Finite and Algorithmic Model Theory

Lecture 2 (Dresden 19.10.22, Short version)

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Today's agenda

Proviso: For simplicity we focus on finite, simple, undirected graphs today!

1. Asymptotic probabilities with examples.

Different perspective: What percentage of graphs verify a given FO sentence?

2. Zero-One Law of FO $=$ Probability that a random structure satisfies φ is always 0 or 1.

3. Proof of the Zero-One Law for FO, based on Grädel's notes [HERE].

- Atomic k-types and extensions axioms. Theory $\mathbb{EA}$ of extension axioms.
- Each extension axiom is almost surely true.
- $\mathbb{EA}$ is ω-categorical, i.e. has exactly one countable model up to $\cong$, the Rado graph (the random graph).
- $\mathbb{EA}$ is complete, i.e. for all $\varphi \in$ FO we have $\mathbb{EA} \models \varphi$ or $\mathbb{EA} \models \neg\varphi$.

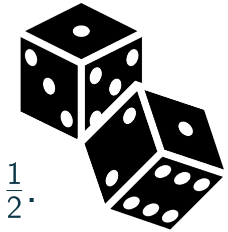
Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

Asymptotic Probabilities

Proviso: For simplicity we focus on finite, simple, undirected graphs today!



We consider random graphs, according to the uniform distribution, i.e. every edge has probability $\frac{1}{2}$.

Let $\mathcal{G}_n$ be the class of simple undirected graphs with n nodes. Of course $|\mathcal{G}_n| = 2^{\frac{n(n-1)}{2}}$.

Let $\mathcal{P}$ be a property of graphs. Let $\mu_n(\mathcal{P})$ = "probability that $\mathcal{P}$ holds in a random graph with n nodes".

$$\mu_n(\mathcal{P}) := \frac{|\{\mathfrak{G} \in \mathcal{G}_n : \mathfrak{G} \models \mathcal{P}\}|}{|\mathcal{G}_n|}$$

Asymptotic probability

$$\mu_\infty(\mathcal{P}) := \lim_{n\to\infty} \mu_n(\mathcal{P})$$

Examples

1. Take $\mathcal{P}$:= "the graph is complete". Then $\mu_\infty(\mathcal{P}) = \lim_{n\to\infty} \frac{|\{\mathfrak{G}\in\mathcal{G}_n : \mathfrak{G}\models\mathcal{P}\}|}{|\mathcal{G}_n|} = \lim_{n\to\infty} \frac{1}{2^{\frac{n(n-1)}{2}}} = 0$.

2. Take $\mathcal{P}$:= "the graph has a triangle". $\mu_3(\mathcal{P}) = \frac{1}{8}$. Since $\mu_{3n}(\mathcal{P}) \geq 1 - (1-\frac{1}{8})^n$, we get $\mu_\infty(\mathcal{P}) = 1$.

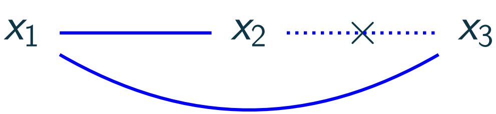
3. Take $\mathcal{P}$:= "the graph has even number of edges". $\mu_\infty(\mathcal{P}) = \frac{1}{2}$. Why?

$$\mu_\infty(\mathcal{P}) = \frac{|\{\mathfrak{G} \in \mathcal{G}_n : \mathfrak{G} \models \mathcal{P}\}|}{2^{\frac{n(n-1)}{2}}} = \frac{\sum_{i\geq 0} \binom{n(n-1)/2}{2i}}{2^{\frac{n(n-1)}{2}}} = [\texttt{Sum of Even Index Binomial Coeff.}] = \frac{2^{\frac{n(n-1)}{2}-1}}{2^{\frac{n(n-1)}{2}}} = \frac{1}{2}$$

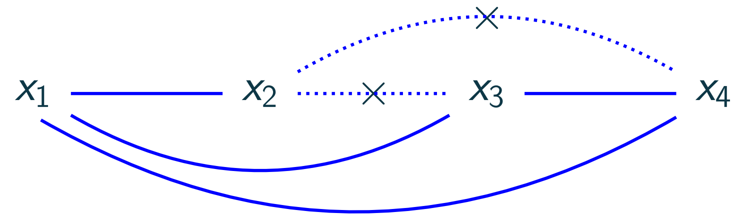
4. Take $\mathcal{P}$:= "the graph has even number of nodes". Then $\mu_\infty(\mathcal{P})$ does not exist.

A k-type is a conjunction of formulae with variables $x_1, \ldots, x_k$ such that for all $i \neq j$ we have

$$x_i \neq x_j \text{ and precisely one of } \mathrm{E}(x_i, x_j) \text{ or } \neg\mathrm{E}(x_i, x_j) \text{ as a conjunct.}$$



$\subseteq$



$s := x_1{\neq}x_2 \wedge x_1{\neq}x_3 \wedge x_2{\neq}x_3 \wedge$
$\mathrm{E}(x_1, x_2) \wedge \neg\mathrm{E}(x_2, x_3) \wedge \mathrm{E}(x_1, x_3)$

$$t := x_1{\neq}x_2 \wedge x_1{\neq}x_3 \wedge x_2{\neq}x_3 \wedge \mathrm{E}(x_1, x_2) \wedge \neg\mathrm{E}(x_2, x_3) \wedge \mathrm{E}(x_1, x_3)$$
$$\wedge x_1{\neq}x_4 \wedge x_2{\neq}x_4 \wedge x_3{\neq}x_4 \wedge \mathrm{E}(x_1, x_4) \wedge \neg\mathrm{E}(x_2, x_4) \wedge \mathrm{E}(x_3, x_4)$$

A $(k+1)$-type t extends a k-type s if conjuncts$(s) \subseteq$ conjuncts(t) (c.f. the above picture).

An (s, t)-extension axiom $\sigma_{s,t}$ is $\forall x_1 \ldots \forall x_k \; s(x_1, \ldots, x_k) \to \exists x_{k+1} t(x_1, \ldots, x_k, x_{k+1})$.

$$\mathbb{EA} := \Big\{ \forall x \; \neg\mathrm{E}(x, x), \; \forall xy \; \mathrm{E}(x, y) \to \mathrm{E}(y, x), \; \sigma_{s,t} \mid s \text{ is } k\text{-type}, \; t \text{ is } (k{+}1)\text{-type}, \; t \text{ extends } s \Big\}$$

Why the theory $\mathbb{EA}$ is important? Zero-One Law for FO[$\{\mathrm{E}\}$].

1. Every extension axiom $\sigma_{s,t}$ from $\mathbb{EA}$ is almost surely true, i.e. $\mu_\infty(\sigma_{s,t}) = 1$ (Exercise).

2. By Compactness, it follows that $\mathbb{EA} \models \varphi$ implies $\mu_\infty(\varphi) = 1$ (TODO).

3. The theory $\mathbb{EA}$ is ω-categorical, i.e. has exactly one countable model up to $\cong$ (TODO).

4. Thus $\mathbb{EA}$ is complete, i.e. for all $\varphi \in$ FO we have $\mathbb{EA} \models \varphi$ or $\mathbb{EA} \models \neg\varphi$ (TODO).

Theorem (Glebskii et al. 1969, Fagin 1976)

For every formula $\varphi \in$ FO[$\{\mathrm{E}\}$] we have that $\mu_\infty(\varphi)$ is either 0 or 1.

Proof

Take any $\varphi \in$ FO[$\{\mathrm{E}\}$]. By (4) either $\mathbb{EA} \models \varphi$ or $\mathbb{EA} \models \neg\varphi$. If $\mathbb{EA} \models \varphi$ then by (2) we have $\mu_\infty(\varphi) = 1$. Otherwise $\mathbb{EA} \models \neg\varphi$, so by (2) we infer $\mu_\infty(\neg\varphi) = 1$, which leads to $\mu_\infty(\varphi) = 1 - \mu_\infty(\neg\varphi) = 0$. ■

Applications?

- Evenness of the number of nodes/edges not FO[$\{\mathrm{E}\}$]-definable.
- No information about connectivity because $\mu_\infty(\texttt{"graph is connected"}) = 0$.

Proof of $\mathbb{EA} \models \varphi$ implies $\mu_\infty(\varphi) = 1$ (assuming that $\forall \sigma \in \mathbb{EA}\ \mu_\infty(\sigma) = 1$).

Handy observations for all $\alpha, \beta, \gamma \in \mathrm{FO}[\{\mathrm{E}\}]$ and all $n \in \mathbb{N}$:

$\mu_n(\neg\alpha) = 1 - \mu_n(\alpha)$

Compactness: $\mathbb{EA} \models \varphi$ implies there is $\mathbb{EA}_0 \subseteq_{\mathsf{fin}} \mathbb{EA}$ implying φ

$\mu_n(\beta \vee \gamma) \leq \mu_n(\beta) + \mu_n(\gamma)$.

Proof

Goal: To show $\mu_\infty(\varphi) = 1$ it suffices to show that $\mu_n(\neg\varphi) \to 0$ when $n \to \infty$.

Assume $\mathbb{EA} \models \varphi$. By compactness, there is a finite $\mathbb{EA}_0 \subseteq \mathbb{EA}$ such that $\mathbb{EA}_0 \models \varphi$.

So $\mu_n(\varphi) \geq \mu_n(\bigwedge \mathbb{EA}_0)$, thus $\mu_n(\neg \bigwedge \mathbb{EA}_0) \geq \mu_n(\neg\varphi)$.

Moreover (by our assumption), $\mu_n(\neg\sigma) = 1 - \mu_n(\sigma)$ tends to 0 when $n \to \infty$.

$$\mu_n(\neg\varphi) \leq \mu_n(\neg \bigwedge \mathbb{EA}_0) = \mu_n\Big(\bigvee_{\sigma \in \mathbb{EA}_0} \neg\sigma\Big) \leq \sum_{\sigma \in \mathbb{EA}_0} \mu_n(\neg\sigma)$$

The sum $\sum_{\sigma \in \mathbb{EA}_0} \mu_n(\neg\sigma)$ converges to 0 for $n \to \infty$, concluding $\mu_\infty(\varphi) = 1$. ∎

$\mathbb{EA}$ is satisfiable and complete (assuming ω-categoricity)

- Note that $\mathbb{EA} \not\models \forall x \bot$ (due to $\mu_\infty(\forall x \bot) = 0$). So $\mathbb{EA}$ have a model (UnSAT theory entails everything).

$\mathbb{EA}$ is complete (assuming ω-categoricity), i.e. for all φ we either have $\mathbb{EA} \models \varphi$ or $\mathbb{EA} \models \neg\varphi$.

Proof

Assume that $\mathbb{EA}$ is not complete. Thus we have $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$ that are both models of $\mathbb{EA}$.

Since $|\mathbb{EA}| = \aleph_0$, by Löwenheim-Skolem we can assume w.l.o.g. that $\mathfrak{A}$ and $\mathfrak{B}$ are also countably-infinite.

But then, by ω-categoricity of $\mathbb{EA}$, we infer $\mathfrak{A} \cong \mathfrak{B}$.

Thus $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$ (since $\mathfrak{B} \models \neg\varphi$). A contradiction!

■

Ad absurdum

Löwenheim-Skolem

ω-categoricity

$\cong$ preserves $\models$

Today's final boss: $\mathbb{EA}$ is ω-categorical

$\mathbb{EA}$ is ω-categorical, i.e. has precisely one countably-infinite model.

Take any two countably-inf models $\mathfrak{A}, \mathfrak{B}$ of $\mathbb{EA}$ with the domains $A := \{a_1, a_2, \ldots\}$ and $B := \{b_1, b_2, \ldots\}$.

Goal: We will show that $\mathfrak{A} \cong \mathfrak{B}$ by producing an infinite growing sequence of partial isomorphisms $\mathfrak{p}_0, \mathfrak{p}_1, \ldots$

The union $\bigcup_{i=0}^{\infty} \mathfrak{p}_i$ will be the desired isomorphism. Start from $\mathfrak{p}_0 := \emptyset$.

Assume that a partial isomorphism $\mathfrak{p}_n = \{a_{i_1} \mapsto b_{i_1}, a_{i_2} \mapsto b_{i_2}, \ldots, a_{i_n} \mapsto b_{i_n}\}$ is given. Goal: define $\mathfrak{p}_{n+1}$.

If $n+1$ is even, we will select some element from $\mathfrak{A}$ (otherwise proceed analogously in $\mathfrak{B}$, proof omitted).

Take $a_k \in A$, for which k is the smallest index so that a_k does not appear in $\mathfrak{p}_n$. What do we know about $\bar{a}$?

There are unique n- and $(n+1)$-types s and t such that $s \subseteq t$, $\mathfrak{A} \models s(a_{i_1}, \ldots, a_{i_n})$, and $\mathfrak{A} \models t(a_{i_1}, \ldots, a_{i_n}, a_k)$.

Since $\mathfrak{p}_n$ is a partial isomorphism, we have $\mathfrak{B} \models s(b_{i_1}, \ldots, b_{i_n})$. But $\sigma_{s,t} \in \mathbb{EA}$ and $\mathfrak{B} \models \mathbb{EA}$!

Thus $\mathfrak{B} \models \sigma_{s,t} := \forall x_1 \ldots \forall x_n \; s(x_1, \ldots, x_n) \rightarrow \exists x_{n+1} \; t(x_1, \ldots, x_n, x_{n+1})$.

So there is an $b \in B$ so that $\mathfrak{B} \models t(b_{i_1}, \ldots, b_{i_n}, b)$. Continue from $\mathfrak{p}_{n+1} := \mathfrak{p}_n \cup \{(a_k \mapsto b)\}$. ∎

induction | back and forth | first not yet covered | exploit types realized by $\bar{a}$ | ind. ass. | $\mathfrak{B} \models \mathbb{EA}$ | Choose a witness

Extra: The Random Graph

We proved that $\mathbb{EA}$ has a model unconstructively. Can we describe the countable model of $\mathbb{EA}$?

Let $\mathfrak{G} = (V, \mathrm{E})$ be a graph such that $V = \mathbb{N}_+$ and $(i, j) \in \mathrm{E}^{\mathfrak{G}}$ iff $p_i \mid j$ or $p_j \mid i$ (p_i is the i-th prime number)

Lemma

$$\mathfrak{G} \models \sigma_{s,t} := \forall x_1 \ldots \forall x_n \; s(x_1, \ldots, x_n) \to \exists x_{n+1} \; t(x_1, \ldots, x_n, x_{n+1})$$

Proof

Take any $a_1, \ldots, a_k$ such that $\mathfrak{G} \models s(a_1, \ldots, a_k)$. Goal: Find a_{k+1} such that $\mathfrak{G} \models t(a_1, \ldots, a_k, a_{k+1})$.

We divide indices $1, 2, \ldots, k$ into Con $:= \{i \mid \mathrm{E}(x_i, x_{k+1}) \in t\}$ and DisC $:= \{i \mid \neg\mathrm{E}(x_i, x_{k+1}) \in t\}$.

Thus, our a_{k+1} must be connected to all a_i with $i \in$ Con and disconnected from all a_i with $i \in$ DisC.

$a_{k+1} := \Pi_{i \in \mathrm{Con}} \; p_{a_i} \cdot q$, where q is any prime number bigger than $\Pi_{i=1}^{k} \; p_{a_i}$

And now it is easy to check our choice of a_{k+1} is correct. ■

Divide $x_1, x_2, \ldots, x_k$ biased on type connections with $k+1$

(Dis)connected with $x \approx$ (non)dividable by the x-th prime number

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