Guarded Fragment - Definition

Definition 10

Guarded fragment (introduced by Andréka, van Benthem, Németi), GF, is the smallest subset of FO such that:

- all atomic formulas belong to GF;
- GF is closed under boolean operations $(\neg, \vee, \wedge, \rightarrow, \leftrightarrow)$;
- quantifiers are relativized by atoms: if $\varphi(\mathbf{x}, \mathbf{y})$ is in GF and $\gamma(\mathbf{x}, \mathbf{y})$ is an atom containing all free variables of φ, then

$$\forall \mathbf{y}(\gamma(\mathbf{x}, \mathbf{y}) \rightarrow \varphi(\mathbf{x}, \mathbf{y}))$$

and

$$\exists \mathbf{y}(\gamma(\mathbf{x}, \mathbf{y}) \wedge \varphi(\mathbf{x}, \mathbf{y}))$$

are in GF. Atoms $\gamma(\mathbf{x}, \mathbf{y})$ are called *guards*. $\mathbf{x}, \mathbf{y}$ denote here some tuples of variables.

Guarded Fragment: Examples

- Examples of formulas in GF
  - $\forall xy(Rxy \rightarrow Ryx)$
  - $\forall x(Px \rightarrow \exists y(Rxy \wedge Qy))$
  - $\forall x(x = x \rightarrow \exists yz(Sxyz \wedge Rxy \wedge Rxz)$

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- Examples of formulas **not** in GF
  - $\exists x(Px \wedge \forall yz(Rxy \rightarrow Rxz)$
  - $\forall xyz(Rxy \wedge Ryz \rightarrow Rxz)$
  - $\forall xy(Px \wedge Py \rightarrow Exy)$

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- Description logic $\mathcal{ALC}$ can be translated to the two-variable guarded fragment GF^2:

  $$\mathtt{Woman} \sqcap \exists \mathtt{hasChild}.(\mathtt{Male} \sqcap \forall \mathtt{hasChild}.(\mathtt{Male} \sqcup \mathtt{Blond}))$$

  translates to GF^2 formula:

  $$Wx \wedge \exists y(Cxy \wedge My \wedge \forall x(Cyx \rightarrow (Mx \vee Bx)))$$

Guarded Fragment: decidability and complexity (review)

- Decidability and complexity of GF (Grädel, 1997):
  - GF has the finite model property
  - SAT(GF) is 2ExpTime-complete
  - SAT(GF^2) (and in fact also SAT(GF^k) for arbitrary fixed k) is ExpTime-complete.
- Many interesting extensions of GF^2, e.g. by fixed point operators, constants, transitive relations in guards are decidable.
- We will see:
  - SAT(GF^2+EG) is NexpTime-complete
  - FINSAT(GF^2+EG) is NexpTime-complete

We can assume w.l.o.g that $\varphi \in GF^2$ has the form

Normal form:

(1) $\exists x \; \alpha(x) \wedge \lambda(x)$

(2) $\forall x \; \alpha(x) \to \lambda(x)$

(3) $\forall xy \; \beta(x,y) \to \lambda(x,y)$

(4) $\forall x \; \alpha(x) \to \exists y \; \beta(x,y) \wedge \lambda(x,y)$

} Conjunction of these statement
λ q-f
α, β atoms

Proof: Routine rewriting.

APspace = ExpTime

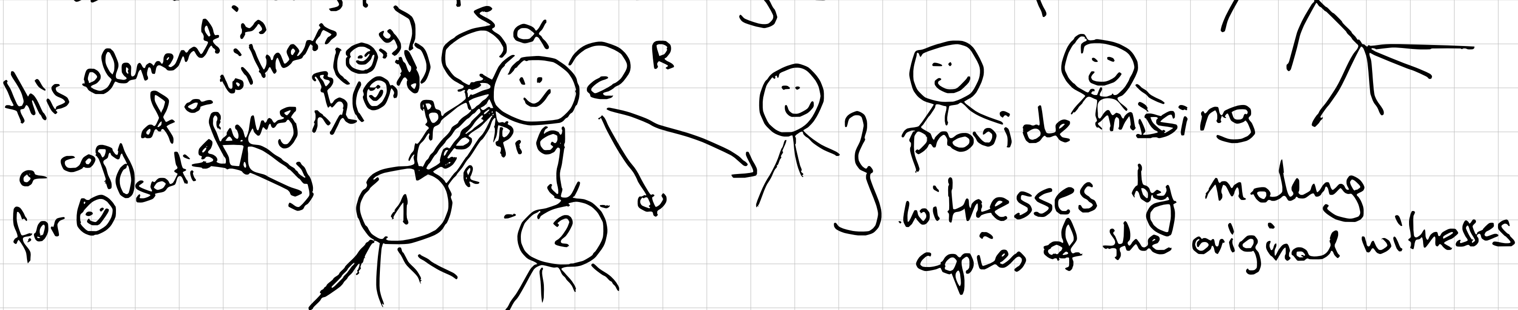
→ usual TM that works in PSpace
+ two extra powers:
1) guessing sth of polynomial size
2) "run things in parallel"

Input: $\exists p \forall q \exists r \exists t \forall s \forall u \; ((p \wedge u) \rightarrow (r \vee u \vee s)) \leftrightarrow (\neg t \wedge \neg q)$

PSpace = APTime

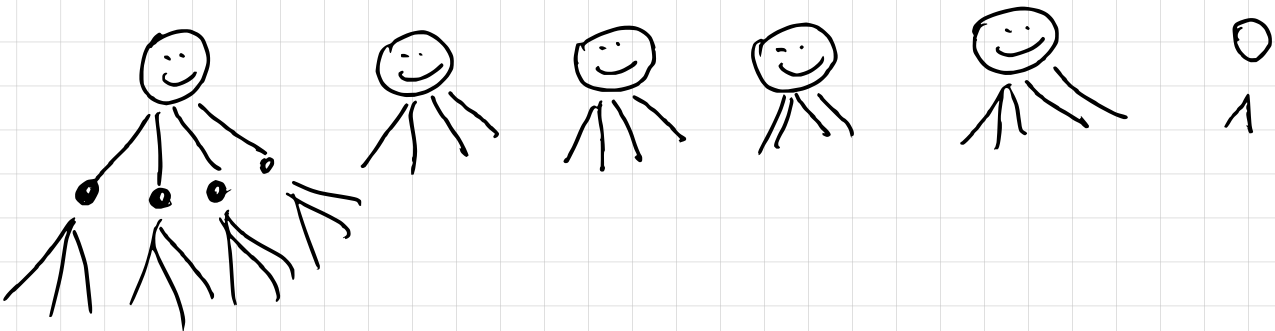
Lemma: If there is a model, then there is a tree-like model for which every node has $\leq |\varphi|$ children.

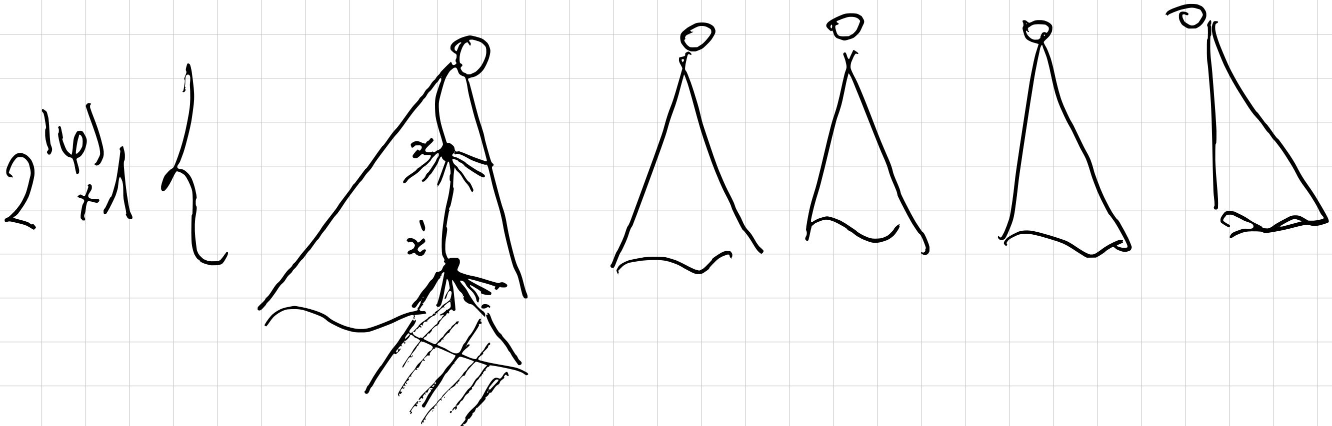
So assume that $\mathfrak{A} \models \varphi$. Take any element of A.



Algorithm for SAT(GF^2)

1. Read φ and turn it into a normal form (poly-time)
2. counter $\leftarrow 0$ (written in binary)
3. Guess a set of roots and check whether they satisfy $\forall x\, \alpha(x) \to \lambda(x)$ conjuncts
   per each $\exists x\, \alpha(x) \wedge \lambda(x)$ conjuncts
   (otherwise reject)
4. Select any of the roots and continue in parallel (call it v)
5. Guess v's children and connections between v and them (+ their types and so on). (in NP, as we have $\leq |\varphi|$ of them)
6. Check if the guess was correct
   i.e. if $\forall x\, \alpha(x) \to \lambda(x)$, $\forall xy\, \beta(x,y) \to \lambda'(x,y)$, $\forall x\, \alpha(x) \to \exists y\, \beta'(x,y) \wedge \lambda''$
   are satisfied.
7. Increment counter by 1. If counter $= 2^{|\varphi|} + 1$ then accept and STOP,
   otherwise choose any child of v and continue in parallel.


1


$2^{|\varphi|}+1$
x
x'