Problem 1.1  Induction

1. Prove by induction that \( n \leq n^2 \) for every \( n \in \mathbb{N} \).

2. Prove by induction that the number of opening parentheses is the same the number of closing parentheses in a formula of propositional logic.

3. Prove by induction that the number of opening parentheses is the same as the number of closing parentheses in a formula of propositional logic.

4. Prove by induction that in a formula of propositional logic as specified in Definition 3.4 occur twice as many parentheses as binary connectives.

5. Refute the following statement: In a formula of propositional logic the number of propositional variables is greater or equal to the number of unary connectives.

Problem 1.2  Existence of \( \mathcal{L}(\mathcal{R}) \)

In definition 3.5 of the course the set \( \mathcal{L}(\mathcal{R}) \) of strings over a given propositional alphabet \( \Sigma_R \) (with \( \mathcal{R} \) a set of propositional variables) was defined as the smallest set of strings over \( \Sigma_R \) which satisfies certain closure properties, to which we will refer in the following as conditions 1–3. However, the proof of existence of this set \( \mathcal{L}(\mathcal{R}) \) was just sketched.

Prove in full detail the existence of the set \( \mathcal{L}(\mathcal{R}) \). Proceed stepwise as follows:

1. Show at first that for every propositional alphabet \( \Sigma_R \) exists a set \( Z \) of strings that fulfils conditions 1–3.

2. We define \( \mathcal{F} := \{ N \mid N \ is \ a \ set \ and \ fulfils \ conditions \ 1–3 \} \).
   
   Show that \( \bigcap \mathcal{F} \) is a set of strings over \( \Sigma_R \).

3. Show that \( \bigcap \mathcal{F} := \{ H \mid H \in N \ for \ all \ N \in \mathcal{F} \} \) fulfils conditions 1–3 as well.

4. Prove the following more general lemma:
   
   For a set \( \mathcal{N} \) of sets holds:
   
   If \( \bigcap \mathcal{N} \in \mathcal{N} \), then \( \bigcap \mathcal{N} \) is the smallest element of \( \mathcal{N} \) w.r.t. the \( \subseteq \) relation.

5. Show by means of the lemmata (1)–(4) given above that the set \( \mathcal{L}(\mathcal{R}) \) exists.

Problem 1.3  Do smallest sets always exist?

An inattentive logician defines his favorite set as the smallest nonempty set of natural numbers. What do you think about this definition?

Problem 1.4  Truth table

Construct the truth table for the propositional formula \( (p \lor \neg q) \rightarrow r \)
Problem 1.5  Semantics

Let $G$ be a propositional formula and $\mathcal{F}$ a set of propositional formulae. Evaluate the following statements.

1. $\mathcal{F} \models G$ holds iff $\mathcal{F}$ and $G$ have the same models.  
   \hspace{1in} true \, \square \, false \, \square

2. $\mathcal{F} \models G$ holds if there is an Interpretation $I$ which maps $G$ to false and which maps all formulae $F \in \mathcal{F}$ to false as well.  
   \hspace{1in} true \, \square \, false \, \square

3. $\mathcal{F} \models G$ implies that there is at least one model of $\mathcal{F}$ and that there is at least one model of $G$.  
   \hspace{1in} true \, \square \, false \, \square

4. $\mathcal{F} \models G$ holds if all formulae in $\mathcal{F}$ are refutable and $G$ is not valid.  
   \hspace{1in} true \, \square \, false \, \square

5. $\mathcal{F} \models G$ implies that $G$ cannot be unsatisfiable.  
   \hspace{1in} true \, \square \, false \, \square

6. $\mathcal{F} \models G$ implies that $G$ is valid if in $\mathcal{F}$ exists at least one valid formula.  
   \hspace{1in} true \, \square \, false \, \square

7. $\mathcal{F} \models G$ holds if $G$ is subformula of a formula $F \in \mathcal{F}$.  
   \hspace{1in} true \, \square \, false \, \square

8. $\mathcal{F} \models G$ holds if a subset of $\mathcal{F}$ is unsatisfiable.  
   \hspace{1in} true \, \square \, false \, \square

9. $\mathcal{F} \models G$ holds iff every model of $\mathcal{F}$ is also a model of $G$.  
   \hspace{1in} true \, \square \, false \, \square

10. $\mathcal{F} \models G$ implies the following:  
    If $\mathcal{F}$ is refutable, then $G$ is refutable as well.  
    \hspace{1in} true \, \square \, false \, \square

11. If $\mathcal{F} \models G$ holds and every formula in $\mathcal{F}$ is satisfiable, then $G$ must be satisfiable as well.  
    \hspace{1in} true \, \square \, false \, \square

12. If $G$ is a generalized disjunction of satisfiable formula $F_1, \ldots, F_n$ ($n > 0$) from $\mathcal{F}$, then $\mathcal{F} \models G$ holds.  
    \hspace{1in} true \, \square \, false \, \square

13. $\mathcal{F} \models G$ implies that if there is at least a model of $\mathcal{F}$, then also exists at least a model of $G$.  
    \hspace{1in} true \, \square \, false \, \square

Problem 1.6  Deduction theorem

Prove Theorem 3.17:
Let $F, F_1, \ldots, F_n$ be propositional formulas. $\{F_1, \ldots, F_n\} \models F$ holds iff $\models ((\ldots (F_1 \land F_2) \land \ldots \land F_n) \rightarrow F)$ holds.

Problem 1.7  Logical consequence and unsatisfiability

Prove the following: $\mathcal{F} \models G$ holds iff $\mathcal{F} \cup \{\neg G\}$ is unsatisfiable.
Problem 1.8  Monotonicity

Given a propositional formula $G$ and sets $F$ and $F'$ of propositional formulae. Prove the following propositions: If $F \models G$ and $F \subseteq F'$, then holds $F' \models G$.

Problem 1.9  Ex falso quodlibet sequitur

Given propositional formulae $F$ and $G$ and a set $F$ of propositional formulae. Prove the following proposition: If $F \in F$ and $\neg F \in F$, then $F \models G$ for arbitrary $G$. 