TECHNISCHE UNIVERSITÄT DRESDEN

Artificial Intelligence, Computational Logic

PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 7 ASP II *slides adapted from Torsten Schaub [Gebser et al.(2012)]

Sarah Gaggl

Dresden, 20th May and 27th June 2019

DRESDEN concept Exzellenz aus Wissenschaft und Kultur

Agenda

1. Introduction
2. Constraint Satisfaction (CSP)
3. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
4. Local Search, Stochastic Hill Climbing, Simulated Annealing
5. Tabu Search
6. Answer-set Programming (ASP)
7. Structural Decomposition Techniques (Tree/Hypertree Decompositions)
8. Evolutionary Algorithms/ Genetic Algorithms

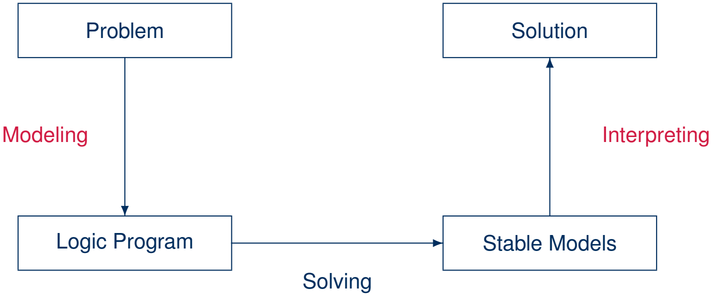
Overview ASP II

- Modeling
  1. Basic Modeling
  2. Methodology
- Language
  3. Motivation
  4. Core language
  5. Extended language
- Language Extensions
  6. Two kinds of negation
  7. Disjunctive logic programs
- Computational Aspects
  9. Complexity

Modeling: Overview

Outline

Modeling and Interpreting



Modeling

- For solving a problem class **C** for a problem instance **I**, encode
  1. the problem instance **I** as a set P_{I} of facts and
  2. the problem class **C** as a set P_{C} of rules

  such that the solutions to **C** for **I** can be (polynomially) extracted from the stable models of $P_{\mathsf{I}} \cup P_{\mathsf{C}}$

- P_{I} is (still) called problem instance
- P_{C} is often called the problem encoding

- An encoding P_{C} is uniform, if it can be used to solve all its problem instances
  That is, P_{C} encodes the solutions to **C** for any set P_{I} of facts

Outline

Basic methodology

Methodology

Generate and Test   (or: Guess and Check)

Generator	Generate potential stable model candidates (typically through non-deterministic constructs)
Tester	Eliminate invalid candidates (typically through integrity constraints)

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Nutshell

Logic program = Data + Generator + Tester (+ Optimizer)

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Satisfiability testing

- Problem Instance: A propositional formula ϕ in CNF
- Problem Class: Is there an assignment of propositional variables to true and false such that a given formula ϕ is true

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- Example: Consider formula

$$(a \vee \neg b) \wedge (\neg a \vee b)$$

- Logic Program:

Generator	Tester	Stable models
$\{a, b\} \leftarrow$	$\leftarrow \quad \mathit{not}\ a, b$	$X_1 = \{a, b\}$
	$\leftarrow \quad a, \mathit{not}\ b$	$X_2 = \{\}$

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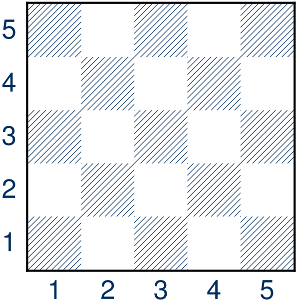
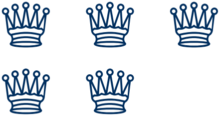
Generator		**Tester**		**Stable models**		
$\{a, b\}$	$\leftarrow$	$\leftarrow$	$not\ a, b$	X_1	=	$\{a, b\}$
		$\leftarrow$	$a, not\ b$	X_2	=	$\{\}$

Outline

The n-Queens Problem



- Place n queens on an $n \times n$ chess board
- Queens must not attack one another



Defining the Field

queens.lp

```
row(1..n).
col(1..n).
```

- Create file `queens.lp`
- Define the field
  - n rows
  - n columns

Defining the Field

Running ...

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
SATISFIABLE

Models       : 1
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Placing some Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
```

- Guess a solution candidate
  by placing some queens on the board

Placing some Queens

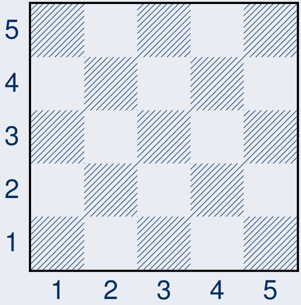
Running . . .

```
$ gringo queens.lp --const n=5 | clasp 3
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(1,1)
Answer: 3
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(2,1)
SATISFIABLE

Models      : 3+
...
```

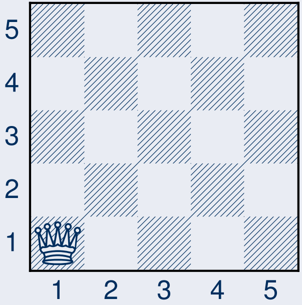
Placing some Queens: Answer 1

Answer 1



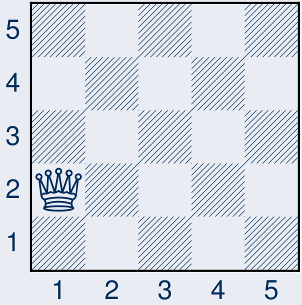
Placing some Queens: Answer 2

Answer 2



Placing some Queens: Answer 3

Answer 3



Placing n Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
```

- Place exactly n queens on the board

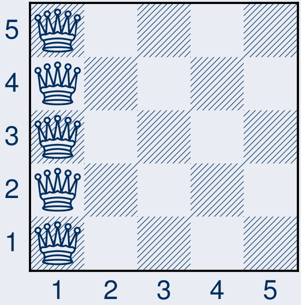
Placing n Queens

Running . . .

```
$ gringo queens.lp --const n=5 | clasp 2
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,1) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(1,2) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
...
```

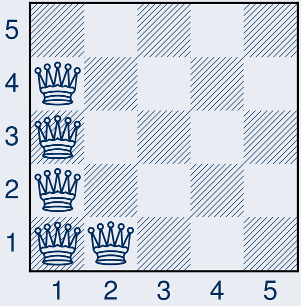
Placing n Queens: Answer 1

Answer 1



Placing n Queens: Answer 2

Answer 2



Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
```

- Forbid horizontal attacks

Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
```

- Forbid horizontal attacks
- Forbid vertical attacks

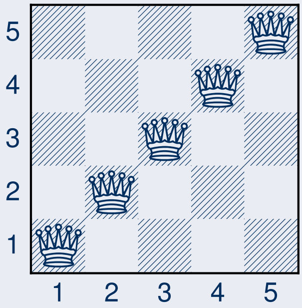
Horizontal and Vertical Attack

Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,5) queen(4,4) queen(3,3) \
queen(2,2) queen(1,1)
...
```

Horizontal and Vertical Attack: Answer 1

Answer 1



Diagonal Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I-J == I'-J'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I+J == I'+J'.
```

- Forbid diagonal attacks

Diagonal Attack

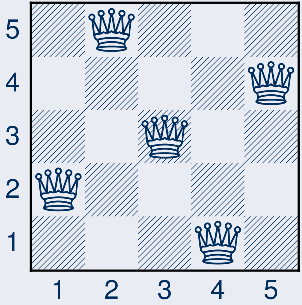
Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(4,5) queen(1,4) queen(3,3) queen(5,2) queen(2,1)
SATISFIABLE

Models       : 1+
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Diagonal Attack: Answer 1

Answer 1



Optimizing

queens-opt.lp

```
1 { queen(I,1..n) } 1 :- I = 1..n.
1 { queen(1..n,J) } 1 :- J = 1..n.
 :- 2 { queen(D-J,J) }, D =   2..2*n.
 :- 2 { queen(D+J,J) }, D = 1-n..n-1.
```

- Encoding can be optimized
- Much faster to solve

And sometimes it rocks

```
$ clingo -c n=5000 queens-opt-diag.lp -config=jumpy -q -stats=3
clingo version 4.1.0
Solving...
SATISFIABLE

Models       : 1+
Time         : 3758.143s (Solving: 1905.22s 1st Model: 1896.20s Unsat: 0.00s)
CPU Time     : 3758.320s

Choices      : 288594554
Conflicts    : 3442   (Analyzed: 3442)
Restarts     : 17     (Average: 202.47 Last: 3442)
Model-Level  : 7594728.0
Problems     : 1      (Average Length: 0.00 Splits: 0)
Lemmas       : 3442   (Deleted: 0)
  Binary     : 0      (Ratio:   0.00%)
  Ternary    : 0      (Ratio:   0.00%)
  Conflict   : 3442   (Average Length: 229056.5 Ratio: 100.00%)
  Loop       : 0      (Average Length:    0.0 Ratio:   0.00%)
  Other      : 0      (Average Length:    0.0 Ratio:   0.00%)

Atoms        : 75084857 (Original: 75069989 Auxiliary: 14868)
Rules        : 100129956 (1: 50059992/100090100 2: 39990/29856 3: 10000/10000)
Bodies       : 25090103
Equivalences : 125029999 (Atom=Atom: 50009999 Body=Body: 0 Other: 75020000)
Tight        : Yes
Variables    : 25024868 (Eliminated: 11781 Frozen: 25000000)
Constraints  : 66664  (Binary: 35.6% Ternary:  0.0% Other: 64.4%)

Backjumps    : 3442   (Average: 681.19 Max: 169512 Sum: 2344658)
  Executed   : 3442   (Average: 681.19 Max: 169512 Sum: 2344658 Ratio: 100.00%)
  Bounded    : 0      (Average:   0.00 Max:      0 Sum:       0 Ratio:   0.00%)
```

Outline

1. Basic Modeling

2. Methodology
   - Satisfiability
   - Queens
   - Traveling Salesperson

Traveling Salesperson

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).
```

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).

cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
cost(3,1,3).  cost(3,4,2).  cost(3,5,2).
cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).
```

Traveling Salesperson

```
node(1..6).



cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
cost(3,1,3).  cost(3,4,2).  cost(3,5,2).
cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).

edge(X,Y) :- cost(X,Y,_).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).

#minimize { C,X,Y : cycle(X,Y), cost(X,Y,C) }.
```

Language: Overview

3. Motivation
4. Core language
5. Extended language

Outline

Basic language extensions

- The expressiveness of a language can be enhanced by introducing new constructs
- To this end, we must address the following issues:
  - What is the syntax of the new language construct?
  - What is the semantics of the new language construct?
  - How to implement the new language construct?

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  - What is the semantics of the new language construct?
  - How to implement the new language construct?
- A way of providing semantics is to furnish a translation removing the new constructs, eg. classical negation
- This translation might also be used for implementing the language extension

Outline

Outline

Integrity constraint

- Idea Eliminate unwanted solution candidates
- Syntax An integrity constraint is of the form

$$\leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$
- Example `:- edge(3,7), color(3,red), color(7,red).`

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- Example `:- edge(3,7), color(3,red), color(7,red).`
- Embedding The above integrity constraint can be turned into the normal rule

$$x \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n, not\ x$$

  where x is a new symbol, that is, $x \notin \mathcal{A}$.

Outline

Choice rule

- Idea Choices over subsets
- Syntax A choice rule is of the form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o$$

where $0 \leq m \leq n \leq o$ and each a_i is an atom for $1 \leq i \leq o$

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- Example

```
{ buy(pizza); buy(wine); buy(corn) } :- at(grocery).
```

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```
- Another Example $P = \{\{a\} \leftarrow b,\ b \leftarrow\}$ has two stable models: $\{b\}$ and $\{a, b\}$

Embedding in normal rules

- A choice rule of form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o$$

can be translated into $2m + 1$ normal rules

$$\begin{array}{rcl} b & \leftarrow & a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o \end{array}$$

$$\begin{array}{rclcrcl} a_1 & \leftarrow & b, not\ a'_1 & \ldots & a_m & \leftarrow & b, not\ a'_m \\ a'_1 & \leftarrow & not\ a_1 & \ldots & a'_m & \leftarrow & not\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \ldots, a'_m$.

Embedding in normal rules

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$$b \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o$$

$$\begin{array}{lllllll} a_1 & \leftarrow & b, not\ a'_1 & \ldots & a_m & \leftarrow & b, not\ a'_m \\ a'_1 & \leftarrow & not\ a_1 & \ldots & a'_m & \leftarrow & not\ a_m \end{array}$$

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Embedding in normal rules

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$$\begin{array}{rcl} b & \leftarrow & a_{m+1}, \dots, a_n, not\ a_{n+1}, \dots, not\ a_o \\ a_1 & \leftarrow & b, not\ a'_1 \quad \dots \quad a_m \leftarrow b, not\ a'_m \\ a'_1 & \leftarrow & not\ a_1 \quad \dots \quad a'_m \leftarrow not\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \dots, a'_m$.

Outline

Cardinality rule

- Idea Control (lower) cardinality of subsets
- Syntax A cardinality rule is the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \}$$

where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
l is a non-negative integer.

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- Note l acts as a lower bound on the body

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- Example

```
pass(c42) :- 2 { pass(a1); pass(a2); pass(a3) }.
```

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- Example
  ```
  pass(c42) :- 2 { pass(a1); pass(a2); pass(a3) }.
  ```
- Another Example $P = \{a \leftarrow 1\{b, c\}, \; b \leftarrow\}$ has stable model $\{a, b\}$

Embedding in normal rules

- Replace each cardinality rule

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

by $a_0 \leftarrow ctr(1, l)$

where atom $ctr(i, j)$ represents the fact that at least j of the literals having an equal or greater index than i, are in a stable model

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$$\begin{array}{rcll} ctr(i, k{+}1) & \leftarrow & ctr(i+1, k), a_i & \\ ctr(i, k) & \leftarrow & ctr(i+1, k) & \text{for } 1 \leq i \leq m \\ ctr(j, k{+}1) & \leftarrow & ctr(j+1, k), not\ a_j & \\ ctr(j, k) & \leftarrow & ctr(j+1, k) & \text{for } m+1 \leq j \leq n \\ ctr(n+1, 0) & \leftarrow & & \end{array}$$

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- Program $\{a \leftarrow,\ c \leftarrow 1\ \{a, b\}\}$ has the stable model $\{a, c\}$

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... and vice versa

- A normal rule

$$a_0 \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n$$

can be represented by the cardinality rule

$$a_0 \leftarrow n\ \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\}$$

Cardinality rules with upper bounds

- A rule of the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, \mathit{not}\; a_{m+1}, \ldots, \mathit{not}\; a_n \; \} \; u \qquad (1)$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
  l and u are non-negative integers

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stands for

$$\begin{array}{rcl} a_0 & \leftarrow & b, \mathit{not} \; c \\ b & \leftarrow & l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \\ c & \leftarrow & u{+}1 \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \end{array}$$

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- Note The single constraint in the body of the cardinality rule (1) is referred to as a cardinality constraint

Cardinality constraints

- Syntax A cardinality constraint is of the form

$$l\ \{\ a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n\ \}\ u$$

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- Informal meaning A cardinality constraint is satisfied by a stable model X, if the number of its contained literals satisfied by X is between l and u (inclusive)

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- In other words, if

$$l \leq | \, (\{a_1, \ldots, a_m\} \cap X) \cup (\{a_{m+1}, \ldots, a_n\} \setminus X) \, | \leq u$$

Cardinality constraints as heads

- A rule of the form

  $$l\ \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, not\ a_{o+1}, \ldots, not\ a_p$$

  where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $1 \leq i \leq p$;
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- Example `1{ color(v42,red); color(v42,green); color(v42,blue) }1.`

Outline

Weight rule

- Syntax A weight rule is the form

$$a_0 \leftarrow l \; \{ \; w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not \; a_{m+1}, \ldots, w_n : not \; a_n \; \}$$

  where $0 \leq m \leq n$ and each a_i is an atom;
  l and w_i are integers for $1 \leq i \leq n$
- A weighted literal $w_i : \ell_i$ associates each literal ℓ_i with a weight w_i

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- Note A cardinality rule is a weight rule where $w_i = 1$ for $0 \leq i \leq n$

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- Meaning A weight constraint is satisfied by a stable model X, if

$$l \leq \left(\sum_{1 \leq i \leq m, a_i \in X} w_i + \sum_{m < i \leq n, a_i \notin X} w_i \right) \leq u$$

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- Example

```
10 { 4:course(db); 6:course(ai); 8:course(project); 3:course(xml) } 20
```

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$$\ell : \ell_1, \ldots, \ell_n$$

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- Informal meaning A conditional literal can be regarded as the list of elements in the set $\{\ell \mid \ell_1, \ldots, \ell_n\}$
- Note The expansion of conditional literals is context dependent
- Example Given '`p(1..3).  q(2).`'

```
r(X):p(X),not q(X) :- r(X):p(X),not q(X); 1{r(X):p(X),not q(X)}.
```

  is instantiated to

```
r(1); r(3) :- r(1), r(3), 1 { r(1), r(3) }.
```

Outline

Optimization statement

- Idea Express (multiple) cost functions subject to minimization and/or maximization
- Syntax A minimize statement is of the form

$$\textit{minimize}\ \{\ w_1@p_1 : \ell_1, \ldots, w_n@p_n : \ell_n\ \}.$$

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  Priority levels, p_i, allow for representing lexicographically ordered minimization objectives

- Meaning A minimize statement is a directive that instructs the ASP solver to compute optimal stable models by minimizing a weighted sum of elements

Optimization statement

- A maximize statement of the form

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- Example When configuring a computer, we may want to maximize hard disk capacity, while minimizing price

```
#maximize { 250@1:hd(1), 500@1:hd(2), 750@1:hd(3), 1000@1:hd(4) }.
#minimize { 30@2:hd(1), 40@2:hd(2), 60@2:hd(3), 80@2:hd(4) }.
```

  The priority levels indicate that (minimizing) price is more important than (maximizing) capacity

Language Extensions: Overview

Outline

Motivation

- Classical versus default negation
  - Symbol $\neg$ and not

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  - Idea
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    - $cross \leftarrow \neg train$
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- Given a program P over $\mathcal{A}$, classical negation is encoded by adding

$$P^{\neg} = \{a \leftarrow b, \neg b \mid a \in (\mathcal{A} \cup \overline{\mathcal{A}}), b \in \mathcal{A}\}$$

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- A set X of atoms is a stable model of a program P over $\mathcal{A} \cup \overline{\mathcal{A}}$, if X is a stable model of $P \cup P^{\neg}$

An example

- The program

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- The stable models of P are given by the ones of $P \cup P^{\neg}$, viz $\{a\}$

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- Note Strictly speaking, an inconsistent set like $\mathcal{A} \cup \overline{\mathcal{A}}$ is not a model
- For a logic program P over $\mathcal{A} \cup \overline{\mathcal{A}}$, exactly one of the following two cases applies:
  1. All stable models of P are consistent or
  2. $X = \mathcal{A} \cup \overline{\mathcal{A}}$ is the only stable model of P

Train spotting

- $P_1 = \{cross \leftarrow not\ train\}$
- $P_2 = \{cross \leftarrow \neg train\}$
- $P_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$
- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$
- $P_5 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train\}$
- $P_6 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train,\ \neg cross \leftarrow\}$

Train spotting

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  - stable model: $\{cross\}$

Train spotting

- $P_2 = \{cross \leftarrow \neg train\}$

Train spotting

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Train spotting

- $P_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$

Train spotting

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  - stable model: $\{cross, \neg train\}$

Train spotting

- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$

Train spotting

- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$
  - stable model: $\{cross, \neg cross, train, \neg train\}$ inconsistent as $\mathcal{A} \cup \bar{\mathcal{A}}$

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- $P_5 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train\}$

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Train spotting

- $P_6 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train,\ \neg cross \leftarrow\}$

Train spotting

- $P_6 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train,\ \neg cross \leftarrow\}$
  - no stable model

Train spotting

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  - stable model: $\{cross\}$
- $P_2 = \{cross \leftarrow \neg train\}$
  - stable model: $\emptyset$
- $P_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$
  - stable model: $\{cross, \neg train\}$
- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$
  - stable model: $\{cross, \neg cross, train, \neg train\}$ inconsistent as $\mathcal{A} \cup \bar{\mathcal{A}}$
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- Given a program P over $\mathcal{A}$, consider the program

$$\begin{aligned} \widetilde{P} \quad = \quad & \{r \in P \mid head(r) \neq not\ a\} \\ & \cup \{\leftarrow body(r) \cup \{not\ \widetilde{a}\} \mid r \in P \text{ and } head(r) = not\ a\} \\ & \cup \{\widetilde{a} \leftarrow not\ a \mid r \in P \text{ and } head(r) = not\ a\} \end{aligned}$$

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$$
\begin{array}{rcl}
\widetilde{P} & = & \{r \in P \mid \mathit{head}(r) \neq \mathit{not}\ a\} \\
& & \cup\ \{\leftarrow \mathit{body}(r) \cup \{\mathit{not}\ \widetilde{a}\} \mid r \in P \text{ and } \mathit{head}(r) = \mathit{not}\ a\} \\
& & \cup\ \{\widetilde{a} \leftarrow \mathit{not}\ a \mid r \in P \text{ and } \mathit{head}(r) = \mathit{not}\ a\}
\end{array}
$$

- A set X of atoms is a stable model of a program P (with default negation in rule heads) over $\mathcal{A}$,
  if $X = Y \cap \mathcal{A}$ for some stable model Y of $\widetilde{P}$ over $\mathcal{A} \cup \widetilde{\mathcal{A}}$

Outline

Disjunctive logic programs

- A disjunctive rule, r, is of the form

$$a_1 \ ; \ldots \ ; a_m \leftarrow a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o$$

  where $0 \leq m \leq n \leq o$ and each a_i is an atom for $0 \leq i \leq o$
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- A disjunctive logic program is a finite set of disjunctive rules
- Notation

$$\begin{array}{rcl}
head(r) & = & \{a_1, \ldots, a_m\} \\
body(r) & = & \{a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o\} \\
body(r)^+ & = & \{a_{m+1}, \ldots, a_n\} \\
body(r)^- & = & \{a_{n+1}, \ldots, a_o\} \\
atom(P) & = & \bigcup_{r \in P} \left(head(r) \cup body(r)^+ \cup body(r)^- \right) \\
body(P) & = & \{body(r) \mid r \in P\}
\end{array}$$

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\mathit{body}(P) & = & \{\mathit{body}(r) \mid r \in P\}
\end{array}$$

- A program is called positive if $\mathit{body}(r)^- = \emptyset$ for all its rules

Stable models

- Positive programs
  - A set X of atoms is closed under a positive program P iff for any $r \in P$, $head(r) \cap X \neq \emptyset$ whenever $body(r)^+ \subseteq X$
    - X corresponds to a model of P (seen as a formula)
  - The set of all $\subseteq$-minimal sets of atoms being closed under a positive program P is denoted by $\min_\subseteq(P)$
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  - A set X of atoms is a stable model of a disjunctive program P, if $X \in \min_\subseteq(P^X)$

A “positive” example

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- The sets $\{a, b\}$, $\{a, c\}$, and $\{a, b, c\}$ are closed under P
- We have $\min_{\subseteq}(P) = \{\{a, b\}, \{a, c\}\}$

Graph coloring (reloaded)

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).


color(X,r) ; color(X,b) ; color(X,g) :- node(X).


:- edge(X,Y), color(X,C), color(Y,C).
```

Graph coloring (reloaded)

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
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col(r).   col(b).   col(g).

color(X,C) : col(C) :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

More Examples

- $P_1 = \{a\,;b\,;c \leftarrow\}$

More Examples

- $P_1 = \{a \,;b \,;c \leftarrow\}$
  - stable models $\{a\}$, $\{b\}$, and $\{c\}$

More Examples

- $P_2 = \{a\,; b\,; c \leftarrow\,,\ \leftarrow a\}$

More Examples

- $P_2 = \{a\,; b\,; c \leftarrow\,,\ \leftarrow a\}$
  - stable models $\{b\}$ and $\{c\}$

More Examples

- $P_3 = \{a \,; b \,; c \leftarrow \,, \,\leftarrow a \,, \; b \leftarrow c \,, \; c \leftarrow b\}$

More Examples

- $P_3 = \{a\,;b\,;c \leftarrow\,,\ \leftarrow a\,,\ b \leftarrow c\,,\ c \leftarrow b\}$
  - stable model $\{b, c\}$

More Examples

- $P_4 = \{a \,; b \leftarrow c \;,\; b \leftarrow not\ a, not\ c \;,\; a \,; c \leftarrow not\ b\}$

More Examples

- $P_4 = \{a\,; b \leftarrow c\ ,\ b \leftarrow not\ a, not\ c\ ,\ a\,; c \leftarrow not\ b\}$
  - stable models $\{a\}$ and $\{b\}$

More Examples

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- $P_2 = \{a \,;b \,;c \leftarrow \,,\ \leftarrow a\}$
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  - stable models $\{a\}$ and $\{b\}$

Some properties

- A disjunctive logic program may have zero, one, or multiple stable models
- If X is a stable model of a disjunctive logic program P, then X is a model of P (seen as a formula)
- If X and Y are stable models of a disjunctive logic program P, then $X \not\subset Y$

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  then X is a model of P (seen as a formula)
- If X and Y are stable models of a disjunctive logic program P,
  then $X \not\subset Y$
- If $A \in X$ for some stable model X of a disjunctive logic program P, then
  there is a rule $r \in P$ such that
  $body(r)^+ \subseteq X$, $body(r)^- \cap X = \emptyset$, and $head(r) \cap X = \{A\}$

An example with variables

$$P = \left\{ \begin{array}{lll} a(1,2) & \leftarrow & \\ b(X)\ ;c(Y) & \leftarrow & a(X,Y), not\ c(Y) \end{array} \right\}$$

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$$\mathit{ground}(P) = \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1), \mathit{not}\ c(1) \\ b(1)\ ; c(2) & \leftarrow & a(1,2), \mathit{not}\ c(2) \\ b(2)\ ; c(1) & \leftarrow & a(2,1), \mathit{not}\ c(1) \\ b(2)\ ; c(2) & \leftarrow & a(2,2), \mathit{not}\ c(2) \end{array} \right\}$$

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For every stable model X of P, we have

- $a(1,2) \in X$ and
- $\{a(1,1), a(2,1), a(2,2)\} \cap X = \emptyset$

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- Consider $X = \{a(1,2), b(1)\}$

An example with variables

$$
ground(P)^X \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1) \\ b(1)\ ; c(2) & \leftarrow & a(1,2) \\ b(2)\ ; c(1) & \leftarrow & a(2,1) \\ b(2)\ ; c(2) & \leftarrow & a(2,2) \end{array} \right\}
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- Consider $X = \{a(1,2), b(1)\}$
- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2), b(1)\},\ \{a(1,2), c(2)\}\ \}$

An example with variables

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- Consider $X = \{a(1,2), b(1)\}$
- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2), b(1)\},\ \{a(1,2), c(2)\}\ \}$
- X is a stable model of P because $X \in \min_{\subseteq}(ground(P)^X)$

An example with variables

$$ground(P) = \left\{ \begin{array}{lll} a(1,2) & \leftarrow & \\ b(1)\ ;c(1) & \leftarrow & a(1,1), not\ c(1) \\ b(1)\ ;c(2) & \leftarrow & a(1,2), not\ c(2) \\ b(2)\ ;c(1) & \leftarrow & a(2,1), not\ c(1) \\ b(2)\ ;c(2) & \leftarrow & a(2,2), not\ c(2) \end{array} \right\}$$

An example with variables

$$ground(P) = \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1), not\ c(1) \\ b(1)\ ; c(2) & \leftarrow & a(1,2), not\ c(2) \\ b(2)\ ; c(1) & \leftarrow & a(2,1), not\ c(1) \\ b(2)\ ; c(2) & \leftarrow & a(2,2), not\ c(2) \end{array} \right\}$$

- Consider $X = \{a(1,2), c(2)\}$

An example with variables

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ground(P)^X = \left\{ \begin{array}{lll} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1) \\ & & \\ b(2)\ ; c(1) & \leftarrow & a(2,1) \\ & & \end{array} \right\}
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- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2)\}\ \}$

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$$

- Consider $X = \{a(1,2), c(2)\}$
- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2)\}\ \}$
- X is no stable model of P because $X \notin \min_{\subseteq}(ground(P)^X)$

Default negation in rule heads

- Consider disjunctive rules of the form

$$a_1 \; ; \ldots \; ; a_m \; ; not \; a_{m+1} \; ; \ldots \; ; not \; a_n \leftarrow a_{n+1}, \ldots, a_o, not \; a_{o+1}, \ldots, not \; a_p$$

where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $0 \leq i \leq p$

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  where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $0 \leq i \leq p$
- Given a program P over $\mathcal{A}$, consider the program

$$\begin{aligned} \widetilde{P} \quad = \quad & \{head(r)^+ \leftarrow body(r) \cup \{not \ \widetilde{a} \mid a \in head(r)^-\} \mid r \in P\} \\ & \cup \{\widetilde{a} \leftarrow not \ a \mid r \in P \text{ and } a \in head(r)^-\} \end{aligned}$$

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- Consider disjunctive rules of the form

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- A set X of atoms is a stable model of a disjunctive program P (with default negation in rule heads) over $\mathcal{A}$, if $X = Y \cap \mathcal{A}$ for some stable model Y of $\widetilde{P}$ over $\mathcal{A} \cup \widetilde{\mathcal{A}}$

An example

- The program

$$P = \{a \; ; \; not \; a \leftarrow\}$$

An example

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$$P = \{a \,;\, not\ a \leftarrow\}$$

yields

$$\widetilde{P} = \{a \leftarrow not\ \widetilde{a}\} \cup \{\widetilde{a} \leftarrow not\ a\}$$

An example

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$$P = \{a \,;\, \mathit{not}\ a \leftarrow\}$$
yields
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- $\widetilde{P}$ has two stable models, $\{a\}$ and $\{\widetilde{a}\}$

An example

- The program
$$P = \{a \; ; \; not \; a \leftarrow\}$$
yields
$$\widetilde{P} = \{a \leftarrow not \; \widetilde{a}\} \cup \{\widetilde{a} \leftarrow not \; a\}$$
- $\widetilde{P}$ has two stable models, $\{a\}$ and $\{\widetilde{a}\}$
- This induces the stable models $\{a\}$ and $\emptyset$ of P

Computational Aspects: Overview

8 Complexity

Outline

8 Complexity

Complexity

Let a be an atom and X be a set of atoms

Complexity

Let a be an atom and X be a set of atoms

- For a positive normal logic program P:
  - Deciding whether X is the stable model of P is P-complete
  - Deciding whether a is in the stable model of P is P-complete

Complexity

Let a be an atom and X be a set of atoms

- For a positive normal logic program P:
  - Deciding whether X is the stable model of P is P-complete
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- For a normal logic program P:
  - Deciding whether X is a stable model of P is P-complete
  - Deciding whether a is in a stable model of P is NP-complete

Complexity

Let a be an atom and X be a set of atoms

- For a positive normal logic program P:
  - Deciding whether X is the stable model of P is P-complete
  - Deciding whether a is in the stable model of P is P-complete
- For a normal logic program P:
  - Deciding whether X is a stable model of P is P-complete
  - Deciding whether a is in a stable model of P is NP-complete
- For a normal logic program P with optimization statements:
  - Deciding whether X is an optimal stable model of P is co-NP-complete
  - Deciding whether a is in an optimal stable model of P is Δ_2^p-complete

Complexity

Let a be an atom and X be a set of atoms

- For a positive disjunctive logic program P:
  - Deciding whether X is a stable model of P is co-NP-complete
  - Deciding whether a is in a stable model of P is NP^{NP}-complete
- For a disjunctive logic program P:
  - Deciding whether X is a stable model of P is co-NP-complete
  - Deciding whether a is in a stable model of P is NP^{NP}-complete
- For a disjunctive logic program P with optimization statements:
  - Deciding whether X is an optimal stable model of P is co-NP^{NP}-complete
  - Deciding whether a is in an optimal stable model of P is Δ_3^p-complete

Complexity

Let a be an atom and X be a set of atoms

- For a positive disjunctive logic program P:
  - Deciding whether X is a stable model of P is co-NP-complete
  - Deciding whether a is in a stable model of P is NP^{NP}-complete
- For a disjunctive logic program P:
  - Deciding whether X is a stable model of P is co-NP-complete
  - Deciding whether a is in a stable model of P is NP^{NP}-complete
- For a disjunctive logic program P with optimization statements:
  - Deciding whether X is an optimal stable model of P is co-NP^{NP}-complete
  - Deciding whether a is in an optimal stable model of P is Δ_3^p-complete
- For a propositional theory Φ:
  - Deciding whether X is a stable model of Φ is co-NP-complete
  - Deciding whether a is in a stable model of Φ is NP^{NP}-complete

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- See also: `http://potassco.sourceforge.net`