TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 1: Introduction and Motivation

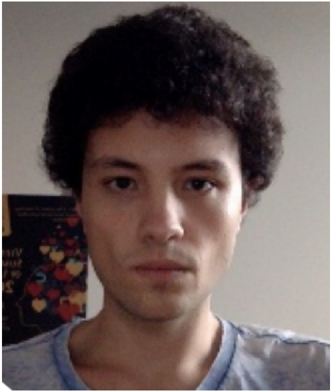
Markus Krötzsch

Knowledge-Based Systems

TU Dresden, 10th Oct 2017

Course Tutors

Markus Krötzsch
Lectures



David Carral
Exercises

Organisation

Lectures

Tuesday, DS 3 (11:10–12:40), APB E005
Wednesday, DS 6 (16:40–18:10), APB E005

Exercise Sessions (starting 17 October)

Tuesday, DS 5 (14:50–16:20), APB E005

Important: No Lectures or Exercises in Week 3 (24 and 25 Oct)

Web Page

`https://iccl.inf.tu-dresden.de/web/Complexity_Theory_(WS2017/18)`

Lecture Notes

Slides of current and past lectures will be online.

Goals and Prerequisites

Goals

- Introduce basic notions of **computational complexity theory**
- Introduce commonly known complexity classes (P, NP, PSpace, . . .) and discuss relationships between them
- Develop tools to classify problems into their corresponding complexity classes
- Introduce (some) advanced topics of complexity theory

(Non-)Prerequisites

- No particular prior courses needed
- Prior acquaintance with Turing Machines and basic topics in formal languages and complexity is helpful
- General mathematical and theoretical computer science skills necessary

Reading List

- **Michael Sipser: Introduction to the Theory of Computation, International Edition; 3rd Edition; Cengage Learning 2013**
- Sanjeev Arora and Boaz Barak: **Computational Complexity: A Modern Approach**; Cambridge University Press 2009
- Michael R. Garey and David S. Johnson: **Computers and Intractability**; Bell Telephone Laboratories, Inc. 1979
- Erich Grädel: **Complexity Theory**; Lecture Notes, Winter Term 2009/10
- John E. Hopcroft and Jeffrey D. Ullman: **Introduction to Automata Theory, Languages, and Computation**; Addison Wesley Publishing Company 1979
- Neil Immerman: **Descriptive Complexity**; Springer Verlag 1999
- Christos H. Papadimitriou: **Computational Complexity**; 1995 Addison-Wesley Publishing Company, Inc

Computational Problems are Everywhere

Example 1.1

- What are the factors of 54,623?
- What is the shortest route by car from Berlin to Hamburg?
- My program now runs for two weeks. Will it ever stop?
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Approach to problems:

> [T]he way is to avoid what is strong, and strike at what is weak.

(Sun Tzu: The Art of War, Chapter 6: Weak Points and Strong)

Examples

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Given a weighted graph and two vertices s, t, find the shortest path between s and t.

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Observation

Difficulty of a problem is hard to assess

Measuring the Difficulty of Problems

Question

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Approach

Estimate the resource requirements of the “best” algorithm that solves this problem.

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- Running Time
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Note

To assess the complexity of a problem, we need to consider **all possible algorithms** that solve this problem.

Problems

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Example 1.4

“Problem: Is a given graph connected?” will be modeled as the word problem of the language

$$\text{GCONN} := \{\, \langle G \rangle \mid G \text{ is a connected graph} \,\}.$$

Then for a graph G we have

$$G \text{ is connected} \iff \langle G \rangle \in \text{GCONN}.$$

Note

The notation $\langle G \rangle$ denotes a suitable encoding of the graph G over some fixed alphabet (e.g., $\{\, 0, 1 \,\}$).

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- Lambda Calculus
- μ-Recursion
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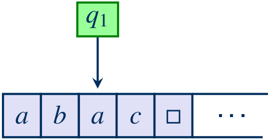

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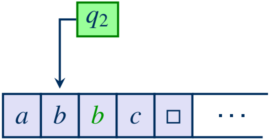

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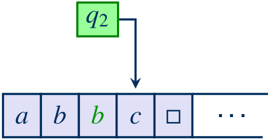

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Avoid: Suppose from now on all problems we consider to be decidable.

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Measuring running time and memory requirements depends highly on the **machine**, and not so much on the **problem**.

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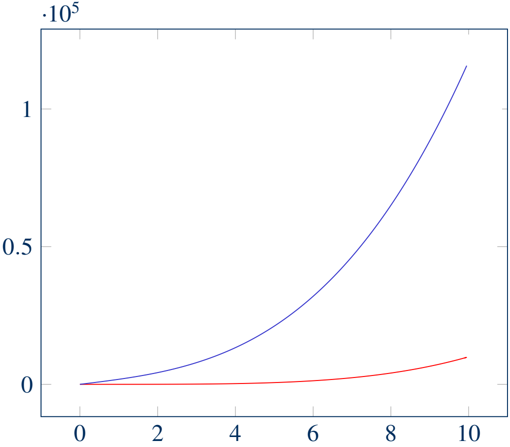
More formally:

$$f(n) = \mathcal{O}(g(n)) \iff \exists c > 0\, \exists n_0 \in \mathbb{N}\, \forall n > n_0 : f(n) \leq c \cdot g(n).$$

Big-$\mathcal{O}$-Notation

Example 1.6

$100n^3 + 1729n = \mathcal{O}(n^4)$:



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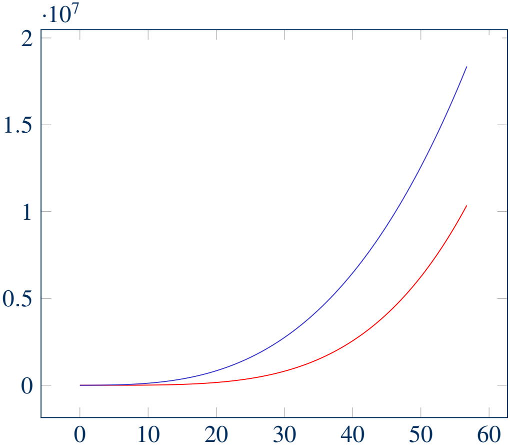

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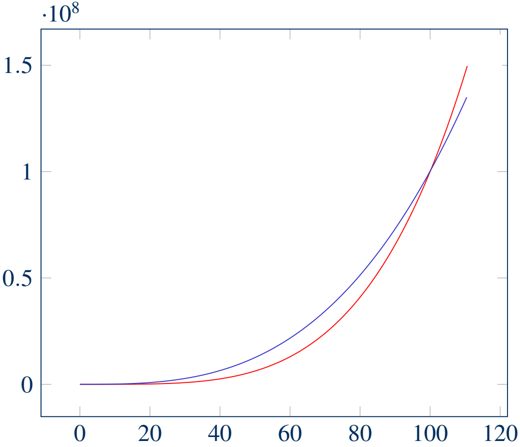

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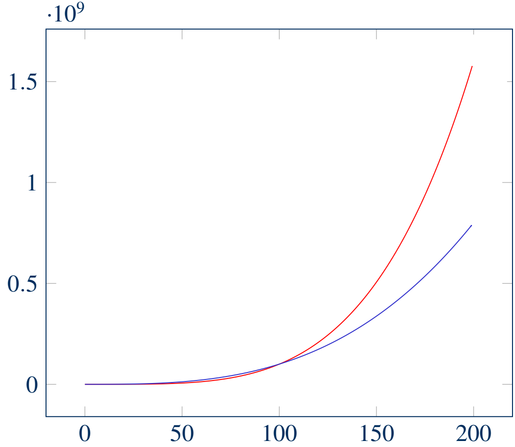

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Exact complexity of TSP unknown.

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$\oplus$P, #P, AC, AC^0, ACC0, AM, AP, APSpace, BPL, BPP, BQP, coNP, E, Exp, FP, IP, MA, MIP, NC, NExpTime, P$/$poly, PH, PP, PSpace, RL, RP, Σ_i^{p}, TISP$(T(n), S(n))$, ZPP, …

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Approach (cf. Cobham–Edmonds Thesis)

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Note

The Cobham-Edmonds-Thesis is only a **rule of thumb**: there are (practically) tractable problems outside of P, and (practically) intractable problems in P.

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Caveat

It is not known how big P is.

In particular, it is unknown whether P $\neq$ NP or not.

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But: we still do not know whether we can or cannot solve satisfiability efficiently.
We only know it will be difficult to find out . . .

Live Survey: Student Haves and Wants

Learning Goals

- Get an overview over the foundations of Complexity Theory
- Gain insights into advanced techniques and results in Complexity Theory
- Understand what it means to “compute” something, and what the strengths and limits of different computing approaches are
- Get a feeling of how hard certain problems are, and where this hardness comes from
- Appreciate how very little we actually know about the computational complexity of many problems

Lecture Outline (1)

- **Turing Machines** (Revision)
  Definition of Turing Machines; Variants; Computational Equivalence; Decidability and Recognizability; Enumeration
- **Undecidability**
  Examples of Undecidable Problems; Mapping Reductions; Rice's Theorem; Recursion Theorem
- **Time Complexity**
  Measuring Time Complexity; Many-One Reductions; Cook-Levin Theorem; Time Complexity Classes (P, NP, ExpTime); NP-completeness; pseudo-NP-complete problems
- **Space Complexity**
  Space Complexity Classes (PSpace, L, NL); Savitch's Theorem; PSpace-completeness; NL-completeness; NL = coNL

Lecture Outline (2)

- **Diagonalisation**
  Hierarchy Theorems (det. Time, non-det. Time, Space); Gap Theorem; Ladner's Theorem; Relativisation; Baker-Gill-Solovay Theorem
- **Alternation**
  Alternating Turing Machines; APTime = PSpace; APSpace = ExpTime; Polynomial Hierarchy; NTIME$(n) \nsubseteq$ TISP$(n^{1.2}, n^{0.2})$
- **Circuit Complexity**
  Boolean Circuits; Alternative Proof of Cook-Levin Theorem; Parallel Computation (NC); P-completeness; P$/$poly; (Karp-Lipton Theorem, Meyer's Theorem)
- **Probabilistic Computation**
  Randomised Complexity Classes (RP, PP, BPP, ZPP); Sipser-Gács-Lautemann Theorem

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