Finite and algorithmic model theory: Lecture 5

Plan for today:

(1) Logical reductions

(2) Rank k-types

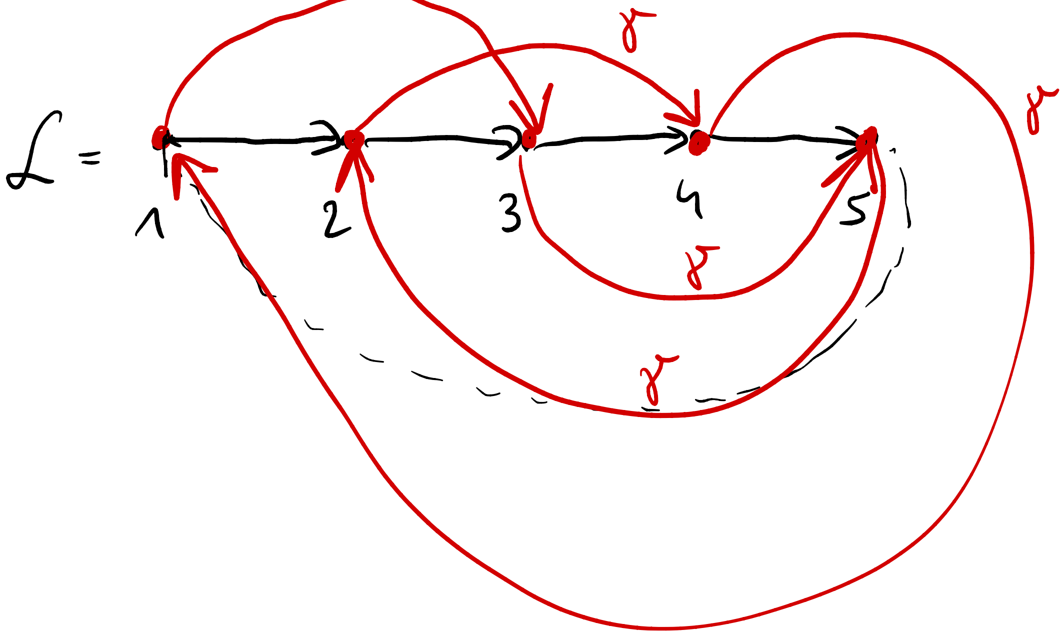
(3) Proof of E-F Games

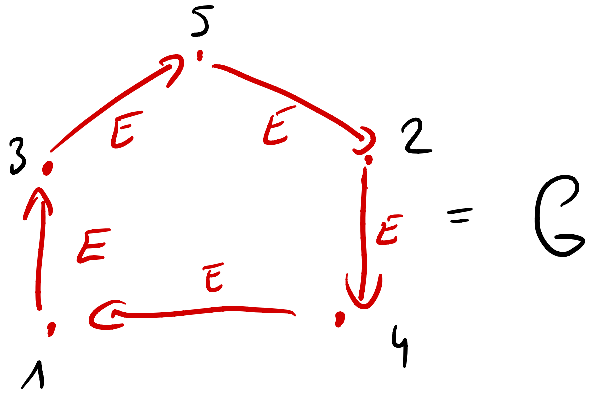
(4) Gaifman graphs and (r,t)-Hanf equivalence

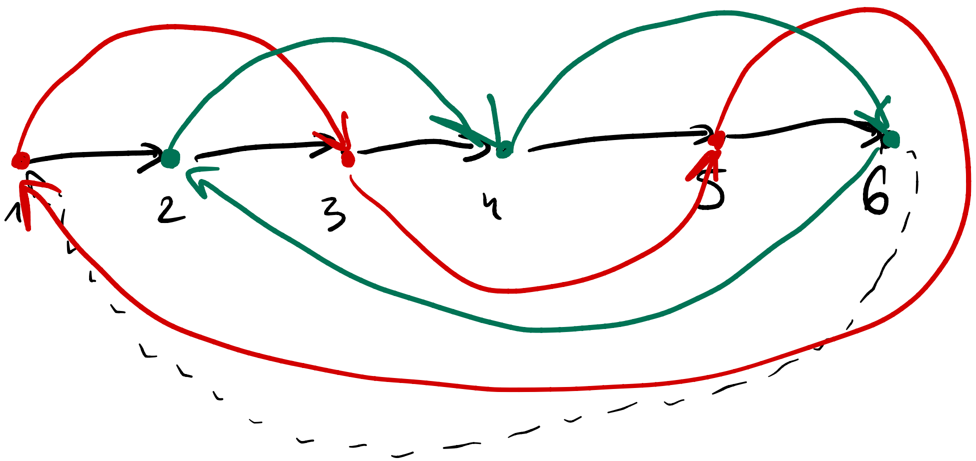
(5) How to use Hanf-locality instead of E-F games.

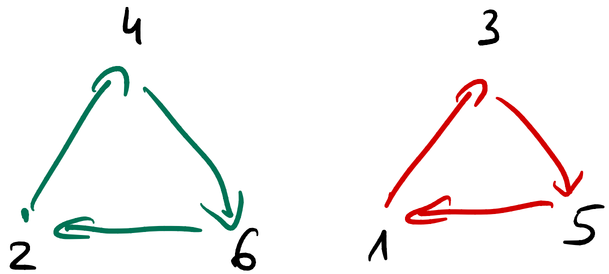
Lemma: Connectivity is not FO[{E}]-definable.

If $\mathcal{L}$ is of odd size <=> G is connected





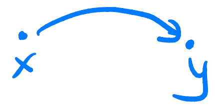




Proof: Assume that there is a formula $\varphi \in FO[\{E\}]$ that expresses connectivity.

Macro:

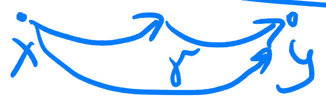
$$Succ(x,y) := x < y \wedge \neg \exists z\ z < y \wedge x < z$$



Define $\boxed{\gamma(x,y)}$ as follow:

- y is a succ of succ of x

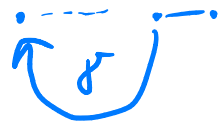
$$\boxed{\exists z\ succ(x,z) \wedge succ(z,y)} \vee$$



or

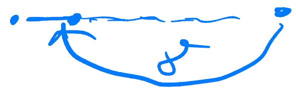
- x is second-to-last and y is the first elem

$$\boxed{\exists z \left(succ(x,z) \wedge \underbrace{\neg \exists t\ z < t}_{z \text{ is the last one}} \right) \wedge \left(\neg \exists t\ t < y \right)} \vee$$



- x is the last elem and y is the second one

$\boxed{\text{(analogous)}}$



Note that

$$\varphi\left[{E(x,y)}/{\gamma(x,y)} \right]$$ expresses that a linear order is of odd size

The formula $\neg \varphi[{E(x,y)}/{\gamma(x,y)}] \equiv \varphi_{even}$, which is not possible ↯

Rank k-type

$qr(\varphi)$ = # nested quantifiers

τ = finite signature $c_1, c_2, \dots, c_n, R_1, R_2, \dots, R_m$, =

What is $FO_0[\tau]$? $c_1 = c_2$, $c_1 = c_3$, $(\neg(c_1 = c_2) \vee c_1 = c_3) \wedge c_2 = c_4$

$c_1 = c_2$

c_1, c_2, S^1, R^2

$S(c_1)$, $S(c_2)$

$R(c_1, c_1)$, $R(c_1, c_2)$

$R(c_2, c_1)$, $R(c_2, c_2)$

$c_1 = c_2$, $c_1 \neq c_2$

$c_1 = c_2 \vee c_1 \neq c_2$ $c_1 = c_2 \wedge c_1 \neq c_2$

Lemma: There are only finitely many $FO_0[\tau]$- formulae.

How many $FO_n[\tau]$ formulae I have? Finitely many.

Assume that $|FO_n[\tau]|$ = finite

$$qr\left(\exists x\ \varphi \leftarrow |\varphi| \leq n\right) \leq n+1$$

Lemma: There are only finitely many $FO_k[\tau]$ formulae [up to equivalence].

$\mathfrak{A}$ - finite structure $\quad a \in A$ - an element

$$tp_k(\mathfrak{A}, a) = \{ \varphi(x) \mid \mathfrak{A} \models \varphi(a),\ qr(\varphi) \leq k \}$$

$\vec{a} \in A^n$ - an n-element tuple

rank k types

$$tp_k(\mathfrak{A}, \vec{a}) = \{ \varphi(\vec{x}) \mid \mathfrak{A} \models \varphi(\vec{a}),\ qr(\varphi) \leq k \}$$

Rank k-types are finite !

$$t = tp_k(\mathfrak{A}, a) \longleftrightarrow \varphi_t = \bigwedge_{\psi \in tp_k(\mathfrak{A}, a)} \psi \ \wedge \bigwedge_{\substack{\psi \notin tp_k(\mathfrak{A}, a) \\ \psi \in FO_k(\tau)}} \neg \psi$$

k-rank type of a in $\mathfrak{A}$ is t iff $\mathfrak{A} \models \varphi_t(a)$

Back - Forth equivalence between $\mathfrak{A}$ and $\mathfrak{B}$

$\mathfrak{A} \simeq_k \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ are back-and-forth k-equivalent.

1) $k = 0$ $\quad \mathfrak{A} \simeq_0 \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic sentences

2) $k > 0$ $\quad \mathfrak{A} \simeq_{k+1} \mathfrak{B}$ iff :

(forth) for all $a \in A$ there is $b \in B$ such that
$$(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$$

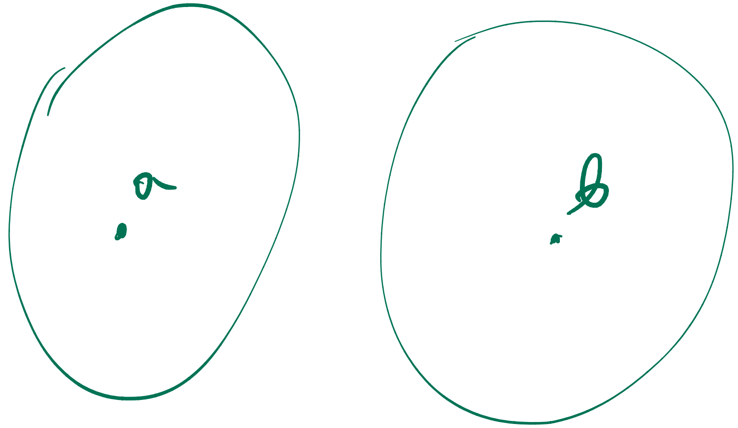
(back) for all $b \in B$ there is $a \in A$ such that
$$(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b).$$

Proof of E-F Games : the following conditions are equivalent :

(1) $\mathfrak{A}$ and $\mathfrak{B}$ agree on all formulae with $qr \leq m$.

(2) $\mathfrak{A} \equiv_m \mathfrak{B}$ (duplicator/$\forall$ has the winning strategy in E-F game)

(3) $\mathfrak{A} \simeq_m \mathfrak{B}$



Proof by induction on m.

1° $m = 0$ ✓

2° $m > 0$ (2) $\Leftrightarrow$ (3)

(3) $\Rightarrow$ (2)

Assume that $\mathfrak{A} \simeq_{m+1} \mathfrak{B}$. Goal: $\mathfrak{A} \equiv_{m+1} \mathfrak{B}$.

We play a game and spoiler selects a in $\mathfrak{A}$ and we need to reply with $b \in B$.

By (forth) we can find $b \in B$ such that $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$.

By inductive assumption $(\mathfrak{A}, a) \equiv_k (\mathfrak{B}, b)$, so we know how to play the game further.

3.3 Games and the Expressive Power of FO

And now it is time to see why games are important. For this, we need a crucial definition of quantifier rank.

Definition 3.8 (Quantifier rank). *The* quantifier rank *of a formula* $\mathsf{qr}(\varphi)$ *is its depth of quantifier nesting. That is:*

- *If* φ *is atomic, then* $\mathsf{qr}(\varphi) = 0$.
- $\mathsf{qr}(\varphi_1 \vee \varphi_2) = \mathsf{qr}(\varphi_1 \wedge \varphi_2) = \max(\mathsf{qr}(\varphi_1), \mathsf{qr}(\varphi_2))$.
- $\mathsf{qr}(\neg\varphi) = \mathsf{qr}(\varphi)$.
- $\mathsf{qr}(\exists x\varphi) = \mathsf{qr}(\forall x\varphi) = \mathsf{qr}(\varphi) + 1$.

We use the notation FO$[k]$ *for all* FO *formulae of quantifier rank up to* k.

In general, quantifier rank of a formula is different from the total of number of quantifiers used. For example, we can define a family of formulae by induction: $d_0(x,y) \equiv E(x,y)$, and $d_k \equiv \exists z\ d_{k-1}(x,z) \wedge d_{k-1}(z,y)$. The quantifier rank of d_k is k, but the total number of quantifiers used in d_k is $2^k - 1$. For formulae in the prenex form (i.e., all quantifiers are in front, followed by a quantifier-free formula), quantifier rank is the same as the total number of quantifiers.

Given a set S of FO sentences (over vocabulary σ), we say that two σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ *agree on* S if for every sentence Φ of S, it is the case that $\mathfrak{A} \models \Phi \Leftrightarrow \mathfrak{B} \models \Phi$.

Theorem 3.9 (Ehrenfeucht-Fraïssé). *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two structures in a relational vocabulary. Then the following are equivalent:*

1. $\mathfrak{A}$ *and* $\mathfrak{B}$ *agree on* FO$[k]$.
2. $\mathfrak{A} \equiv_k \mathfrak{B}$.

We will prove this theorem shortly, but first we discuss how this is useful for proving inexpressibility results.

Characterizing the expressive power of FO via games gives rise to the following methodology for proving inexpressibility results.

Corollary 3.10. *A property* $\mathcal{P}$ *of finite* σ*-structures is* not expressible *in* FO *if for every* $k \in \mathbb{N}$, *there exist two finite* σ*-structures,* $\mathfrak{A}_k$ *and* $\mathfrak{B}_k$, *such that:*

- $\mathfrak{A}_k \equiv_k \mathfrak{B}_k$, *and*
- $\mathfrak{A}_k$ *has property* $\mathcal{P}$, *and* $\mathfrak{B}_k$ *does not.*

Proof. Assume to the contrary that $\mathcal{P}$ is definable by a sentence Φ. Let $k = \mathsf{qr}(\Phi)$, and pick $\mathfrak{A}_k$ and $\mathfrak{B}_k$ as above. Then $\mathfrak{A}_k \equiv_k \mathfrak{B}_k$, and thus if $\mathfrak{A}_k$ has property $\mathcal{P}$, then so does $\mathfrak{B}_k$, which contradicts the assumptions. $\square$

We shall see in the next section that the *if* of Corollary 3.10 can be replaced by *iff*; that is, Ehrenfeucht-Fraïssé games are complete for first-order definability.

The methodology above extends from sentences to formulas with free variables.

Corollary 3.11. *An m-ary query Q on σ-structures is* not expressible *in* FO *iff for every $k \in \mathbb{N}$, there exist two finite σ-structures, $\mathfrak{A}_k$ and $\mathfrak{B}_k$, and two m-tuples $\vec{a}$ and $\vec{b}$ in them such that:*

- $(\mathfrak{A}_k, \vec{a}) \equiv_k (\mathfrak{B}_k, \vec{b})$, *and*
- $\vec{a} \in Q(\mathfrak{A}_k)$ *and* $\vec{b} \notin Q(\mathfrak{B}_k)$. $\square$

We next see some simple examples of using games; more examples will be given in Sect. 3.6. An immediate application of the Ehrenfeucht-Fraïssé theorem is that EVEN is not FO-expressible when σ is empty: we take $\mathfrak{A}_k$ to contain k elements, and $\mathfrak{B}_k$ to contain $k+1$ elements. However, we have already proved this by a simple compactness argument in Sect. 3.1. But we could not prove, by the same argument, that EVEN is not expressible over finite linear orders. Now we get this for free:

Corollary 3.12. EVEN *is not* FO*-expressible over linear orders.*

Proof. Pick $\mathfrak{A}_k$ to be a linear order of length 2^k, and $\mathfrak{B}_k$ to be a linear order of length $2^k + 1$. By Theorem 3.6, $\mathfrak{A}_k \equiv_k \mathfrak{B}_k$. The statement now follows from Corollary 3.10. $\square$

3.4 Rank-k Types

We now further analyze FO[k] and introduce the concept of types (more precisely, rank-k types).

First, what is FO[0]? It contains Boolean combinations of atomic formulas. If we are interested in sentences in FO[0], these are precisely *atomic* sentences: that is, sentences without quantifiers. In a relational vocabulary, such sentences are Boolean combinations of formulae of the form $c = c'$ and $R(c_1, \ldots, c_k)$, where $c, c', c_1, \ldots, c_k$ are constant symbols from σ.

Next, assume that φ is an FO[$k+1$] formula. If $\varphi = \varphi_1 \vee \varphi_2$, then both φ_1, φ_2 are FO[$k+1$] formulae, and likewise for $\wedge$; if $\varphi = \neg\varphi_1$, then $\varphi_1 \in$ FO[$k+1$]. However, if $\varphi = \exists x \psi$ or $\varphi = \forall x \psi$, then ψ is an FO[k] formula. Hence, every formula from FO[$k+1$] is equivalent to a Boolean combination of formulae of the form $\exists x \psi$, where $\psi \in$ FO[k]. Using this, we show:

Lemma 3.13. *If σ is finite, then up to logical equivalence,* FO[k] *over σ contains only finitely many formulae in m free variables $x_1, \ldots, x_m$.*

Proof. The proof is by induction on k. The base case is FO[0]; there are only finitely many atomic formulae, and hence only finitely many Boolean combinations of those, up to logical equivalence. Going from k to $k+1$, recall that each formula $\varphi(x_1, \ldots, x_m)$ from FO[$k+1$] is a Boolean combination of $\exists x_{m+1}\psi(x_1, \ldots, x_m, x_{m+1})$, where $\psi \in$ FO[k]. By the hypothesis, the number of FO[k] formulae in $m+1$ free variables $x_1, \ldots, x_{m+1}$ is finite (up to logical equivalence) and hence the same can be concluded about FO[$k+1$] formulas in m free variables. $\square$

In model theory, a *type* (or m-type) of an m-tuple $\vec{a}$ over a σ structure $\mathfrak{A}$ is the set of all FO formulae φ in m free variables such that $\mathfrak{A} \models \varphi(\vec{a})$. This notion is too general in our setting, as the type of $\vec{a}$ over a finite $\mathfrak{A}$ describes $(\mathfrak{A}, \vec{a})$ up to isomorphism.

Definition 3.14 (Types). *Fix a relational vocabulary σ. Let $\mathfrak{A}$ be a σ-structure, and $\vec{a}$ an m-tuple over A. Then the* rank-k m-type *of $\vec{a}$ over $\mathfrak{A}$ is defined as*

$$\mathrm{tp}_k(\mathfrak{A}, \vec{a}) = \{\varphi \in \mathrm{FO}[k] \mid \mathfrak{A} \models \varphi(\vec{a})\}.$$

A rank-k m-type *is any set of formulae of the form* $\mathrm{tp}_k(\mathfrak{A}, \vec{a})$*, where* $|\vec{a}| = m$*. When m is clear from the context, we speak of rank-k types.*

In the special case of $m = 0$ we deal with $\mathrm{tp}_k(\mathfrak{A})$, defined as the set of FO[k] sentences that hold in $\mathfrak{A}$. Also note that rank-k types are maximally consistent sets of formulae: that is, each rank-k type S is consistent, and for every $\varphi(x_1, \ldots, x_m) \in$ FO[k], either $\varphi \in S$ or $\neg\varphi \in S$.

At this point, it seems that rank-k types are inherently infinite objects, but they are not, because of Lemma 3.13. We know that up to logical equivalence, FO[k] is finite, for a fixed number m of free variables. Let $\varphi_1(\vec{x}), \ldots, \varphi_M(\vec{x})$ enumerate all the nonequivalent formulae in FO[k] with free variables $\vec{x} = (x_1, \ldots, x_m)$. Then a rank-$k$ type is uniquely determined by a subset K of $\{1, \ldots, M\}$ specifying which of the φ_i's belong to it. Moreover, testing that $\vec{x}$ satisfies all the φ_i's with $i \in K$ and does not satisfy all the φ_j's with $j \notin K$ can be done by a single formula

$$\alpha_K(\vec{x}) \equiv \bigwedge_{i \in K} \varphi_i \wedge \bigwedge_{j \notin K} \neg\varphi_j. \tag{3.3}$$

Note that $\alpha_K(\vec{x})$ itself is an FO[k] formula, since no new quantifiers were introduced.

Furthermore, all the α_K's are mutually exclusive: for $K \neq K'$, if $\mathfrak{A} \models \alpha_K(\vec{a})$, then $\mathfrak{A} \models \neg\alpha_{K'}(\vec{a})$. Every FO[$k$] formula is a disjunction of some of the α_K's: indeed, every FO[k] formula is equivalent to some φ_i in the above enumeration, which is the disjunction of all α_K's with $i \in K$.

Summing up, we have the following.

Theorem 3.15. *a) For a finite relational vocabulary σ, the number of different rank-k m-types is finite.*

b) Let $T_1, \ldots, T_r$ enumerate all the rank-k m-types. There exist FO[k] *formulae $\alpha_1(\vec{x}), \ldots, \alpha_r(\vec{x})$ such that:*

- *for every $\mathfrak{A}$ and $\vec{a} \in A^m$, it is the case that $\mathfrak{A} \models \alpha_i(\vec{a})$ iff $\mathrm{tp}_k(\mathfrak{A}, \vec{a}) = T_i$, and*
- *every* FO[k] *formula $\varphi(\vec{x})$ in m free variables is equivalent to a disjunction of some α_i's.*

Thus, in what follows we normally associate types with their defining formulae α_i's (3.3). It is important to remember that these defining formulae for rank-k types have the same quantifier rank, k.

From the Ehrenfeucht-Fraïssé theorem and Theorem 3.15, we obtain:

Corollary 3.16. *The equivalence relation $\equiv_k$ is of finite index (that is, has finitely many equivalence classes).*

As promised in the last section, we now show that games are complete for characterizing the expressive power of FO: that is, the *if* of Corollary 3.10 can be replaced by *iff*.

Corollary 3.17. *A property $\mathcal{P}$ is expressible in* FO *iff there exists a number k such that for every two structures $\mathfrak{A}, \mathfrak{B}$, if $\mathfrak{A} \in \mathcal{P}$ and $\mathfrak{A} \equiv_k \mathfrak{B}$, then $\mathfrak{B} \in \mathcal{P}$.*

Proof. If $\mathcal{P}$ is expressible by an FO sentence Φ, let $k = \mathsf{qr}(\Phi)$. If $\mathfrak{A} \in \mathcal{P}$, then $\mathfrak{A} \models \Phi$, and hence for $\mathfrak{B}$ with $\mathfrak{A} \equiv_k \mathfrak{B}$, we have $\mathfrak{B} \models \Phi$. Thus, $\mathfrak{B} \in \mathcal{P}$.

Conversely, if $\mathfrak{A} \in \mathcal{P}$ and $\mathfrak{A} \equiv_k \mathfrak{B}$ imply $\mathfrak{B} \in \mathcal{P}$, then any two structures with the same rank-k type agree on $\mathcal{P}$, and hence $\mathcal{P}$ is a union of types, and thus definable by a disjunction of some of the α_i's defined by (3.3). $\square$

Thus, a property $\mathcal{P}$ is *not* expressible in FO *iff* for every k, one can find two structures, $\mathfrak{A}_k \equiv_k \mathfrak{B}_k$, such that $\mathfrak{A}_k$ has $\mathcal{P}$ and $\mathfrak{B}_k$ does not.

3.5 Proof of the Ehrenfeucht-Fraïssé Theorem

We shall prove the equivalence of 1 and 2 in the Ehrenfeucht-Fraïssé theorem, as well as a new important condition, the *back-and-forth* equivalence. Before stating this condition, we briefly analyze the equivalence relation $\equiv_0$.

When does the duplicator win the game without even starting? This happens iff $(\emptyset, \emptyset)$ is a partial isomorphism between two structures $\mathfrak{A}$ and $\mathfrak{B}$. That is, if $\vec{c}$ is the tuple of constant symbols, then $c_i^{\mathfrak{A}} = c_j^{\mathfrak{A}}$ iff $c_i^{\mathfrak{B}} = c_j^{\mathfrak{B}}$ for every i, j, and for each relation symbol R, the tuple $(c_{i_1}^{\mathfrak{A}}, \ldots, c_{i_k}^{\mathfrak{A}})$ is in $R^{\mathfrak{A}}$ iff the tuple $(c_{i_1}^{\mathfrak{B}}, \ldots, c_{i_k}^{\mathfrak{B}})$ is in $R^{\mathfrak{B}}$. In other words, $(\emptyset, \emptyset)$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic sentences.

We now use this as the basis for the inductive definition of back-and-forth relations on $\mathfrak{A}$ and $\mathfrak{B}$. More precisely, we define a family of relations $\simeq_k$ on pairs of structures of the same vocabulary as follows:

- $\mathfrak{A} \simeq_0 \mathfrak{B}$ iff $\mathfrak{A} \equiv_0 \mathfrak{B}$; that is, $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic sentences.
- $\mathfrak{A} \simeq_{k+1} \mathfrak{B}$ iff the following two conditions hold:

forth: for every $a \in A$, there exists $b \in B$ such that $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$;
back: for every $b \in B$, there exists $a \in A$ such that $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$.

We now prove the following extension of Theorem 3.9.

Theorem 3.18. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures in a relational vocabulary σ. Then the following are equivalent:*

1. *$\mathfrak{A}$ and $\mathfrak{B}$ agree on* FO$[k]$.
2. *$\mathfrak{A} \equiv_k \mathfrak{B}$.*
3. *$\mathfrak{A} \simeq_k \mathfrak{B}$.*

Proof. By induction on k. The case of $k = 0$ is obvious. We first show the equivalence of 2 and 3. Going from k to $k+1$, assume $\mathfrak{A} \simeq_{k+1} \mathfrak{B}$; we must show $\mathfrak{A} \equiv_{k+1} \mathfrak{B}$. Assume for the first move the spoiler plays $a \in A$; we find $b \in \mathfrak{B}$ with $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$, and thus by the hypothesis $(\mathfrak{A}, a) \equiv_k (\mathfrak{B}, b)$. Hence the duplicator can continue to play for k moves, and thus wins the $k+1$-move game. The other direction is similar.

With games replaced by the back-and-forth relation, we show the equivalence of 1 and 3. Assume $\mathfrak{A}$ and $\mathfrak{B}$ agree on all quantifier-rank $k+1$ sentences; we must show $\mathfrak{A} \simeq_{k+1} \mathfrak{B}$. We prove the *forth* case; the *back* case is identical. Pick $a \in A$, and let α_i define its rank-k 1-type. Then $\mathfrak{A} \models \exists x \alpha_i(x)$. Since $\mathsf{qr}(\alpha_i) = k$, this is a sentence of quantifier-rank $k+1$; hence $\mathfrak{B} \models \exists x \alpha_i(x)$. Let b be the witness for the existential quantifier; that is, $\mathrm{tp}_k(\mathfrak{A}, a) = \mathrm{tp}_k(\mathfrak{B}, b)$. Hence for every σ_1 sentence Ψ of $\mathsf{qr}(\Psi) = k$, we have $(\mathfrak{A}, a) \models \Psi$ iff $(\mathfrak{B}, b) \models \Psi$, and thus $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ agree on quantifier-rank k sentences. By the hypothesis, this implies $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$.

For the implication $3 \to 1$, we need to prove that $\mathfrak{A} \simeq_{k+1} \mathfrak{B}$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ agree on FO$[k+1]$. Every FO$[k+1]$ sentence is a Boolean combination of $\exists x \varphi(x)$, where $\varphi \in$ FO$[k]$, so it suffices to prove the result for sentences of the form $\exists x \varphi(x)$. Assume that $\mathfrak{A} \models \exists x \varphi(x)$, so $\mathfrak{A} \models \varphi(a)$ for some $a \in A$. By **forth**, find $b \in B$ such that $(\mathfrak{A}, a) \simeq_k (\mathfrak{B}, b)$; hence $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ agree on FO$[k]$ by the hypothesis. Hence, $\mathfrak{B} \models \varphi(b)$, and thus $\mathfrak{B} \models \exists x \varphi(x)$. The converse (that $\mathfrak{B} \models \exists x \varphi(x)$ implies $\mathfrak{A} \models \exists x \varphi(x)$) is identical, which completes the proof. $\square$

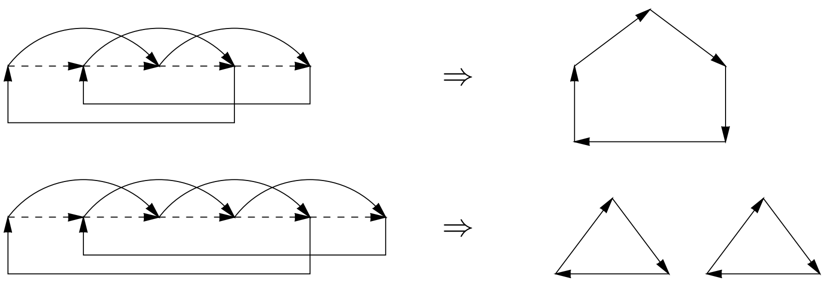


Fig. 3.3. Reduction of parity to connectivity

3.6 More Inexpressibility Results

So far we have used games to prove that EVEN is not expressible in FO, in both ordered and unordered settings. Next, we show inexpressibility of graph connectivity over finite graphs. In Sect. 3.1 we used compactness to show that connectivity of arbitrary graphs is inexpressible, leaving open the possibility that it may be FO-definable over finite graphs. We now show that this cannot happen. It turns out that no new game argument is needed, as the proof uses a reduction from EVEN over linear orders.

Assume that connectivity of finite graphs is definable by an FO sentence Φ, in the vocabulary that consists of one binary relation symbol E. Next, given a linear ordering, we define a directed graph from it as described below. First, from a linear ordering $<$ we define the successor relation

$$\mathrm{succ}(x, y) \;\equiv\; (x < y) \wedge \forall z\big((z \leq x) \vee (z \geq y)\big).$$

Using this, we define an FO formula $\gamma(x, y)$ such that $\gamma(x, y)$ is true iff one of the following holds:

- y is the successor of the successor of x: $\exists z\ \big(\mathrm{succ}(x, z) \wedge \mathrm{succ}(z, y)\big)$, or
- x is the predecessor of the last element, and y is the first element: $\big(\exists z\ (\mathrm{succ}(x, z) \wedge \forall u(u \leq z))\big) \wedge \forall u(y \leq u)$, or
- x is the last element and y is the successor of the first element (the FO formula is similar to the one above).

Thus, $\gamma(x, y)$ defines a new graph on the elements of the linear ordering; the construction is illustrated in Fig. 3.3.

Now observe that the graph defined by γ is connected iff the size of the underlying linear ordering is odd. Hence, taking $\neg\Phi$, and substituting γ for every occurrence of the predicate E, we get a sentence that tests EVEN for linear orderings. Since this is impossible, we obtain the following.

Corollary 3.19. *Connectivity of finite graphs is not* FO*-definable.*