TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 5: Time Complexity and Polynomial Time

Markus Krötzsch

Knowledge-Based Systems

TU Dresden, 1st Nov 2017

Time Complexity

Measuring Complexity

Complexity Theory

Study the fine structure of decidable languages.

Goal

Classify languages by the amount of resources needed to solve them.

Resources

When dealing with Turing machines, we will primarily consider

- time: the running time of algorithms (steps on a Turing-machine)
- space: the amount of additional memory needed (cells on the Turing-tapes)

Time and Space Bounded Turing Machines

Definition 5.1: Consider a Turing machine $\mathcal{M}$ and a function $f : \mathbb{N} \to \mathbb{R}^+$.

(1) $\mathcal{M}$ is f-time bounded if it halts on every input $w \in \Sigma^*$ after $\leq f(|w|)$ steps.

(2) $\mathcal{M}$ is f-space bounded if it halts on every input $w \in \Sigma^*$ using $\leq f(|w|)$ cells on its tapes.

(Here we typically assume that Turing machines have a separate input tape that we do not count in measuring space complexity.)

Notation 5.2: Sometimes notations like "$f(n)$-time bounded" are used, assuming inputs to be of length n
$\leadsto$ we use this when convenient, e.g., to write "n^3-bounded"

Big-O and Small-o

Algorithms are often judged by their asymptotic complexity, i.e., their behaviour in the limit.

We recall and extend the definition from Lecture 1:

Definition 5.3: The Big-O notation classifies functions using asymptotic upper bounds:

$$f(n) = O(g(n)) \quad \text{iff} \quad \exists c > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0 \colon f(n) \leq c \cdot g(n)$$

Then f is asymptotically bounded by g up to a constant factor.

Big-O and Small-o

Algorithms are often judged by their asymptotic complexity, i.e., their behaviour in the limit.

We recall and extend the definition from Lecture 1:

Definition 5.3: The Big-O notation classifies functions using asymptotic upper bounds:

$$f(n) = O(g(n)) \quad \text{iff} \quad \exists c > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0 : f(n) \leq c \cdot g(n)$$

Then f is asymptotically bounded by g up to a constant factor.

Definition 5.4: The small-o notation classifies by a function that dominates them:

$$f(n) = o(g(n)) \quad \text{iff} \quad \forall c > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0 : f(n) \leq c \cdot g(n)$$

Then f is asymptotically dominated by g.

Relatives of the O Notation

There are a number of further asymptotic notations besides Big-O and small-o. Their essence and underlying intuition is as follows:

Notation	$C = \lim_{n\to\infty} \frac{f(n)}{g(n)}$	Intuition
$f \in O(g)$	$C < \infty$	"“$f \leq g$"”
$f \in \Omega(g)$	$C > 0$	"“$f \geq g$"”
$f \in \Theta(g)$	$0 < C < \infty$	"“$f = g$"”
$f \in o(g)$	$C = 0$	"“$f < g$"”
$f \in \omega(g)$	$C = \infty$	"“$f > g$"”

Relaxed Time and Space Bounds

We can use Big-O notation to generalise bounded TMs:

Definition 5.5: A Turing machine $\mathcal{M}$ is

(1) $O(g(n))$-time bounded if it is f-time bounded for some f with $f(n) = O(g(n))$

(2) $O(g(n))$-space bounded if it is f-space bounded for some f with $f(n) = O(g(n))$

Notation 5.6: We generally allow the use of $O(g(n))$ in place of a function $f(n)$ with analogous meaning.

Deterministic Complexity Classes

Bounding TMs is the basis for both complexity theory and for studies of algorithmic complexity.

Definition 5.7: Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

(1) DTime($f(n)$) is the class of all languages **L** for which there is an $O(f(n))$-time bounded Turing machine deciding **L**.

(2) DSpace($f(n)$) is the class of all languages **L** for which there is an $O(f(n))$-space bounded Turing machine deciding **L**.

Notation 5.8: Sometimes Time($f(n)$) is used instead of DTime($f(n)$).

Is Complexity Theory Impossible in Practice?

The classes DTIME(f) and DSPACE(f) depend on

- details of the computational model
- details of the input encoding
- details of the implementation

An exact specification of such bounds is often extremely hard.

Is Complexity Theory Impossible in Practice?

The classes DTIME(f) and DSPACE(f) depend on

- details of the computational model
- details of the input encoding
- details of the implementation

An exact specification of such bounds is often extremely hard.

Example 5.9: A naive algorithm can perform matrix multiplication in DTIME(n^3).

Is Complexity Theory Impossible in Practice?

The classes DTIME(f) and DSPACE(f) depend on

- details of the computational model
- details of the input encoding
- details of the implementation

An exact specification of such bounds is often extremely hard.

Example 5.9: A naive algorithm can perform matrix multiplication in DTIME(n^3). Since many decades, researchers have been searching for better solutions: DTIME($n^{2.808}$) [Strassen, 1969], DTIME($n^{2.796}$) [Pan, 1978], DTIME($n^{2.780}$) [Bini et al., 1979], DTIME($n^{2.522}$) [Schönhage, 1981], DTIME($n^{2.517}$) [Romani, 1982], DTIME($n^{2.496}$) [Coppersmith & Winograd, 1981], DTIME($n^{2.479}$) [Strassen, 1986], DTIME($n^{2.376}$) [Coppersmith & Winograd, 1990], DTIME($n^{2.374}$) [Stothers, 2010], and DTIME($n^{2.373}$) [Williams, 2011].

Is Complexity Theory Impossible in Practice?

The classes DTIME(f) and DSPACE(f) depend on

- details of the computational model
- details of the input encoding
- details of the implementation

An exact specification of such bounds is often extremely hard.

Example 5.9: A naive algorithm can perform matrix multiplication in DTIME(n^3). Since many decades, researchers have been searching for better solutions: DTIME$(n^{2.808})$ [Strassen, 1969], DTIME$(n^{2.796})$ [Pan, 1978], DTIME$(n^{2.780})$ [Bini et al., 1979], DTIME$(n^{2.522})$ [Schönhage, 1981], DTIME$(n^{2.517})$ [Romani, 1982], DTIME$(n^{2.496})$ [Coppersmith & Winograd, 1981], DTIME$(n^{2.479})$ [Strassen, 1986], DTIME$(n^{2.376})$ [Coppersmith & Winograd, 1990], DTIME$(n^{2.374})$ [Stothers, 2010], and DTIME$(n^{2.373})$ [Williams, 2011]. Conjectured optimal solution: DTIME(n^2).

Defining Complexity Classes

Solution: Make complexity classes big enough to hide such details.

$$\mathsf{P} = \mathsf{PTime} = \bigcup_{d\geq 1} \mathsf{DTime}(n^d)$$ polynomial time

$$\mathsf{Exp} = \mathsf{ExpTime} = \bigcup_{d\geq 1} \mathsf{DTime}(2^{n^d})$$ exponential time

$$\mathsf{2Exp} = \mathsf{2ExpTime} = \bigcup_{d\geq 1} \mathsf{DTime}(2^{2^{n^d}})$$ double-exponential time

$$\mathsf{E} = \mathsf{ETime} = \bigcup_{d\geq 1} \mathsf{DTime}(2^{dn})$$ exp. time with linear exponent

$$\mathsf{L} = \mathsf{LogSpace} = \mathsf{DSpace}(\log n)$$ logarithmic space

$$\mathsf{PSpace} = \bigcup_{d\geq 1} \mathsf{DSpace}(n^d)$$ polynomial space

$$\mathsf{ExpSpace} = \bigcup_{d\geq 1} \mathsf{DSpace}(2^{n^d})$$ exponential space

Time Complexity Classes

$$\mathsf{P} = \mathsf{PTime} = \bigcup_{d\geq 1} \mathsf{DTime}(n^d) \qquad \text{polynomial time}$$

$$\mathsf{Exp} = \mathsf{ExpTime} = \bigcup_{d\geq 1} \mathsf{DTime}(2^{n^d}) \qquad \text{exponential time}$$

$$\mathsf{2Exp} = \mathsf{2ExpTime} = \bigcup_{d\geq 1} \mathsf{DTime}(2^{2^{n^d}}) \qquad \text{double-exponential time}$$

Note: Complexity classes are classes of languages.

Observation: The following relationships are clear from the definition:

$$\mathsf{P} \subseteq \mathsf{ExpTime} \subseteq \mathsf{2ExpTime} \subseteq \mathsf{3ExpTime} \subseteq \mathsf{4ExpTime} \subseteq \ldots$$

A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?

A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?
- If not, how much more resources do we need to be able to solve strictly more problems?

A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?
- If not, how much more resources do we need to be able to solve strictly more problems?
- How do the complexity classes relate to each other?

A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?
- If not, how much more resources do we need to be able to solve strictly more problems?
- How do the complexity classes relate to each other?
- Are there any tools by which we can show that a problem is in any of these classes but not in another?

$\leadsto$ discussed in future lectures

A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?
- If not, how much more resources do we need to be able to solve strictly more problems?
- How do the complexity classes relate to each other?
- Are there any tools by which we can show that a problem is in any of these classes but not in another?

$\rightsquigarrow$ discussed in future lectures

- How do we classify "efficient" in terms of complexity classes?

$\rightsquigarrow$ coming up next

Different Definitions of Complexity Classes?

How is complexity affected by the chosen model of computation?

- Is DTime(f) the same for multi-tape TMs?
- And how about non-deterministic TMs?
- Or TMs with a two-way infinite tape?
- Or random access machines?
- …

Different Definitions of Complexity Classes?

How is complexity affected by the chosen model of computation?

- Is DTime(f) the same for multi-tape TMs?
- And how about non-deterministic TMs?
- Or TMs with a two-way infinite tape?
- Or random access machines?
- ...

Many complexity classes are robust against many such variations
$\leadsto$ coming up next

Polynomial Time

Polynomial Time

An "intuitive" definition of "efficient":

- Any linear time computation is "efficient".
- Any program that
  - performs "efficient" operations (e.g. linear number of iterations) and
  - only uses "efficient" subprograms

  is "efficient".

Polynomial Time

An "intuitive" definition of "efficient":

- Any linear time computation is "efficient".
- Any program that
  - performs "efficient" operations (e.g. linear number of iterations) and
  - only uses "efficient" subprograms

  is "efficient".

This turns out to be equivalent to PTime.

$$\text{PTime} := \bigcup_{d \geq 1} \text{DTime}(n^d)$$

PTime serves as a mathematical model of "efficient" computation.

Robustness of the Definition

If PTime is to be the mathematical model of efficient computation,
it should not depend on

- the exact computation-model we are using,
- or how we encode the input (within reason).

Multi-Tape Turing Machines

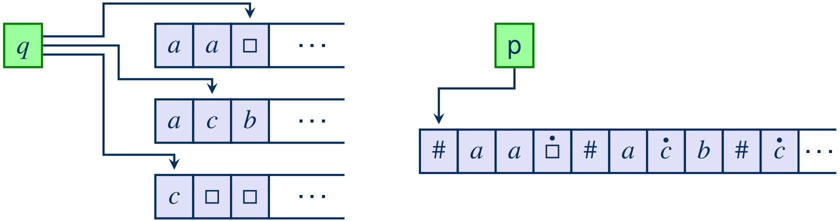
Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof: Simulate a multi-tape TM with a single-tape TM as shown in Lecture 2:



Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do:
  $O(f(n))$ steps

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do:
  $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do: $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps
- Sometimes the whole tape is shifted to make space: at most k times $O(f(n))$ steps

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do: $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps
- Sometimes the whole tape is shifted to make space: at most k times $O(f(n))$ steps
- Overall: one step is simulated in $O(f(n))$ steps

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do: $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps
- Sometimes the whole tape is shifted to make space: at most k times $O(f(n))$ steps
- Overall: one step is simulated in $O(f(n))$ steps
- Simulating $f(n)$ such steps takes $f(n) \cdot O(f(n)) = O(f^2(n))$ steps

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do: $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps
- Sometimes the whole tape is shifted to make space: at most k times $O(f(n))$ steps
- Overall: one step is simulated in $O(f(n))$ steps
- Simulating $f(n)$ such steps takes $f(n) \cdot O(f(n)) = O(f^2(n))$ steps
- Tape initialisation takes another $O(f(n))$ (irrelevant)

Multi-Tape Turing Machines

Theorem 5.10 (Sipser, Theorem 7.8): Consider a function f with $f(n) \geq n$. Then, for every $f(n)$-time bounded k-tape Turing machine ($k > 1$), there is an equivalent $O(f^2(n))$-time bounded single-tape Turing machine.

Proof (cont.): Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than $f(n)$ symbols on them
- The simulation scans the whole tape once to find out what to do: $O(f(n))$ steps
- Then it updates the tapes whole tape in one pass: $O(f(n))$ steps
- Sometimes the whole tape is shifted to make space: at most k times $O(f(n))$ steps
- Overall: one step is simulated in $O(f(n))$ steps
- Simulating $f(n)$ such steps takes $f(n) \cdot O(f(n)) = O(f^2(n))$ steps
- Tape initialisation takes another $O(f(n))$ (irrelevant)

Total simulation possible in $O(f^2(n))$. $\square$

P is Robust for Multi-Tape TMs

Let $\mathsf{DTime}_k(f(n))$ denote "$\mathsf{DTime}(f(n))$ for a k-tape TM".

Theorem 5.11:

$$\bigcup_{d\in\mathbb{N}} \mathsf{DTime}(n^d) = \bigcup_{d\in\mathbb{N}} \mathsf{DTime}_k(n^d) \text{ for every } k \geq 1$$

Proof: The inclusion $\subseteq$ is clear.
The inclusion $\supseteq$ follows from the previous Theorem 5.10. $\square$

Robustness Against Other Models of Computation

P is robust against further models of computation:

(1) We can simulate $f(n)$ steps of a two-way infinite k-tape Turing-machine with an equivalent standard k-tape TM in $O(f(n))$ steps.

(2) We can simulate $f(n)$ steps of a RAM-machine with a 3-tape TM in $O(f^3(n))$ steps. Vice-versa in $O(f(n))$ steps.

Robustness Against Other Models of Computation

P is robust against further models of computation:

(1) We can simulate $f(n)$ steps of a two-way infinite k-tape Turing-machine with an equivalent standard k-tape TM in $O(f(n))$ steps.

(2) We can simulate $f(n)$ steps of a RAM-machine with a 3-tape TM in $O(f^3(n))$ steps. Vice-versa in $O(f(n))$ steps.

Consequences:

- PTime is the same for all these models (unlike linear time)
- The exponential time complexity classes are as robust as P

How about non-deterministic TMs?

Robustness Against Other Models of Computation

P is robust against further models of computation:

(1) We can simulate $f(n)$ steps of a two-way infinite k-tape Turing-machine with an equivalent standard k-tape TM in $O(f(n))$ steps.

(2) We can simulate $f(n)$ steps of a RAM-machine with a 3-tape TM in $O(f^3(n))$ steps. Vice-versa in $O(f(n))$ steps.

Consequences:

- PTime is the same for all these models (unlike linear time)
- The exponential time complexity classes are as robust as P

How about non-deterministic TMs?

It is unknown if PTime is robust against this, but most think it is not
$\leadsto$ see next lectures

Linear Speed-Up

The Big-O notation in DTime hides arbitrary linear factors.
Is it justified to rely on this for defining P?

Linear Speed-Up

The Big-O notation in DTime hides arbitrary linear factors.
Is it justified to rely on this for defining P?

Yes, it turns out that we can make multi-tape TMs "arbitrarily fast":

Theorem 5.12 (Linear Speed-Up Theorem): Consider an $f(n)$-time bounded k-tape Turing machine $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ with $k > 1$.

Then, for every constant $c > 0$, there is a $(\frac{1}{c} \cdot f(n) + n + 2)$-time bounded k-tape TM $\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', q'_0, q'_{\text{accept}}, q'_{\text{reject}})$ that accepts the same language.

Linear Speed-Up (Proof)

Proof (sketch): Let $\Gamma' := \Sigma \cup \Gamma^m$ where $m := \lceil 6c \rceil$. We construct $\mathcal{M}'$ as follows:

Linear Speed-Up (Proof)

Proof (sketch): Let $\Gamma' := \Sigma \cup \Gamma^m$ where $m := \lceil 6c \rceil$. We construct $\mathcal{M}'$ as follows:

Step 1: Compress $\mathcal{M}$'s input.

Copy the input to tape 2, compressing m symbols into one (i.e., each symbol corresponds to an m-tuple from Γ^m). This takes $n + 2$ steps.

Linear Speed-Up (Proof)

Proof (sketch): Let $\Gamma' := \Sigma \cup \Gamma^m$ where $m := \lceil 6c \rceil$. We construct $\mathcal{M}'$ as follows:

Step 1: Compress $\mathcal{M}$'s input.

Copy the input to tape 2, compressing m symbols into one (i.e., each symbol corresponds to an m-tuple from Γ^m). This takes $n + 2$ steps.

Step 2: Simulate $\mathcal{M}$'s computation, m steps at once.

(1) Read (in 4 steps) symbols to the left, right and the current position and "store" in Q', using $|Q \times \{1, \ldots, m\}^k \times \Gamma^{3mk}|$ extra states.

(2) Simulate (in 2 steps) the next m steps of $\mathcal{M}$ (as $\mathcal{M}$ can only modify the current position and one of its neighbours)

(3) $\mathcal{M}'$ accepts (rejects) if $\mathcal{M}$ accepts (rejects)

For further details see Papadimitriou, Theorem 2.2. $\square$

Different Encodings

Some simple observations:

(1) For any $n \in \mathbb{N}$, the length of the encoding of n in base b_1 and base b_2 are related by a constant factor, for all $b_1, b_2 \geq 2$.

Different Encodings

Some simple observations:

(1) For any $n \in \mathbb{N}$, the length of the encoding of n in base b_1 and base b_2 are related by a constant factor, for all $b_1, b_2 \geq 2$.

(2) For any graph G, the length of its encoding as an
   - adjacency matrix
   - list of nodes + list of edges
   - adjacency list
   - ...

   are all polynomially related.

Different Encodings

Some simple observations:

(1) For any $n \in \mathbb{N}$, the length of the encoding of n in base b_1 and base b_2 are related by a constant factor, for all $b_1, b_2 \geq 2$.

(2) For any graph G, the length of its encoding as an

- adjacency matrix
- list of nodes + list of edges
- adjacency list
- ...

are all polynomially related.

Consequence:

PTime is the same for all these encodings (unlike linear time).

PTime = tractable?

The class Ptime is a reasonable mathematical model of the class of problems which are tractable or solvable in practice.

PTime = tractable?

The class Ptime is a reasonable mathematical model of the class of problems which are tractable or solvable in practice.

However: This correspondence is not exact.

- When the degree of polynomials is very high, the time grows so quickly that in practice the problem is not solvable.
- The constants may also be very large

PTime = tractable?

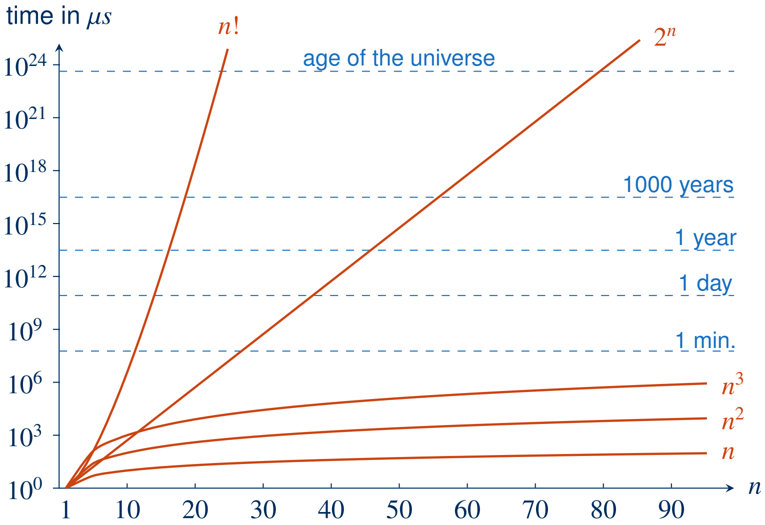
The class Ptime is a reasonable mathematical model of the class of problems which are tractable or solvable in practice.

However: This correspondence is not exact.

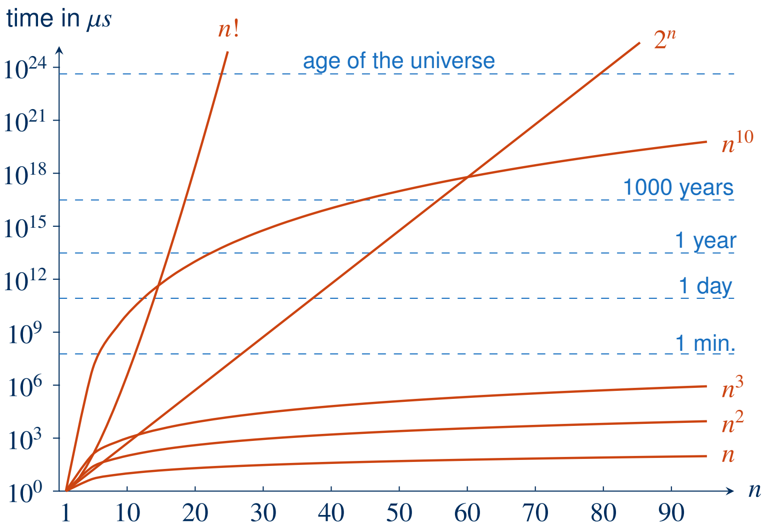
- When the degree of polynomials is very high, the time grows so quickly that in practice the problem is not solvable.
- The constants may also be very large

And yet: For many concrete PTime-problems arising in practice, algorithms with moderate exponents and constants have been found.

Growth Rate of Some Functions



Growth Rate of Some Functions



Problems in P

Proving a Problem is in PTime

- The most direct way to show that a problem is in PTime is to exhibit a polynomial time algorithm that solves it.
- Even a naive polynomial-time algorithm often provides a good insight into how the problem can be solved efficiently.
- Because of robustness, we do not generally need to specify all the details of the machine model or the encoding.

  $\leadsto$ pseudo-code is sufficient

Example: Satisfiability

Some of the most important problems concern logical formulae

Definition 5.13 (Propositional Logic Syntax): Formulae of propositional logic are built up inductively

- (Propositional) Variables: $X_i \qquad i \in \mathbb{N}$
- Boolean connectives: If φ, ψ are propositional formulae then so are
  - $(\psi \vee \varphi)$
  - $(\psi \wedge \varphi)$
  - $\neg\varphi$

Example: Satisfiability

Some of the most important problems concern logical formulae

Definition 5.13 (Propositional Logic Syntax): Formulae of propositional logic are built up inductively

- (Propositional) Variables: $X_i \qquad i \in \mathbb{N}$
- Boolean connectives: If φ, ψ are propositional formulae then so are
  - $(\psi \lor \varphi)$
  - $(\psi \land \varphi)$
  - $\neg\varphi$

Example 5.14: The following is a propositional logic formula:

$$(X_1 \lor X_2 \lor \neg X_5) \;\land\; (\neg X_2 \lor \neg X_4 \lor \neg X_5) \;\land\; (X_2 \lor X_3 \lor X_4)$$

Conjunctive Normal Form

Definition 5.15 (Conjunctive Normal Form): A propositional logic formula φ is in conjunctive normal form (CNF) if

$$\varphi = C_1 \wedge \cdots \wedge C_m$$

where each C_i is a clause, that is, a disjunction of literals

$$C_i = (L_{i1} \vee \cdots \vee L_{ik})$$

and a literal is a variable X_i or a negation $\neg X_i$ thereof.

A CNF φ is in k-CNF is it has at most k literals per clause.

Conjunctive Normal Form

Definition 5.15 (Conjunctive Normal Form): A propositional logic formula φ is in conjunctive normal form (CNF) if

$$\varphi = C_1 \wedge \cdots \wedge C_m$$

where each C_i is a clause, that is, a disjunction of literals

$$C_i = (L_{i1} \vee \cdots \vee L_{ik})$$

and a literal is a variable X_i or a negation $\neg X_i$ thereof.

A CNF φ is in k-CNF is it has at most k literals per clause.

Example 5.16: The following formula is in 3-CNF:

$$(X_1 \vee X_2 \vee \neg X_5) \;\wedge\; (\neg X_2 \vee \neg X_4 \vee \neg X_5) \;\wedge\; (X_2 \vee X_3 \vee X_4)$$

Propositional Logic Semantics

Definition 5.17: A formula φ is satisfiable if it is satisfied by an assignment that maps each variable in φ to either 0 or 1 (and recursively defined for larger fomulae as usual).

Specifically: A formula in CNF is satisfiable if there is an assignment β for variables of φ so that every clause contains at least

- one variable to which β assigns 1, or
- one negated variable to which β assigns 0.

Propositional Logic Semantics

Definition 5.17: A formula φ is satisfiable if it is satisfied by an assignment that maps each variable in φ to either 0 or 1 (and recursively defined for larger fomulae as usual).

Specifically: A formula in CNF is satisfiable if there is an assignment β for variables of φ so that every clause contains at least

- one variable to which β assigns 1, or
- one negated variable to which β assigns 0.

Example 5.18: The formula

$$(X_1 \vee X_2 \vee \neg X_5) \ \wedge \ (\neg X_2 \vee \neg X_4 \vee \neg X_5) \ \wedge \ (X_2 \vee X_3 \vee X_4)$$

is satisfied by $\{X_1 \mapsto 1, \ X_2 \mapsto 0, \ X_3 \mapsto 1, \ X_4 \mapsto 0, \ X_5 \mapsto 1\}$.

The Satisfiability Problem

Related to propositional formulae, the following two problems are the most important:

SAT

Input:	Propositional formula φ in CNF
Problem:	Is φ satisfiable?

k**-SAT**

Input:	Propositional formula φ in k-CNF
Problem:	Is φ satisfiable?

2-Sat is Polynomial

Theorem 5.19: 2-Sat $\in$ PTime.

2-Sat is Polynomial

Theorem 5.19: 2-Sat ∈ PTime.

Proof: The following algorithm solves the problem in polynomial time.

Main: Input Γ in CNF

bcp(Γ)
if conflict **return** UNSAT

while $\Gamma \neq \emptyset$ **do**
  choose var. X from Γ
  set $\Gamma' := \Gamma$
  assign($\Gamma, X, 1$)
  bcp(Γ)
  if conflict
    $\Gamma := \Gamma'$
    assign($\Gamma, X, 0$)
    bcp(Γ)
    if conflict
      return UNSAT

bcp(Γ) (boolean constraint propagation)

while Γ contains unit-clause C **do**
  if $C = \{X\}$ assign($\Gamma, X, 1$)
  if $C = \{\neg X\}$ assign($\Gamma, X, 0$)
if Γ contains empty clause **return** conflict

assign(Γ, X, c)

if $c = 1$
  remove from Γ all clauses C with $X \in C$
  remove $\neg X$ from all remaining clauses
if $c = 0$
  remove from Γ all clauses C with $\neg X \in C$
  remove X from all remaining clauses

□

Polynomial-Time Reductions

As for decidability we can use reductions to show membership in PTime.

Definition 5.20: A language $\mathbf{L_1} \subseteq \Sigma^*$ is polynomially many-one reducible to $\mathbf{L_2} \subseteq \Sigma^*$, denoted $\mathbf{L_1} \leq_p \mathbf{L_2}$, if there is a polynomial-time computable function f such that for all $w \in \Sigma^*$

$$w \in \mathbf{L_1} \qquad \text{if and only if} \qquad f(w) \in \mathbf{L_2}.$$

Polynomial-Time Reductions

As for decidability we can use reductions to show membership in PTime.

Definition 5.20: A language $\mathbf{L_1} \subseteq \Sigma^*$ is polynomially many-one reducible to $\mathbf{L_2} \subseteq \Sigma^*$, denoted $\mathbf{L_1} \leq_p \mathbf{L_2}$, if there is a polynomial-time computable function f such that for all $w \in \Sigma^*$

$$w \in \mathbf{L_1} \qquad \text{if and only if} \qquad f(w) \in \mathbf{L_2}.$$

Theorem 5.21: If $\mathcal{L}_1 \leq_p \mathcal{L}_2$ and $\mathcal{L}_2 \in$ PTime then $\mathcal{L}_1 \in$ PTime.

Proof: The sum and composition of polynomials is a polynomial. □

Reductions in PTime

All non-trivial members of PTime can be reduced to each other:

Theorem 5.22: If **B** is any language in P, $\mathbf{B} \neq \emptyset$, and $\mathbf{B} \neq \Sigma^*$, then $\mathbf{A} \leq_p \mathbf{B}$ for any $\mathbf{A} \in$ P.

Reductions in PTime

All non-trivial members of PTime can be reduced to each other:

Theorem 5.22: If **B** is any language in P, $\mathbf{B} \neq \emptyset$, and $\mathbf{B} \neq \Sigma^*$, then $\mathbf{A} \leq_p \mathbf{B}$ for any $\mathbf{A} \in$ P.

Proof: Choose $w \in \mathbf{B}$ and $w' \notin \mathbf{B}$.

Reductions in PTime

All non-trivial members of PTime can be reduced to each other:

Theorem 5.22: If **B** is any language in P, **B** $\neq \emptyset$, and **B** $\neq \Sigma^*$, then **A** $\leq_p$ **B** for any **A** $\in$ P.

Proof: Choose $w \in$ **B** and $w' \notin$ **B**.

Define the function f by setting

$$f(x) := \begin{cases} w & \text{if } x \in \mathbf{A} \\ w' & \text{if } x \notin \mathbf{A} \end{cases}$$

Since **A** $\in$ P, this function f is computable in polynomial time, and it is a reduction from **A** to **B**. □

Example: Colourability

Definition 5.23 (Vertex Colouring): A vertex colouring of G with k colours is a function

$$c : V(G) \longrightarrow \{1, \ldots, k\}$$

such that adjacent nodes have different colours, that is:

$$\{u, v\} \in E(G) \text{ implies } c(u) \neq c(v)$$

k-Colouring

Input:	Graph G, $k \in \mathbb{N}$
Problem:	Does G have a vertex colouring with k colours?

For $k = 2$ this is the same as **Bipartite**.

Reducing 2-Colourability to 2-Sat

Theorem 5.24: 2-**Colourability** $\leq_p$ 2-**Sat**, and therefore 2-**Colourability** $\in$ P.

Reducing 2-Colourability to 2-Sat

Theorem 5.24: 2-Colourability $\leq_p$ 2-Sat, and therefore 2-Colourability $\in$ P.

Proof: We define a reduction as follows: Given graph G

- For each vertex $v \in V(G)$ of the graph introduce new variable X_v
- For each $\{u, v\} \in E(G)$ add clauses $(X_u \vee X_v)$ and $(\neg X_u \vee \neg X_v)$

This is obviously computable in polynomial time.

Reducing 2-Colourability to 2-Sat

Theorem 5.24: 2-Colourability $\leq_p$ 2-Sat, and therefore 2-Colourability $\in$ P.

Proof: We define a reduction as follows: Given graph G

- For each vertex $v \in V(G)$ of the graph introduce new variable X_v
- For each $\{u, v\} \in E(G)$ add clauses $(X_u \vee X_v)$ and $(\neg X_u \vee \neg X_v)$

This is obviously computable in polynomial time.

We check that it is a reduction:

- If G is 2-colourable, use colouring to assign truth values.
  (One colour is true, the other false)
- If the formula is satisfiable, the truth assignment defines valid 2-colouring.
  For every edge $\{u, v\} \in E(G)$, one variable X_u, X_v must be set to true, the other to false.

□

Trivially Tractable Problems

A large class of languages is generally tractable:

Theorem 5.25: If **L** is a finite language, then it is decided by an $O(1)$-time bounded TM. In other words, all finite languages are decidable in constant time (and hence also in polynomial time).

Trivially Tractable Problems

A large class of languages is generally tractable:

Theorem 5.25: If **L** is a finite language, then it is decided by an $O(1)$-time bounded TM. In other words, all finite languages are decidable in constant time (and hence also in polynomial time).

Proof:

- As **L** is finite, there is a maximum length m of words in **L**.
- Read the input up to the first m letters.
- The state space contains a table containing the correct result for all such inputs.
- All other inputs are rejected. $\square$

A Note on Constructiveness

The next result is an example of a theorem that proves the existence of a P algorithm in cases where we do not know what this algorithm is.

Example 5.26: Let **L** be the language that contains all correct sentences from the following set:

$$\{\text{“P is the same as NP”}, \text{“P is not the same as NP”}\}$$

Then **L** is decidable in constant time.

A Note on Constructiveness

The next result is an example of a theorem that proves the existence of a P algorithm in cases where we do not know what this algorithm is.

Example 5.26: Let **L** be the language that contains all correct sentences from the following set:

$$\{\text{“P is the same as NP”}, \text{“P is not the same as NP”}\}$$

Then **L** is decidable in constant time.
However, we don't know which constant-time algorithm decides it.

A Note on Constructiveness

The next result is an example of a theorem that proves the existence of a P algorithm in cases where we do not know what this algorithm is.

Example 5.26: Let **L** be the language that contains all correct sentences from the following set:

$$\{\text{“P is the same as NP”}, \text{“P is not the same as NP”}\}$$

Then **L** is decidable in constant time.
However, we don't know which constant-time algorithm decides it.

Non-constructiveness:

- We can prove that there is a correct polynomial time algorithm.
- We cannot construct such an algorithm.

Such solutions are called non-constructive.

An Interesting Problem in P

Theorem 5.27: It is decidable in polynomial-time ($O(n^3)$) if a graph can knotlessly be embedded into 3-dimensional space.

An Interesting Problem in P

Theorem 5.27: It is decidable in polynomial-time ($O(n^3)$) if a graph can knotlessly be embedded into 3-dimensional space.

Proof (sketch):

- Robertson & Seymour proved a general result that implies the existence of a finite set of forbidden structures in knotlessly embeddable graphs.
- For each of these forbidden structures we can test whether a graph contains one of them in time $O(n^3)$.
- Hence, to decide if a graph is knotlessly embeddable, we only need to test for each of the finitely many forbidden structures, whether they occur in the graph.

This yields a cubic time decision procedure. $\square$

An Interesting Problem in P

Theorem 5.27: It is decidable in polynomial-time ($O(n^3)$) if a graph can knotlessly be embedded into 3-dimensional space.

Proof (sketch):

- Robertson & Seymour proved a general result that implies the existence of a finite set of forbidden structures in knotlessly embeddable graphs.
- For each of these forbidden structures we can test whether a graph contains one of them in time $O(n^3)$.
- Hence, to decide if a graph is knotlessly embeddable, we only need to test for each of the finitely many forbidden structures, whether they occur in the graph.

This yields a cubic time decision procedure. □

However: We do not currently know what these structures are.

Summary and Outlook

Complexity classes are based on asymptotic resource estimates, further generalised by considering general classes of bounds (e.g., all polynomial functions)

Ignoring constant factors is justified due to Linear Speedup

P is the most common approximation of "efficient"

Polynomial many-one reductions are used show membership in P

What's next?

- NP
- Hardness and completeness
- More examples of problems