

COMPLEXITY THEORY

Lecture 14: P vs. NP: Ladner's Theorem

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Knowledge-Based Systems

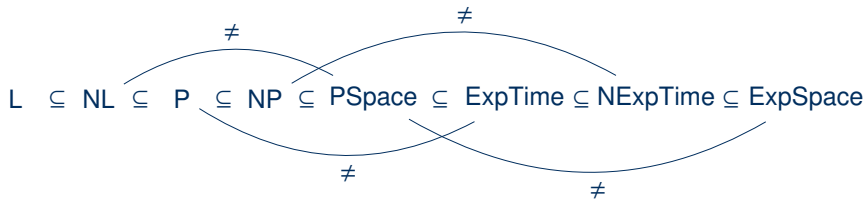
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For the most current version of this course, see
https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en

Review

Review: Hierarchies and Gaps

Hierarchy theorems tell us that more time/space leads to more power:



Gap theorems tell us that, for non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources may not lead to more power

Any natural problems in the hierarchy?

To show that complexity classes are different

- we have defined concrete diagonalisation languages that can show the difference (i.e., our argument was **constructive**),
- but these diagonalisation languages are rather artificial (i.e., not **natural**).

Are there, e.g., any natural ExpTime problems that are not in P?

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Yes, many:

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Yes, many:

Theorem 14.1: If L is ExpTime-hard, then $L \notin P$.

Proof: We have shown that there is a language $D \in \text{ExpTime} \setminus P$. If L is ExpTime-hard, then there is a polynomial many-one reduction $D \leq_p L$. Therefore, if L were in P , then so would D – contradiction. \square

Similar results hold for other classes we separated: A problem that is hard for the larger class cannot be included in the smaller.

Ladner's Theorem

P vs. NP revisited

We have seen that a great variety of difficult problems in NP turn out to be NP-complete.

A natural question to ask is whether this apparent **dichotomy** is a law of nature:

Hypothesis: Every problem in NP is either in P or NP-complete.

P vs. NP revisited

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A natural question to ask is whether this apparent **dichotomy** is a law of nature:

Hypothesis: Every problem in NP is either in P or NP-complete.

In 1975, Richard E. Ladner showed that this is wrong, unless $P = NP$

(in the latter case, uninterestingly, the non-trivial problems in P would turn out to be exactly the set of NP-complete problems)

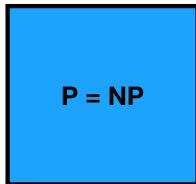
Theorem 14.2 (Ladner, 1975): If $P \neq NP$, then there are problems in NP that are neither in P nor NP-complete.

Such problems are called **NP-intermediate**.

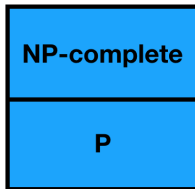
Illustration

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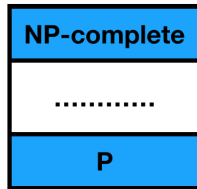
In other words, given the following illustrations of the possible relationships between P and NP:



a)



b)



c)

Ladner tells us that the middle cannot be correct.

Proving the Theorem

Theorem 14.2 (Ladner, 1975): If $P \neq NP$, then there are problems in NP that are neither in P nor NP -complete.

Proof idea: We will directly define an NP -intermediate language by defining an NTM \mathcal{K} that recognises it.

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We want to construct $L(\mathcal{K})$ to be:

- (1) different from all problems in P
- (2) different from all problems that **SAT** can be reduced to

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Moreover, the sets we diagonalise against are effectively enumerable:

- There is an effective enumeration $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \dots$ of all polynomially time-bounded DTMs, each together with a suitable bounding function

For example, enumerate all pairs of TMs and polynomials, and make the enumeration consist of the TMs obtained by artificially restricting the run of a TM with a suitable countdown.

- There is an effective enumeration $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \dots$ of all polynomial many-one reductions, each together with a suitable bounding function

This is similar to enumerating polytime TMs; we can restrict to one input alphabet that we also use for **SAT**

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How can we do two diagonalisations at once? — Simply interleave the enumerations:

- On each even number $2i$, show that the i th polytime TM \mathcal{M}_i is not equivalent to \mathcal{K} : there is w such that $\mathcal{M}_i(w) \neq \mathcal{K}(w)$
- For each odd number $2i + 1$, show that the i th reduction \mathcal{R}_i does not reduce **SAT** to \mathcal{K} : there is w such that $\mathcal{K}(\mathcal{R}_i(w)) \neq \mathbf{SAT}(w)$

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Nevertheless, there is a problem: How can we flip the output of **SAT**?

- \mathcal{K} is required to run in NP
- Computing the actual result of **SAT** is NP-hard
- To show $\mathcal{K}(\mathcal{R}_i(w)) \neq \mathbf{SAT}(w)$, one might have to show $w \notin \mathbf{SAT}$, which is presumably not in NP

↪ the required computation seems too hard!

Solution: Lazy diagonalisation

Idea: Do not attempt to show too much on small inputs, but wait patiently until inputs are large enough to show the required differences

Main ingredients:

- A **very** slow growing but polynomially computable function f
- A problem in NP that is NP-hard: **SAT**
- A problem in NP that is not NP-hard:

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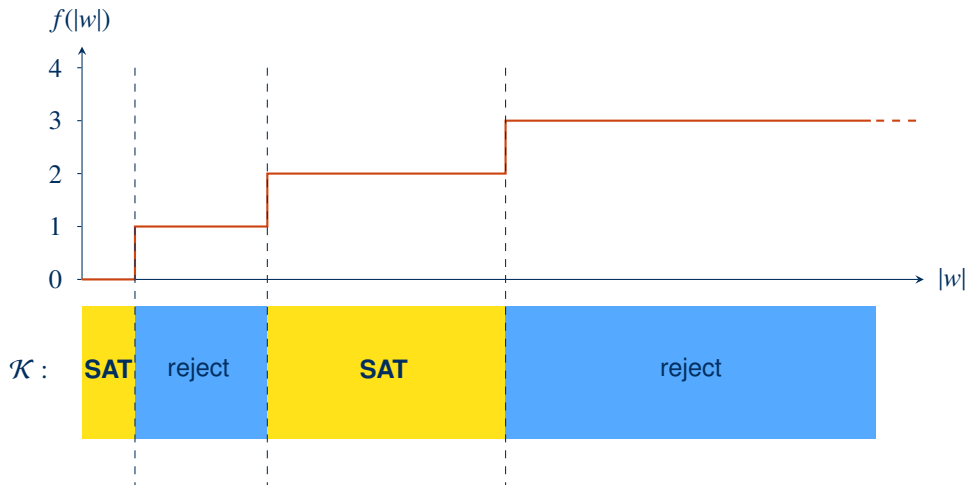
We will define a TM \mathcal{K} that does the following on input w :

- (1) Compute the value $f(|w|)$
- (2) If $f(|w|)$ is even: return whether $w \in \mathbf{SAT}$
- (3) If $f(|w|)$ is odd: return whether $w \in \emptyset$, i.e., reject

Intuition: the NP-intermediate language $\mathbf{L}(\mathcal{K})$ is **SAT** with “holes punched out of it” (namely for all inputs where f is odd)

Illustration of \mathcal{K} 's behaviour

We can sketch the behaviour of \mathcal{K} as follows:



What is f ?

Reminder: $\mathcal{K}(w)$ is **SAT**(w) if $f(|w|)$ is even, and *false* if $f(|w|)$ is odd.

The key to the proof is the definition of f – this is where the diagonalisation happens.

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Intuition: Keep the current value of f until progress has been made in diagonalisation

- Keep an even value $f(|w|) = 2i$ until you can show in polynomial time (in $|w|$) that there is v such that $\mathcal{M}_i(v) \neq \mathcal{K}(v)$
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If we can do this in NP, it will be enough already:

- If \mathcal{K} were equivalent to any \mathcal{M}_i , then f would eventually become an even constant, and \mathcal{K} would solve **SAT** on all but finitely many instances
 $\leadsto \mathcal{K}$ would be NP-hard, and equivalent to a polytime TM $\leadsto \mathbf{P} = \mathbf{NP}$

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- If \mathcal{K} would allow **SAT** to be reduced to it by some reduction \mathcal{R}_i , then f would eventually become an odd constant, and $\mathbf{L}(\mathcal{K})$ would be a finite language
 $\leadsto \mathcal{K}$ would be in P, and **SAT** would reduce to it $\leadsto \mathbf{P} = \mathbf{NP}$

In either case, this contradicts our assumption that $\mathbf{P} \neq \mathbf{NP}$

What is f ?

We consider some fixed **deterministic** TM S with $\mathbf{L}(S) = \mathbf{SAT}$, and an enumeration v_0, v_1, \dots of all words ordered by length, and lexicographic for words of equal length.

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- (1) Perform the computations of $f(0), f(1), f(2), \dots$ in order until n computing steps have been performed in total. Store the largest value $f(\ell) = k$ that could be computed in this time (set $k = 0$ if no value was computed).

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 - (2.a) If $k = 2i$ is even: Iterate over all words v , simulate $\mathcal{M}_i(v)$, $S(v)$, and (recursively) compute $f(|v|)$. Terminate this effort after n steps. If a word is found such that $\mathcal{K}(v) \neq \mathcal{M}_i(v)$, then return $k + 1$; else return k

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 - (2.b) If $k = 2i + 1$ is odd: Iterate over all words v , simulate $\mathcal{R}_i(v)$ (this produces a word), $S(v)$, $S(\mathcal{R}_i(v))$, and (recursively) compute $f(|\mathcal{R}_i(v)|)$. Terminate this effort after n steps. If a word is found such that $\mathcal{K}(\mathcal{R}_i(v)) \neq S(v)$, then return $k + 1$; else return k .

Is f well-defined?

Our definition of f computes values for f recursively. Is this ok?

- Yes, the computation that needs to be done for each $f(n)$ is fully defined
- All the simulated TMs are known or computable
- Since computation is time-limited to the input value n , there is no danger of endless recursion
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Indeed, f grows **very** slowly!

- A large input n might be needed to find the next counterexample word v needed in diagonalisation
- Even if such v was found in n steps (making progress from n to $n + 1$), it will be only much later that $f(n)$ can be computed in step (1) and f will even start to look for a way of getting to $n + 2$.
- In fact, already the requirement to recompute all previous values of f before considering an increase ensures that $f \in O(\log \log n)$.

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$L(\mathcal{K})$ is not in P: As argued before: if it were in P, it would be equivalent to some polytime TM \mathcal{M}_i , and f would eventually be constant at $2i$, making \mathcal{K} equivalent to **SAT** (up to finite variations), which contradicts $P \neq NP$.

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$L(\mathcal{K})$ is not in NP-hard: As argued before: if it were NP-hard, there would be a polynomial many-one reduction \mathcal{R}_i from **SAT**, and f would eventually be constant at $2i + 1$, making \mathcal{K} equivalent to \emptyset (up to finite variations), which contradicts $P \neq NP$. \square

Discussion: Proof of Ladner's Theorem

Note 1: It is interesting to meditate on the following facts:

- We have defined a rather “busy” computation of f that checks that diagonalisation (over two different sets) must happen
- This definition of computation is essential to prove the result
- Nevertheless, diagonalisation remained “internal”: from the outside, \mathcal{K} is just a TM that sometimes solves **SAT** (for a long range of inputs), and at other times just rejects every input (again for very long ranges of inputs)

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Note 3: Are there any natural problems that are known to be NP-intermediate?

- No: finding one would prove $P \neq NP$
- Candidate problems (link) include, e.g., **GRAPH ISOMORPHISM** and **FACTORING**

Beware: the latter is not about deciding if a number is prime, but about checking something specific about its factors, e.g., whether the largest factor contains at least one 7 when written in decimal

Summary and Outlook

Ladner's theorem tells us that, in the intuitive case that $P \neq NP$, there must be (counterintuitively?) many problems in NP that are neither polynomially solvable nor NP-complete

The proof is based on a technique of lazy diagonalisation

What's next?

- Generalising Ladner's Theorem
- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation