Exercise 11.1. Show $\text{BPP} = \text{coBPP}$.

Exercise 11.2. Find the error in the following proof that $\text{PP} = \text{BPP}$.

Let $L \in \text{PP}$. Then there exists a PTM accepting $L$ with error probability $< 1/2$. Using error amplification, we can make this error arbitrarily small, and in particular smaller than $1/3$. Thus $L \in \text{BPP}$.

Exercise 11.3. Let $M$ be a polynomial-time probabilistic Turing machine. We say that $M$ has error probability $< 1/3$ if and only if

$$P(M \text{ accepts } w) < \frac{1}{3} \quad \text{or} \quad P(M \text{ accepts } w) \geq \frac{2}{3}$$

for all inputs $w$. Show that deciding whether a polynomial-time probabilistic Turing machine has error probability $< 1/3$ is undecidable.

* Exercise 11.4. Show $\text{BPP}^{\text{BPP}} = \text{BPP}$.

Exercise 11.5. Let $0 < p < 1$ and let $(X_i \mid i \in \mathbb{N})$ be a sequence of independent random variables $X_i : \Omega \to \{0, 1\}$ such that $P(X_i = 1) = p$ for all $i \in \mathbb{N}$. Describe a way how to transform the sequence $(X_i \mid i \in \mathbb{N})$ into a sequence $(Y_i \mid i \in \mathbb{N})$ such that $P(Y_i = 1) = P(Y_i = 0) = 1/2$.

* Exercise 11.6. Let $p(x_1, \ldots, x_n)$ be a non-zero integer polynomial such that every $x_i$ has degree at most $d$ in $p$. Show that then for every $m \in \mathbb{N}$ it is true that

$$\left| \left\{ (a_1, \ldots, a_n) \in \{0, \ldots, m\}^n \mid p(a_1, \ldots, a_n) = 0 \right\} \right| \leq ndm^{n-1}.$$ 

Use induction over $n$. 