=$	$\exists^*\mathbb{FO}^+_=$
$\forall^*\mathbb{FO}_=$	NExp	yes max(C,1)	no	yes	NExp	AC^0	coNExp	$\exists\mathbb{FO}^+$	$\varnothing\mathbb{FO}^+_=$
$\exists\exists\exists\forall\mathbb{FO}$	NP	yes (C+3)	no	no	NP	AC^0	und.	$\exists\exists\exists\mathbb{FO}^+$	$\exists\exists\exists\mathbb{FO}^+$
$\exists\exists\forall\forall\mathbb{FO}$	NP	yes (C+2)	no	no	NP	AC^0	coNExp	$\exists\exists\mathbb{FO}^+$	$\exists\exists\mathbb{FO}^+$
$\exists^*\mathbb{FO}_=$	NP	yes (C+Ex)	no	yes	NP	AC^0	coNExp	$\exists^*\mathbb{FO}^+_=$	$\exists^*\mathbb{FO}^+_=$
$\exists^*\mathbb{FO}^+_=$	const.	yes (C+Ex)	no	yes	NP	AC^0	trivial	$\exists^*\mathbb{FO}^+_=$	$\exists^*\mathbb{FO}^+_=$
$\mathbb{SO}$	und.	no	yes	yes	und.	und.	und.	none	$\mathbb{HSO}$