Formal Concept Analysis

I Contexts, Concepts, and Concept Lattices

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slides based on a lecture by Prof. Gerd Stumme

Agenda

Multi-valued Contexts

- so far: "attribute" denotes properties which an object may have or not (so-called *one-valued attributes*)
- but: attributes like "color", "weight", "sex", or "grade" have *values*
- we denote such attributes *many-valued attributes* (DIN 2330 calls many-valued attributes *Merkmalarten*.)

Def.: A *many-valued context* (G, M, W, I) consists of sets G, M and W and a ternary relation I between G, M and W (i.e., $I \subseteq G \times M \times W$) for which it holds that

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \quad \text{always implies} \quad w = v.$$

Multi-valued Contexts

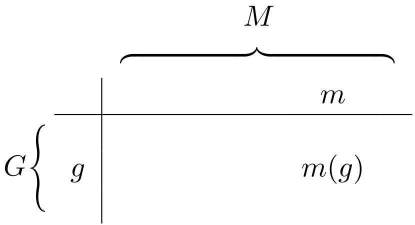
- The elements of
  - G are called *objects*, those of
  - M *(many-valued) attributes* and those of
  - W *attribute values*.
- $(g, m, w) \in I$ is read as "the attribute m has the value w for the object g".
- The many-valued attributes can be regarded as partial maps from G in W. Therefore, it seems reasonable to write $m(g) = w$ instead of $(g, m, w) \in I$.
- The *domain* of an attribute m is defined to be

$$\operatorname{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}$$

- An attribute m is called *complete*, if $\operatorname{dom}(m) = G$. A many-valued context is *complete*, if all its attributes are complete.

Multi-valued Contexts

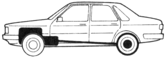
Like one-valued contexts, also many-valued contexts can be represented by tables, the rows of which are labelled by the objects and the columns labelled by the attributes:

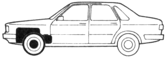


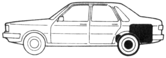
The entry in row g and column m then represents the attribute value $m(g)$. If the attribute m does not have a value for the object g, there will be no entry.

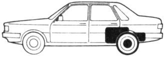
Multi-valued Contexts: "Drive Concepts for Motorcars"

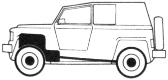
multi-valued context showing a comparison of the different possibilities of arranging the engine and the drive mechanism of a motorcar[1]


conventional


front-wheel


rear-wheel


mid-engine


all-wheel

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
all-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

De := drive efficiency empty; Dl := drive efficiency loaded; R := road holding/handling properties; S := self-steering efficiency; E := economy of space; C := cost of construction; M := maintainability;

[1]Source: Schlag nach! 100 000 Tatsachen aus allen Wissensgebieten. BI Verlag Mannheim, 1982

Conceptual Scaling

How can we assign concepts to a many-valued context?

We do his in the following way:

- the many-valued context is *transformed* into a one-valued one (explained later), the *derived* context
- from this derived one-valued context concepts can be obtained
- this process is called *conceptual scaling*
- not uniquely determined: depends on the initial transformation (may at first seem confusing but has proven to be feasible)

Process of scaling: each attribute of a many-valued context is interpreted by means of a(nother) context, called *conceptual scale*.

Conceptual Scaling

Def.: A *scale* for the attribute m of a many-valued context is a (one-valued) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called *scale values*, the attributes are called *scale attributes*.

$\mathbb{S}_{\mathsf{R}} :=$

	$++$	$+$	$--$
excellent	×	×	
good		×	
very poor			×

- every context can be used as a scale
- formally, no difference between a scale and a context
- we use the term "scale" only for contexts having a clear, meaningful conceptual structure
- some (particularly simple) contexts used as scales very frequently

Conceptual Scaling: “Drive Concepts for Motorcars”

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
all-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

one-valued context obtained as derived context of the multi-valued context above, using following scales:

$\mathbb{S}_{\mathrm{De}} := \mathbb{S}_{\mathrm{Dl}} :=$

	++	+	−
excellent	×	×	
good		×	
poor			×

$\mathbb{S}_{\mathrm{R}} :=$

	++	+	−−
excellent	×	×	
good		×	
very poor			×

$\mathbb{S}_{\mathrm{S}} :=$

	u	o	n	u/n
understeering	×			
oversteering		×		
neutral			×	
understeering/neutral				×

$\mathbb{S}_{\mathrm{C}} :=$

	vl	l	m	h
very low	×	×		
low		×		
medium			×	
high				×

$\mathbb{S}_{\mathrm{E}} := \mathbb{S}_{\mathrm{M}} :=$

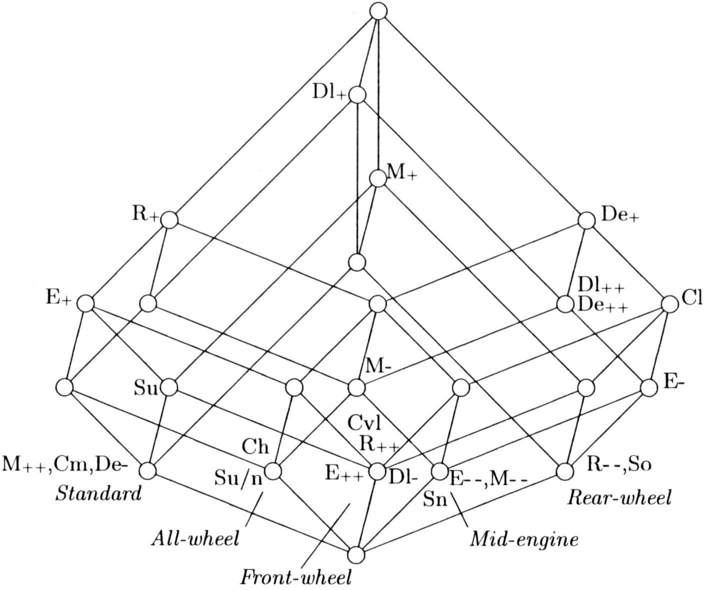
	++	+	−	−−
excellent	×	×		
good		×		
poor			×	
very poor			×	×

Conceptual Scaling: “Drive Concepts for Motorcars”

	De			Dl			R			S				E				C				M			
	++	+	−	++	+	−	++	+	−−	u	o	n	u/n	++	+	−	−−	vl	l	m	h	++	+	−	−−
conventional			×		×			×		×					×					×		×	×		
front-wheel		×				×	×	×		×				×	×			×	×				×		
rear-wheel	×	×		×	×				×		×					×			×				×		
mid-engine	×	×		×	×			×				×				×	×		×					×	×
all-wheel	×	×		×	×			×					×		×						×			×	

(If scale $\mathbb{S}_E$ were used for the attributes De, Dl, and R as well, the derived context would have turned out only slightly different.)

Conceptual Scaling: "Drive Concepts for Motorcars"



De := drive efficiency empty
Dl := drive efficiency loaded
R := road holding/handling properties
S := self-steering efficiency
E := economy of space
C := cost of construction
M := maintainability

Conceptual Scaling

Derived one-valued context obtained from many-valued context (G, M, W, I) and scale contexts $\mathbb{S}_m, m \in M$ as follows:

- object set G remains unchanged
- every many-valued attribute m is replaced by the scale attributes of the scale $\mathbb{S}_m$
- if we imagine the many-valued context as table, this means to replace each attribute value $m(g)$ by the $m(g)$-row of the scale context $\mathbb{S}_m$

	De	Dl	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good				very poor
all-wheel	excellent	excellent	good	under			poor

	++	+	−
excellent	×	×	
good		×	
poor			×

	De			Dl			R												C			M			
	++	+	−	++	+	−	++	+	−−										l	m	h	++	+	−	−−
conventional			×		×			×		×					×					×		×	×		
front-wheel		×				×	×	×		×				×	×			×	×				×		
rear-wheel	×	×		×	×				×		×					×			×				×		
mid-engine	×	×		×	×			×				×				×	×		×					×	×
all-wheel	×	×		×	×			×					×		×						×			×	

Conceptual Scaling

We now give a formal definition of scaling, using the following abbreviation:

$$\dot{M}_m := \{m\} \times M_m.$$

Def.: If (G, M, W, I) is a many-valued context and $\mathbb{S}_m, m \in M$ are scale contexts, then the *derived context with respect to plain scaling* is the context (G, N, J) with

$$N := \bigcup_{m \in M} \dot{M}_m,$$

and

$$gJ(m, n) :\Longleftrightarrow m(g) = w \text{ and } wI_m n.$$

Note: The attribute set of the derived context is the disjoint union of the attribute sets of the scales involved. In order to make sure that the sets are disjoint, we replace the attribute set M_m of the scale $\mathbb{S}_m$ by $\dot{M}_m$.

Conceptual Scaling

Frequently used scales:

- nominal scales
- ordinal scales
- interordinal scales
- biordinal scales
- dichotomic scale

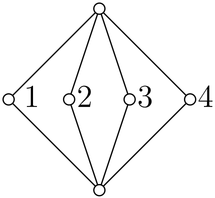
They will be introduced in detail on the following slides.
We will use the abbreviation $\mathbf{n} := \{1, \ldots, n\}$.

Elementary Scales

Nominal Scales: $\mathbb{N}_n := (\mathbf{n}, \mathbf{n}, =)$

Nominal scales are used to scale attributes, the values of which *mutually exclude* each other. If an attribute, for example, has the values $\{$*masculine, feminine, neuter*$\}$, the use of a nominal scale suggests itself. We thereby obtain a *partition* of the objects into extents. In this case, the classes correspond to the values of the attribute.

	1	2	3	4
1	×			
2		×		
3			×	
4				×



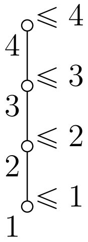
The Nominal Scale $\mathbb{N}_4$.

Elementary Scales

Ordinal Scales: $\mathbb{O}_n := (\mathbf{n}, \mathbf{n}, \leqslant)$

Ordinal scales scale many-valued attributes, the values of which are *ordered* and where each value implies the weaker ones. If an attribute has for instance the values {*loud, very loud, extremely loud*} ordinal scaling suggests itself. The attribute values then result in a chain of extents, interpreted as a *hierarchy*.

	1	2	3	4
1	×	×	×	×
2		×	×	×
3			×	×
4				×



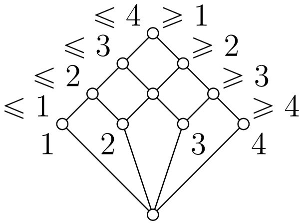
The Ordinal Scale $\mathbb{O}_4$.

Elementary Scales

Interordinal Scales: $\mathbb{I}_n := (\mathbf{n}, \mathbf{n}, \leqslant) \mid (\mathbf{n}, \mathbf{n}, \geqslant)$

Questionnaires often offer opposite pairs as possible answers, as for example *active–passive*, *talkative–taciturn*, etc., allowing a choice of intermediate values. In this case, we have a *bipolar* ordering of the values. This kind of attributes lend themselves to scaling by means of an interordinal scale. The extents of the interordinal scale are precisely the intervals of values, in this way, the *betweenness relation* is reflected conceptually.

	$\leqslant$1	$\leqslant$2	$\leqslant$3	$\leqslant$4	$\geqslant$1	$\geqslant$2	$\geqslant$3	$\geqslant$4
1	×	×	×	×	×			
2		×	×	×	×	×		
3			×	×	×	×	×	
4				×	×	×	×	×



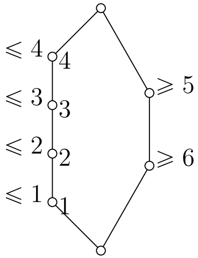
The Interordinal Scale $\mathbb{I}_4$.

Elementary Scales

Biordinal Scales: $\mathbb{M}_{n,m} := (\mathbf{n}, \mathbf{n}, \leqslant) \dot\cup (\mathbf{m}, \mathbf{m}, \geqslant)$

Colloquially we often use opposite pairs not in the sense of an interordinal scale, but simpler: each object is assigned one of the two poles, allowing graduations. The values {*very low, low, loud, very loud*} for example suggest this way of scaling: *loud* and *low* mutually exclude each other, *very loud* implies *loud*, *very low* implies *low*. We also find this kind of *partition with a hierarchy* in the names of the school marks: an *excellent* performance obviously is also *very good*, *good*, and *satisfactory*, but not *unsatisfactory* or a *fail*.

	⩽1	⩽2	⩽3	⩽4	⩾5	⩾6
1	×	×	×	×		
2		×	×	×		
3			×	×		
4				×		
5					×	
6					×	×



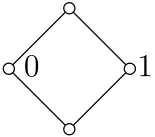
The Biordinal Scale $\mathbb{M}_{4,2}$.

Elementary Scales

The **Dichotomic Scale:** $\mathbb{D} := (\{0,1\},\{0,1\},=)$

The dichotomic scale constitutes a special case, since it is isomorphic to the scales $\mathbb{N}_2$ amd $\mathbb{M}_{1,1}$ and closely related to $\mathbb{I}_2$. It is frequently used to scale attributes with the values of the kind $\{$*yes*, *no*$\}$.

	0	1
0	×	
1		×



The Dichotomic Scale $\mathbb{D}$.

Elementary Scales

- frequent special case of scaling: all many-valued attributes are interpreted with respect to same scale (or family of scales)
- thus we speak of a *nominally scaled context*, if all scales $\mathbb{S}_m$ are nominal scales, etc.
- we call a many-valued context *nominal*, if the nature of the data suggests nominal scaling
- a many-valued context is called an *ordinal context* if for each attribute, the set of values is ordered in a natural way

Elementary Scales: Example "Forum Romanum"

	Forum Romanum	B	GB	M	P
1	Arch of Septimus Severus	*	*	**	*
2	Arch of Titus	*	**	**	
3	Basilica Julia			*	
4	Basilica of Maxentius	*			
5	Phocas column		*	**	
6	Curia				*
7	House of the Vestals			*	
8	Portico of Twelve Gods		*	*	*
9	Temple of Antonius and Fausta	*	*	***	*
10	Temple of Castor and Pollux	*	**	***	*
11	Temple of Romulus		*		
12	Temple of Saturn			**	*
13	Temple of Vespasian			**	
14	Temple of Vesta		**	**	*

Example of an ordinal context: Ratings of monuments on the Forum Romanum in different travel guides (B = Baedecker, GB = Les Guides Bleus, M = Michelin, P = Polyglott). The context becomes ordinal through the number of stars awarded. If no star has been awarded, this is rated zero.

Elementary Scales: Example "Forum Romanum"

	Forum Romanum	B	GB		M			GB
		*	*	**	*	**	***	*
1	Arch of Septimus Severus	×	×		×	×		×
2	Arch of Titus	×	×	×	×	×		
3	Basilica Julia				×			
4	Basilica of Maxentius	×						
5	Phocas column		×		×	×		
6	Curia							×
7	House of the Vestals				×			
8	Portico of Twelve Gods		×		×			×
9	Temple of Antonius and Fausta	×	×		×	×	×	×
10	Temple of Castor and Pollux	×	×	×	×	×	×	×
11	Temple of Romulus		×					
12	Temple of Saturn				×	×		×
13	Temple of Vespasian				×	×		
14	Temple of Vesta		×	×	×	×		×

Elementary Scales: Example "Forum Romanum"