# Finite and Algorithmic Model Theory

## Lecture 3 (Dresden 26.10.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

TECHNISCHE UNIVERSITÄT DRESDEN

Uniwersytet Wrocławski

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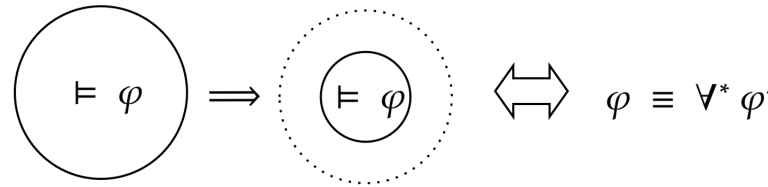

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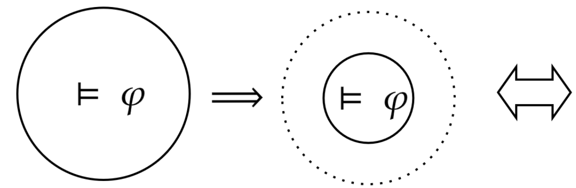


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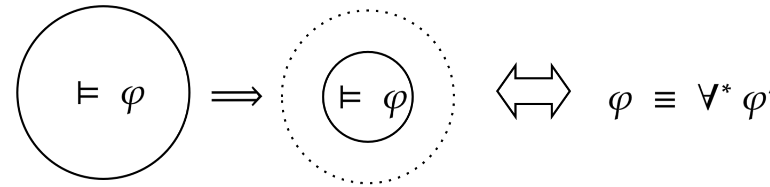

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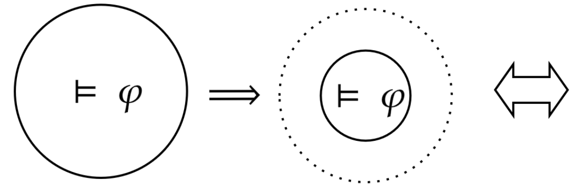


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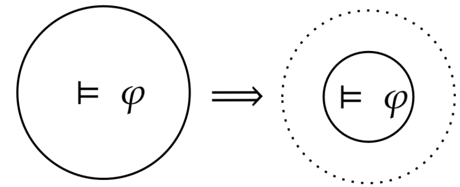


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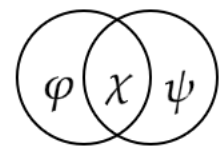


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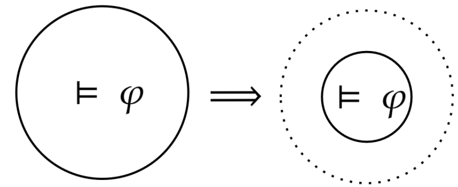


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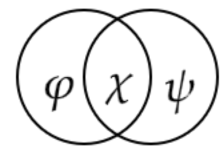


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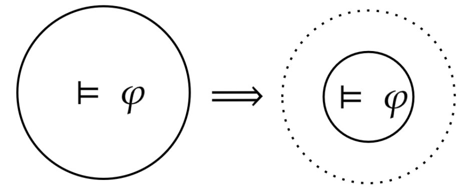


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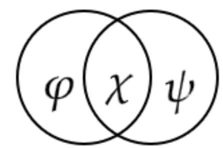


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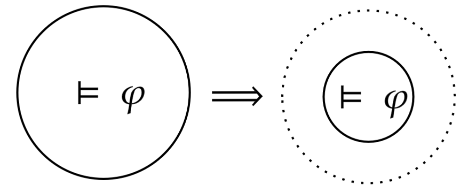


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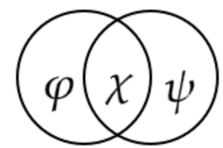


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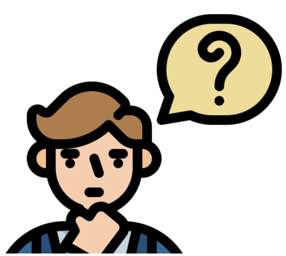


**Feel free to ask questions and interrupt me!**

**Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!**

**Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!**

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Alternative definition: $\mathrm{D}(\mathfrak{A}) := \{\varphi \in \mathrm{FO}[\tau_A] \mid \mathfrak{A}_A \models \varphi,\ \varphi \text{ is quantifier free}\}$

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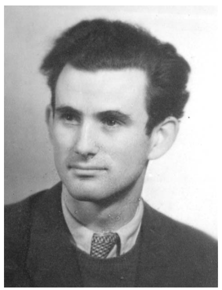

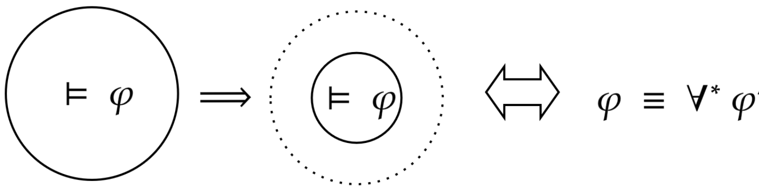


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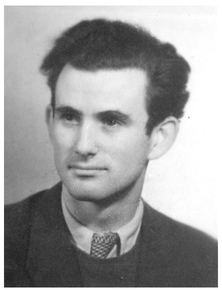

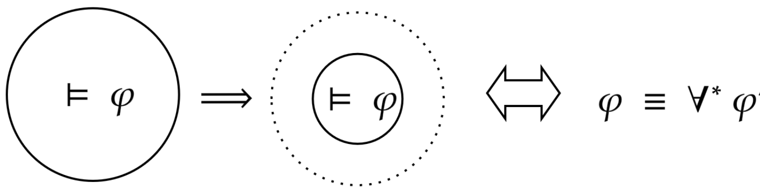


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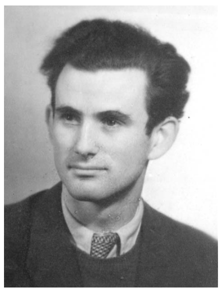

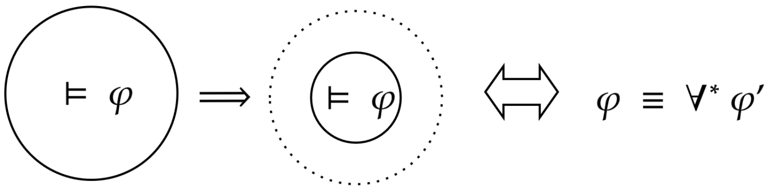


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- Łoś-Tarski over restricted classes, e.g. Sankaran et al. [`MFCS 2014`] or Atserias et al. [`SIAM 2008`].

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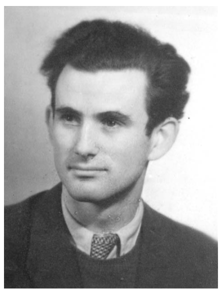

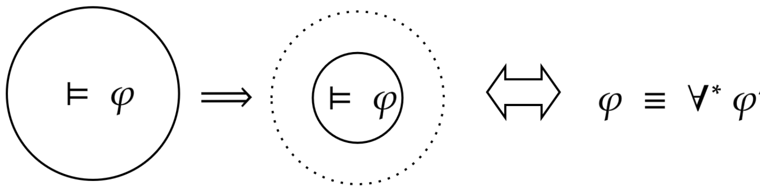


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# Preservation Theorems

Common theme: Formulae having semantic properties are precisely these of a syntactic fragment of FO

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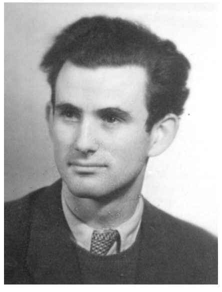

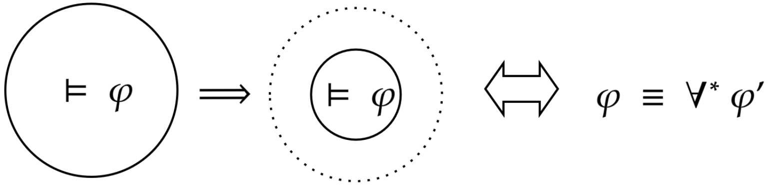


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- Open problem: Is there a non-trivial $\mathcal{L} \subseteq$ FO (without equality) without Łoś-Tarski? [B. 2022]

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But then $\bigwedge_{\psi \in \Psi_0} \psi$ is the desired universal formula equivalent to $\varphi$.

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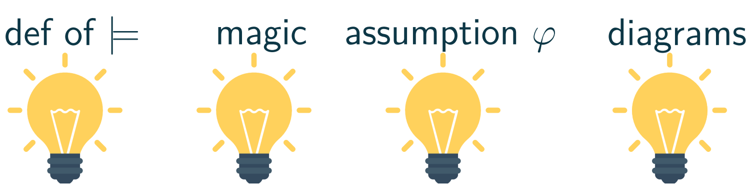

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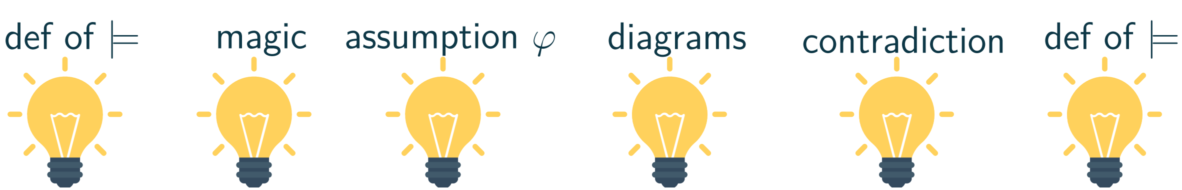

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universals are preserved under $\subseteq$

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universals are preserved under $\subseteq$

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**Theorem** (Tait 1959)

There is an FO formula that is preserved under substructures of finite structures but it is not equivalent (in the finite) to any universal formula.

**Proof**

Consider $\tau = \{\mathtt{min}^{(0)}, \mathtt{max}^{(0)}, <^{(2)}, \mathrm{Next}^{(2)}, \mathrm{P}^{(1)}\}$. Let $\varphi_0$ be a universal stating that

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If $\mathfrak{B} \models \varphi_1$ then $\mathfrak{A} = \mathfrak{B}$, concluding $\mathfrak{B} \models \varphi$. $\square$

universals are preserved under $\subseteq$ finiteness

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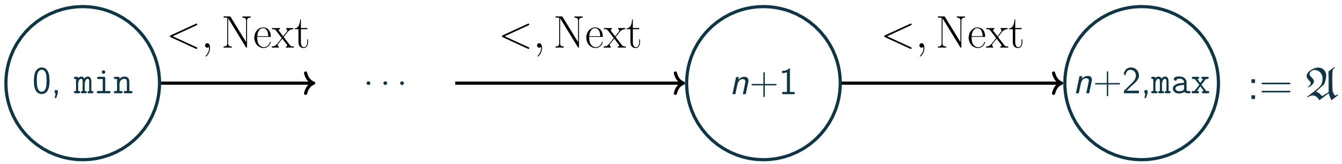


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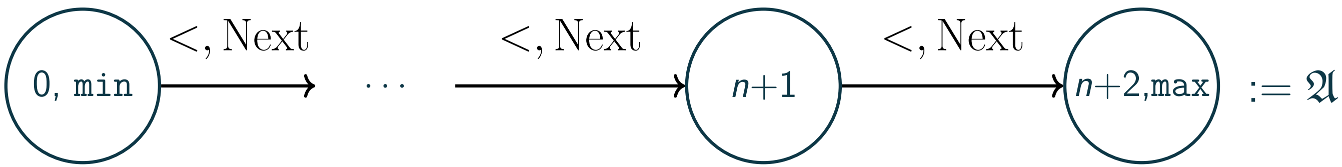


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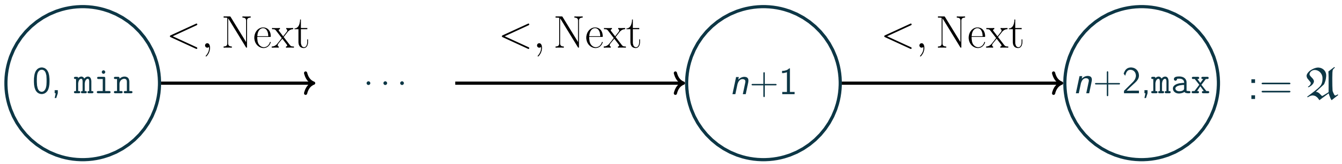


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contradiction

def of $\mathrm{P}$

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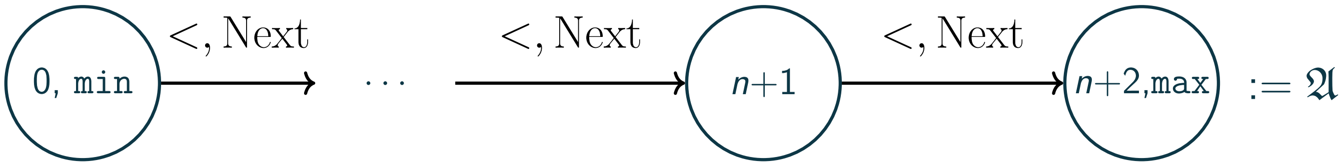


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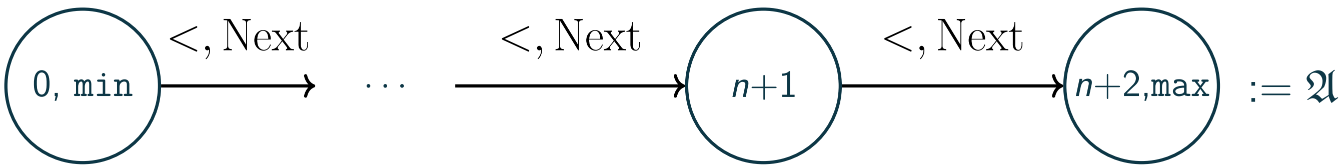


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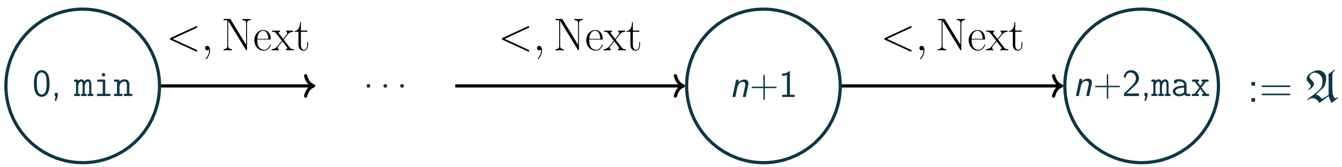


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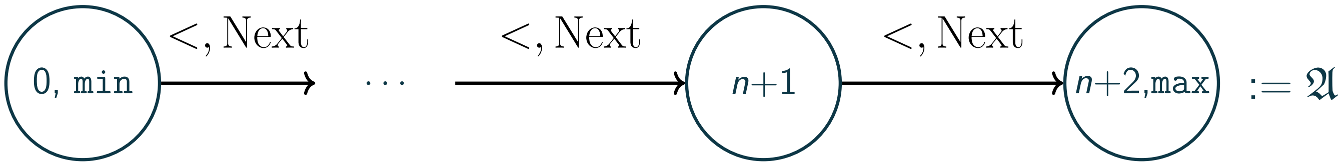


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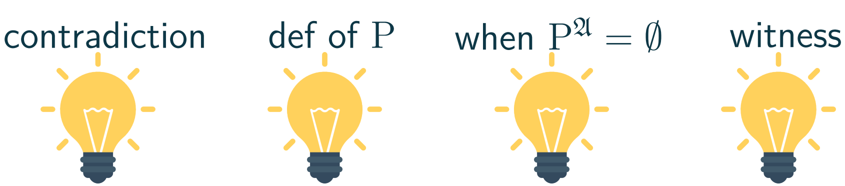

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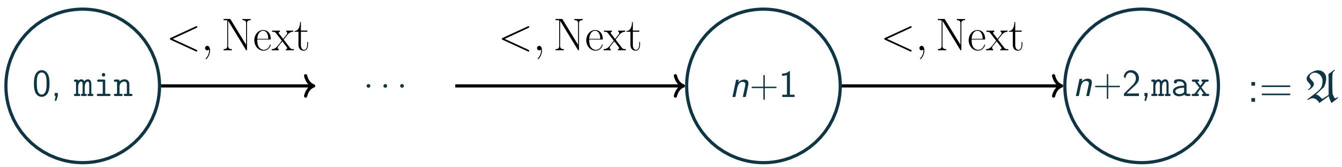


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contradiction | def of $\mathrm{P}$ | when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ | witness

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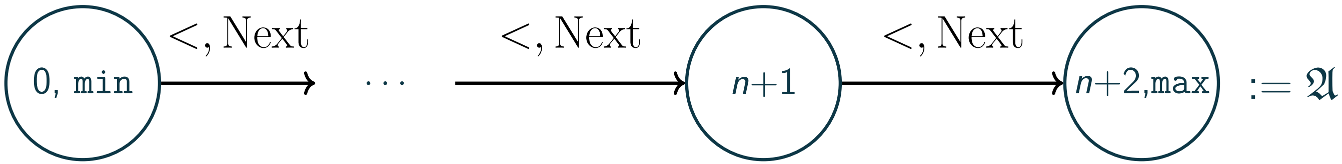


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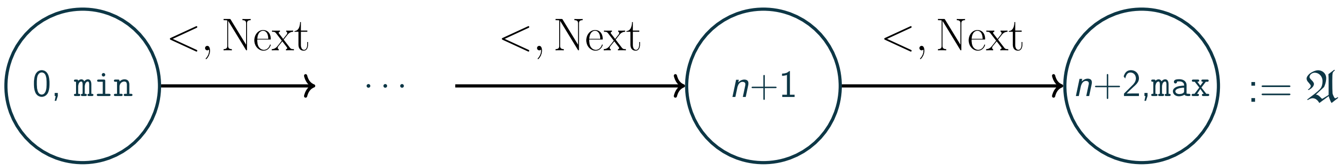


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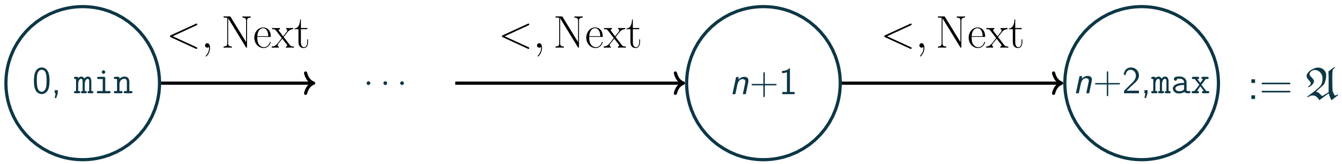


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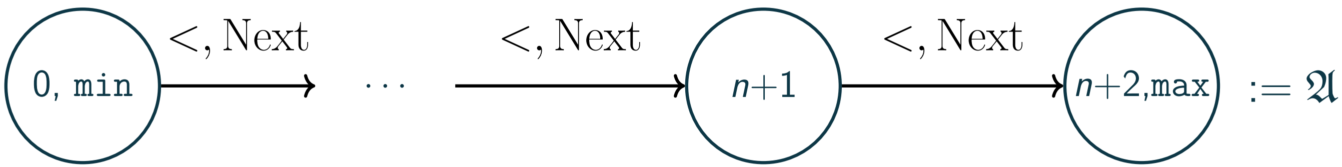


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## Failure of Łoś-Tarski in the finite. (Part II)

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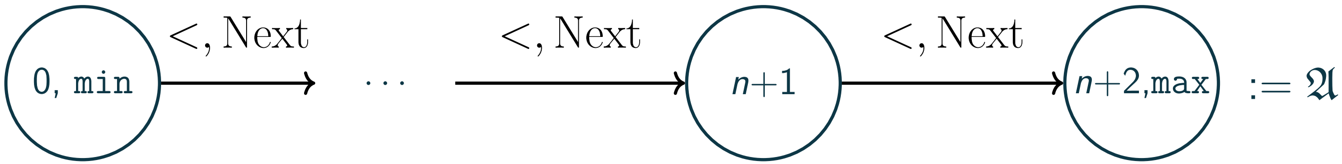
$\varphi_1 := \forall x \forall y\ \mathrm{Next}(x, y) \leftrightarrow (x < y \land \neg(\exists z\ x < z \land z < y))$ and $\varphi := \varphi_0 \land (\varphi_1 \to \exists x\ \mathrm{P}(x))$.

**Lemma** ($\varphi$ is not equivalent (in the finite) to any universal formula.)

Ad absurdum, there exists quantifier-free $\chi(\bar{x})$ with $n$ variables so that $\varphi \equiv_{\mathsf{fin}} \forall \bar{x}\ \chi(\bar{x})$. Take $\mathfrak{A}$ as below.




By construction $\mathfrak{A} \models \varphi_0 \land \varphi_1$. Moreover, observe that $(\mathfrak{A}, \mathrm{P}^{\mathfrak{A}}) \models \varphi$ iff $\mathrm{P}^{\mathfrak{A}} \neq \emptyset$.

Then $(\mathfrak{A}, \emptyset) \not\models \varphi$ implies $(\mathfrak{A}, \emptyset) \not\models \forall \bar{x}\ \chi(\bar{x})$. Thus $(\mathfrak{A}, \emptyset) \models \neg\chi(\bar{a})$ for suitable $\bar{a}$.

Take $b$ to be different from $\bar{a}$, $\mathtt{max}^{\mathfrak{A}}$ and $\mathtt{min}^{\mathfrak{A}}$ (we have enough elements!). Then $(\mathfrak{A}, \{b\}) \models \varphi$.

But $(\mathfrak{A}, \{b\}) \models \neg\chi(\bar{a})$ ($\mathfrak{A} \restriction \bar{a}$ was not touched!).

contradiction | def of $\mathrm{P}$ | when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ | witness | select suitable $b$ and make it satisfy $\mathrm{P}$ | def of $\varphi$

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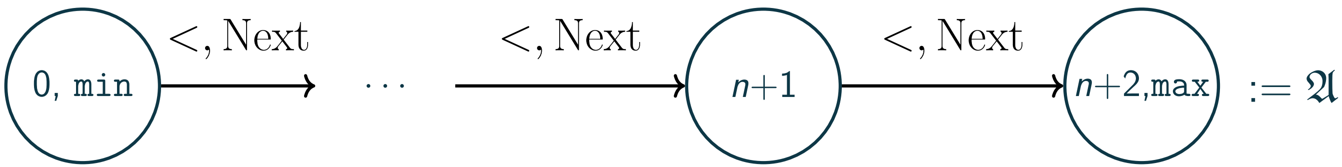


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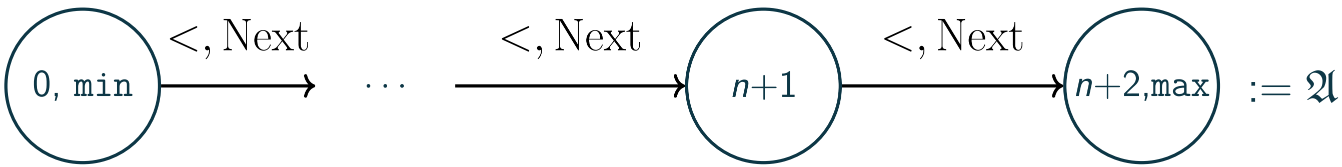


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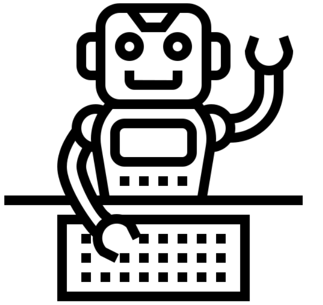
# Can we make Łoś-Tarski theorem computable?



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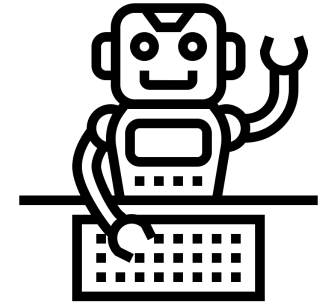

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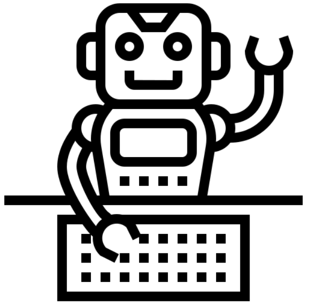

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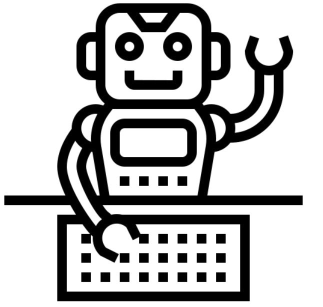

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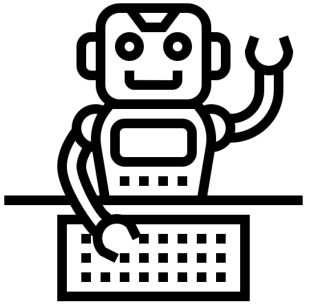

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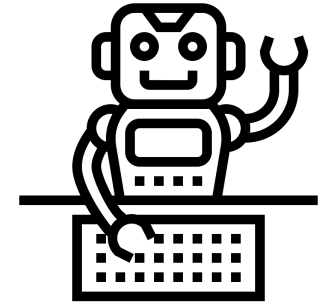

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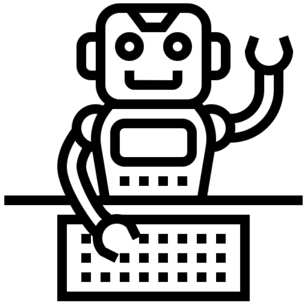

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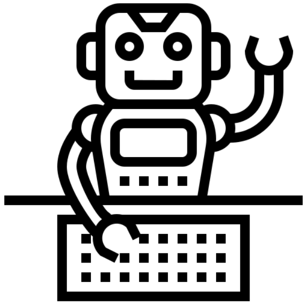

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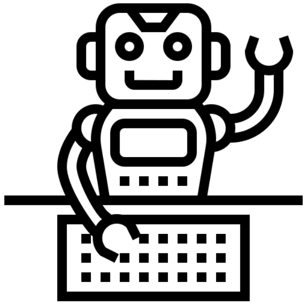

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**Theorem** (Lyndon–Tarski 1956, Rossmann 2005)

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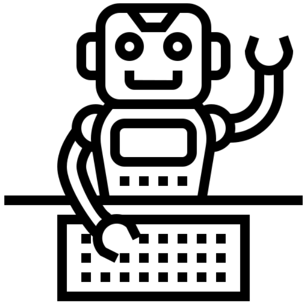

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- A notable example of classical MT theorem that works in the finite, c.f. [Rossmann's paper]

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