

# **Exercise 7: Query Optimisation and First-Order Query Expressivity**

Database Theory

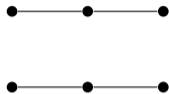
2023-05-23

Maximilian Marx, Markus Krötzsch

## Exercise 1

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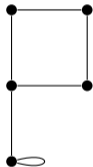
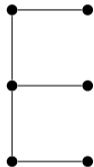
(i)



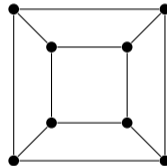
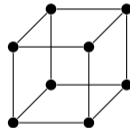
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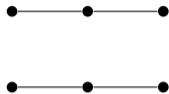
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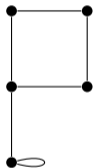
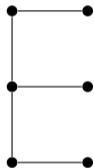
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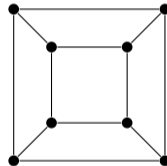
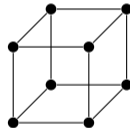
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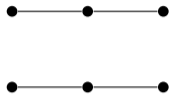


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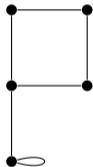
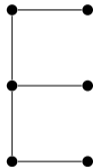
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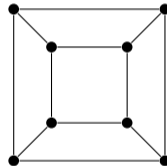
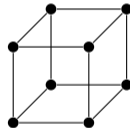
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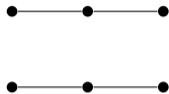
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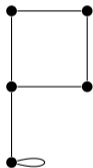
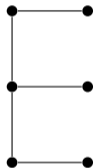
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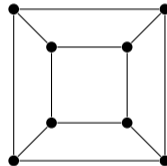
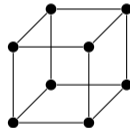
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**Solution.**

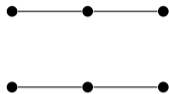
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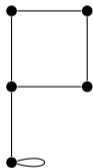
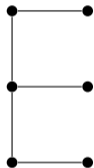
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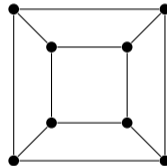
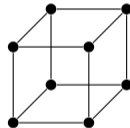
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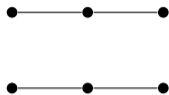
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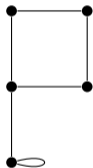
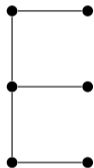
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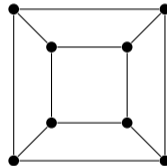
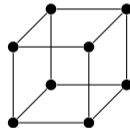
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**Solution.**

(i)  $r \leq 1$ ,

(ii)  $r \leq 2$ ,

(iii)  $r = 0$ , and

(iv)  $r \geq 0$ .

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**Exercise.** A *linear order* is a relational structure with one binary relational symbol  $\leq$  that is interpreted as a reflexive, asymmetric, transitive and total relation over the domain. Up to renaming of domain elements there is exactly one linear order for every finite domain, which can be depicted as a chain of elements. We denote the linear order of size  $n$  by  $\mathcal{L}_n$ . For example:

$$\mathcal{L}_6 : 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6$$

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*The following are equivalent:*

- ▶  $\mathcal{L}_m \sim_r \mathcal{L}_n$ , and
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### Solution.

1.  $r \leq 2$ .
2.  $n \geq 2^r - 1 \implies r \leq \lfloor \log_2(n + 1) \rfloor$ .

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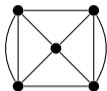


Figure: A

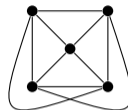


Figure: B

1. Can the graphs A and B be distinguished by a FO query?
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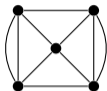


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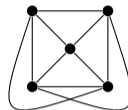


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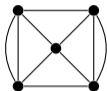


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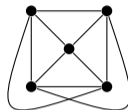


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1. This query matches B but not A:

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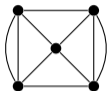


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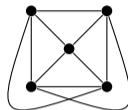


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2. For  $\varphi$  with quantifier rank  $r$ , consider counterexamples of size  $d = 3^r$ :

