
Exercise 1

Show that there are only finitely many formulae (up to equivalence) in $\mathsf{FO}_0[\sigma]$ for a finite σ.

Exercise 2

By induction, employing the previous exercise, show that for any n, the set $\mathsf{FO}_n[\sigma]$ is also finite (up to equivalence).

Exercise 3

Provide an inductive proof that $\mathfrak{A} \simeq_m \mathfrak{B}$ iff $\mathfrak{A}, \mathfrak{B}$ agree on all formulae from $\mathsf{FO}_m[\sigma]$. Hint: we need to use the fact that rank m-types can be described by a single formula of quantifier-rank m.

Exercise 4

The next exercise is about showing that if $\mathfrak{L}_1, \mathfrak{L}_2$ are linear orders with endpoints (i.e. $\{\leq, min, max\}$-structures in which $\leq$ is interpreted as a linear order, and min, max are constant symbols interpreted as the first and the last element according to $\leq$) of length $\geq 2^m$ then $\mathfrak{L}_1 \equiv_m \mathfrak{L}_2$. We will use the so-called *composition* method.

For a linear order $\mathcal{L}$ and an element $a \in L$ we will use a notation $\mathfrak{L}^{\leq a}$ and $\mathfrak{L}^{\geq a}$ to denote the substructure of $\mathfrak{L}$ obtained by taking all the elements smaller than or equal to a (resp. $\geq a$). First, prove the following lemma.

Lemma 1. *Let $a \in L_1, b \in L_2$ be such that $\mathfrak{L}_1^{\leq a} \equiv_k \mathfrak{L}_2^{\leq b}$ and $\mathfrak{L}_1^{\geq a} \equiv_k \mathfrak{L}_2^{\geq b}$. Then $(\mathfrak{L}_1, a) \equiv_k (\mathfrak{L}_2, b)$.*

And going back to the proof of that $|L_1| \geq 2^m, |L_2| \geq 2^m$ implies $\mathfrak{L}_1 \equiv_m \mathfrak{L}_2$: prove it by induction. The base case is obvious. For the inductive step use our strategy "play far whenever spoiler plays far" and employ the above lemma.

A game with 2 pebbles is a simple variant of E-G games. We again have two players (spoiler/duplicator) and each of the players have 2 distinct pebbles to play with (call them x, y) as well as two structures $\mathfrak{A}$ and $\mathfrak{B}$. The game takes r rounds. During each round, Spoiler selects one of the structures (say $\mathfrak{A}$) and one of its elements (call it a) and places one of his pebbles on such an element. Note that the pebble disappears from its previous position, in stark contrast to E-F games, where we remember the whole history of the game. Then Duplicator responds and he looses if the function mapping the x, y Spoiler's pebbles to x, y Duplicator's pebbbles is not a partial isomorphism (saying it easier, x, y in one structure must satisfy the same atomic formulae as in the second structure and vice versa and they must agree on constants). Duplicator wins if he can survive r rounds.

With FO^2 we denote the fragment of FO in which we can use only 2 variables, namely x, y (note that variables may be reused). It can be shown that if duplicator has a winning strategy in r-round 2-pebble game, then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulae from FO^2 with quantifier rank $\leq r$.

Exercise 5

Show that you can express in $\mathsf{FO}^2[\{E\}]$ that there is an E-path from some element of length at least 5.

Exercise 6

Show that you cannot express in $\mathsf{FO}^2[\{E\}]$ that E is functional (i.e. each node has at most one outgoing E-edge).

Exercise 7

Show that you cannot express in $\mathsf{FO}^2[\{E\}]$ that E is a linear order.
