The Fuzzy Linguistic Description Logic $\mathcal{ALC}_{\mathcal{FL}}$

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Abstract

We present the fuzzy linguistic description logic $\mathcal{ALC}_{\mathcal{FL}}$, an instance of the description logic framework $\mathcal{L}$–$\mathcal{ALC}$ with the certainty lattice characterized by a hedge algebra. Beside constructors of $\mathcal{L}$–$\mathcal{ALC}$, $\mathcal{ALC}_{\mathcal{FL}}$ allows the modification by hedges.

Keywords: Description logics, hedge algebras, uncertainty.

1 Introduction

Description Logics (DLs) have been studied and applied successfully in quite a lot of fields (see e.g. [1]). To deal with vague and imprecise information in real-world applications, fuzzy $\mathcal{ALC}$ [7] introduces fuzzy concepts. As a more general case of fuzzy $\mathcal{ALC}$, a DL framework $\mathcal{L}$–$\mathcal{ALC}$ based on certainty lattices is presented in [8].

Humans typically use linguistic modifiers (hedges) like "very", "more or less" etc. to distinguish, e.g. between an old man and a very old one. In [9] Zadeh uses exponent functions to represent hedges modifying fuzzy sets, e.g. $\mu_{veryA}(u) = \mu_A(u)^2$. In many human languages, there is almost a continuum of phrases like "more or less", "much less", "possibly rather" and so forth expressing different levels of emphasis. Hedge Algebras (HAs), introduced in [5], give an algebraic characterization of such linguistic hedges. $\mathcal{ALC}_{\mathcal{FH}}$ [3] and $\mathcal{ALC}_{\mathcal{FLH}}$ [2] extend fuzzy $\mathcal{ALC}$ by allowing the modification by hedges of HAs.

In general, the domain of a HA can be represented as a lattice. Thus, an instance of $\mathcal{L}$–$\mathcal{ALC}$, where the certainty lattice is a truth domain represented by a HA, is a DL in which the truth degree of an assertion is a linguistic value, e.g., $John$ is an element of a concept $Young$ with degree $VeryTrue$. The idea is meaningful because in daily life, when being asked to assess the degree of a person being $Young$, it is usually easier to give a verbal answer like, for example, $Very\_High$ or $Quite\_True$, rather than to give a numerical answer like, for example, 0.5 or 0.7.

In this paper, we present the fuzzy linguistic DL $\mathcal{ALC}_{\mathcal{FL}}$, which is such an instance of $\mathcal{L}$–$\mathcal{ALC}$. Beside constructors of $\mathcal{L}$–$\mathcal{ALC}$, $\mathcal{ALC}_{\mathcal{FL}}$ allows the modification by hedges. Because the certainty lattice is characterized by a HA, the modification by hedges becomes more natural than that in $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$. Moreover, we show that $\mathcal{ALC}_{\mathcal{FL}}$ overcomes the following drawback in $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$.

In $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$, the hedge application is ambiguous. For example, the concept $VeryMolYoung$[1] can be interpreted as $(VeryMol)Young$ in which $VeryMol$ is a modifier, or $Very(MolYoung)$ in which $Very$ and Mol are two different modifiers. Unfortunately, in both $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$ $(VeryMol)Young \neq Very(MolYoung)$. Therefore, we may have

$$\begin{array}{l} \langle a : Very(MolYoung) < 0.7\rangle \\ \not\models\ \langle a : (VeryMol)Young < 0.7\rangle, \end{array}$$

which is surprising.

The paper is structured as follows. In Section 2, we discuss linear symmetric HAs and restrict ourselves to monotonic HAs. Then, $\mathcal{ALC}_{\mathcal{FL}}$ is introduced in Section 3 and the satisfiability problem is discussed in Section 4. A brief conclusion in Section 5 concludes the paper.

2 Logical Basis

2.1 Linear Symmetric Hedge Algebras

In order to define inverse mappings of hedges later on with ease, we consider linear symmetric HAs only. The readers are referred to [4, 5, 6] for general HAs.

Consider a truth domain consisting of linguistic values, e.g., $VeryVeryTrue$, $PossiblyMoreFalse$, etc. In such a truth domain the value $VeryVeryTrue$ is obtained by applying the modifier $Very$ twice to the generator $True$. Thus, given a set of generators $G = \{True, False\}$ and a nonempty finite set H of hedges, the set X of linguistic values is $\{\delta c \mid c \in G,\ \delta \in H^*\}$. Furthermore, if we consider $True > False$, then this order relation also holds for other pairs, e.g., $VeryTrue > MoreTrue$. It means that there exists a partial order $>$ on X.

In general, given nonempty finite sets G and H of generators and hedges resp., the set of values generated from G and H is defined as $X = \{\delta c \mid c \in G,\ \delta \in H^*\}$. Given a strictly partial order $>$ on X, we define $u \geq v$ iff $u > v$ or $u = v$. Thus, X is described by an abstract algebra $AX = (X, G, H, >)$.

[1] Mol is an abbreviation for more-or-less

Each hedge $h \in H$ can be regarded as a unary function $h : X \to X$, $x \mapsto hx$. Moreover, suppose that each hedge is an *ordering operation*, i.e., $\forall h \in H$. $\forall x \in X$. $hx > x$ xor $hx < x$. Let $I \notin H$ be the identity hedge, i.e., $Ix = x$ for all $x \in X$. Let us define some properties of hedges in the following definition.

Definition 1. *A hedge chain σ is a word over H, $\sigma \in H^*$. In the hedge chain $h_p \dots h_1$, h_1 is called the first hedge whereas h_p is called the last one.*

Given two hedges h, k, we say that

- *h and k are* converse *if $\forall x \in X.hx > x$ iff $kx < x$;*
- *h and k are* compatible *if $\forall x \in X.hx > x$ iff $kx > x$;*
- *h modifies terms stronger or equal than k, denoted by $h \geq k$, if $\forall x \in X.(hx \leq kx \leq x)$ or $(hx \geq kx \geq x)$[2]. $h > k$ if $h \geq k$ and $h \neq k$;*
- *h is* positive w.r.t. *k if $\forall x \in X.(hkx < kx < x)$ or $(hkx > kx > x)$;*
- *h is* negative w.r.t. *k if $\forall x \in X.(kx < hkx < x)$ or $(kx > hkx > x)$.*

The most commonly used HAs are symmetric ones, in which there are exactly two generators, like e.g., $G = \{True, False\}$. In this paper, we only consider symmetric HAs. Let $G = \{c^+, c^-\}$ where $c^+ > c^-$. c^+ and c^- are called *positive* and *negative generators* respectively. The set H is decomposed into the subsets $H^+ = \{h \in H \mid hc^+ > c^+\}$ and $H^- = \{h \in H \mid hc^+ < c^+\}$. For each value $x \in X$, let $H(x) = \{\sigma x \mid \sigma \in H^*\}$.

Definition 2. *An abstract algebra $AX = (X, G, H, >)$, where $H \neq \emptyset$, $G = \{c^+, c^-\}$ and $X = \{\sigma c \mid c \in G, \sigma \in H^*\}$, is called a* linear symmetric HA *if it satisfies the following conditions:*

(A1) For all $h \in H^+$ and $k \in H^-$, h and k are converse.

(A2) The sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered with the least element I.

(A3) For each pair $h, k \in H$, either h is positive or negative wrt k.

(A4) If $h \neq k$ and $hx < kx$ then $h'hx < k'kx$, for all $h, k, h', k' \in H$ and $x \in X$.

(A5) If $u \notin H(v)$ and $u < v$ $(u > v)$ then $u < hv$ $(u > hv$, resp.), for any $h \in H$.

[2]The orders used here are not strict since we want to compare any $h \in H$ with I, e.g. $Very \geq I$.

Example 3. *Consider a HA* $AX = (X, \{True, False\}, H, >)$*, where* $H = \{Very, More, \ Probably, Mol\}$*, and (i)* $Very$ *and* $More$ *are positive wrt* $Very$ *and* $More$*, negative wrt* $Probably$ *and* Mol*; (ii)* $Probably$ *and* Mol *are negative wrt* $Very$ *and* $More$*, positive wrt* $Probably$ *and* Mol*.*

H is decomposed into $H^+ = \{Very, More\}$ *and* $H^- = \{Probably, Mol\}$*. In* $H^+ \cup \{I\}$ *we have* $Very > More > I$*, whereas in* $H^- \cup \{I\}$ *we have* $Mol > Probably > I$*.*

The following proposition shows how to compare elements in X.

Proposition 4 ([5])**.** *Consider a linear symmetric HA* $AX = (X, G, H, >)$*, an element* $u \in X$*, and two values generated from* u $x = h_n \dots h_1 u, y = k_m \dots k_1 u$*. There exists* $j \leq min(m, n) + 1$ *such that for every* $i < j$*, we have* $h_i = k_i$*, and:*

(i) $x < y$ *iff* $h_j x_j < k_j x_j$*, where* $x_j = h_{j-1} \dots h_1 u$*;*

(ii) $x = y$ *iff* $n = m = j$ *and* $h_j x_j = k_j x_j$*;*

Because each of $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ is linearly ordered, together with the fact that G is linearly ordered, it is quite straight to prove the following.

Proposition 5 ([5])**.** *For a linear symmetric HA* $AX = (X, G, H, >)$*,* X *is linearly ordered.*

Definition 6. *Given* $x = \sigma c$*, where* $\sigma \in H^*, c \in \{c^+, c^-\}$*, we call* $y = \sigma c'$ *the* contradictory element *of* x*, denoted by* $y = -x$*, if* $\{c, c'\} = \{c^+, c^-\}$*.*

Let $x, y \in X$*, we define* $\vee$*,* $\wedge$*, and* $\rightarrow$ *as:* $x \vee y = \max(x, y); x \wedge y = \min(x, y)$*;* $x \rightarrow y = -x \vee y$*.*

In [6], HAs are extended by adding two artificial hedges inf and sup defined as $\inf(x) = \text{infimum}(H(x))$, $\sup(x) = \text{supremum}(H(x))$. If $H \neq \emptyset$, $H(c^+)$ and $H(c^-)$ are infinite, according to [6] $\inf(c^+) = \sup(c^-)$. Let $W = \inf(c^+) = \sup(c^-)$, $\sup(True) = 1$, $\inf(False) = 0$, i.e., 1 and 0 resp. are the greatest and the least elements of X. The following properties show that X can be used as the truth domain for a non-classical logic.

Proposition 7 ([6])**.** *For every symmetric extended HA, the following properties hold:*

1. $-hx = h(-x)$*, for any* $h \in H$;
2. $- - x = x$; $-1 = 0$, $-0 = 1$, $-W = W$;
3. $-(x \vee y) = (-x \wedge -y)$,
   $-(x \wedge y) = (-x \vee -y)$;
4. $x \wedge -x < W < y \vee -y$;
5. $x > y$ *iff* $-x < -y$;
6. $x \rightarrow y = -y \rightarrow -x$;

7. $x \to (y \to z) = y \to (x \to z)$;
8. $x \to y \geq x' \to y'$
   if $x \leq x'$ *and/or* $y \geq y'$;
9. $1 \to x = x,\ x \to 1 = 1,$
   $0 \to x = 1,\ x \to 0 = -x$;
10. $x \to y > W$ *iff* $x < W$ *or* $y > W$;
11. $x \to y < W$ *iff* $y < W$ *and* $x > W$;
12. $x \to y = 1$ *iff* $x = 0$ *or* $y = 1$.

To define the semantics of the hedge modification in our logic, we define the so-called *inverse mapping* of a hedge. In order to define it with ease, let us consider some restrictions for the HAs representing the truth domain. In the rest of the paper, without stating otherwise, "hedge algebra" means "linear symmetric hedge algebra".

2.2 Hedge Algebras as Truth Domains

Definition 8. *A HA $AX = (X, G, H, >)$ is called* monotonic *if each $h \in H^+$ (H^-) is positive wrt all $k \in H^+$ (H^-), and negative wrt all $k \in H^-$ (H^+).*

As defined, both sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered. However, $H \cup \{I\}$ is not, e.g., in Example 3 $Very \in H^+$ and $Mol \in H^-$ are not comparable. Let us extend the order relation on $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ to one on $H \cup \{I\}$ as follows.

Definition 9. *Given $h, k \in H \cup \{I\}$, $h \geq_h k$ iff*

- $h \in H^+, k \in H^-$; *or*
- $h, k \in H^+ \cup \{I\}$ *and* $h \geq k$; *or*
- $h, k \in H^- \cup \{I\}$ *and* $h \leq k$.

$h >_h k$ *iff* $h \geq_h k$ *and* $h \neq k$.

Example 10. *The HA in Example 3 is monotonic. The order relation $>_h$ in $H \cup \{I\}$ is $Very >_h More >_h I >_h Probably >_h Mol$.*

Then, in monotonic HAs, hedges are "context-free", i.e., a hedge modifies the meaning of a linguistic value independently of preceding hedges in the hedge chain.

Proposition 11. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then,*

$$h >_h k \quad \Leftrightarrow \quad h\sigma c^+ > k\sigma c^+ \tag{1}$$

Proof. By induction on the length of σ:

Base step: $\sigma = \epsilon$: obvious.

Induction step: Assume that (1) holds for $\sigma = h_n \dots h_1$ and consider $\sigma' = h_{n+1}h_n \dots h_1$.

($\Rightarrow$) Assuming $h >_h k$, we consider 2 cases:

(i) $h_{n+1}h_n \dots h_1c^+ > h_n \dots h_1c^+$: because (1) holds for σ, $h_{n+1}h_n \dots h_1c^+ > h_n \dots h_1c^+ = Ih_n \dots h_1c^+$ implies $h_{n+1} >_h I$ and thus $h_{n+1} \in H^+$. There are 3 cases for h and k:

- $h \in H^+, k \in H^-$: because AX is monotonic, h is positive w.r.t h_{n+1} whereas k is negative w.r.t h_{n+1}. Therefore, $hh_{n+1} \dots h_1c^+ > h_{n+1} \dots h_1c^+ > kh_{n+1} \dots h_1c^+$, i.e., ($\Rightarrow$) holds for σ'.
- $h, k \in H^+ \cup \{I\}$ and $h > k$, i.e., h, k are positive w.r.t h_{n+1}. Because $h > k$, $hh_{n+1} \dots h_1c^+ > kh_{n+1} \dots h_1c^+ \geq h_{n+1} \dots h_1c^+$, i.e., ($\Rightarrow$) holds for σ'.
- $h, k \in H^- \cup \{I\}$ and $h < k$: similar to the previous case.

(ii) $h_{n+1}h_n \dots h_1c^+ < h_n \dots h_1c^+$: similar to the previous case.

($\Leftarrow$) Assume $h\sigma c^+ > k\sigma c^+$. Suppose $h \leq_h k$, then either $h <_h k$ or $h = k$. If $h <_h k$, the proof of ($\Rightarrow$) implies $h\sigma c^+ < k\sigma c^+$. If $h = k$ then $h\sigma c^+ = k\sigma c^+$. Thus, $h\sigma c^+ \leq k\sigma c^+$, which contradicts the assumption.

Hence, (1) holds for σ of length n implies (1) holds for σ' of length $n+1$. Consequently, (1) holds for all $\sigma \in H^*$. □

The following property follows Proposition 11 immediately.

Corollary 12. *Given a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$, we have*

1. $\forall h \in H^+, k \in H^-. h\sigma c^+ > \sigma c^+$ *and* $k\sigma c^+ < \sigma c^+$.
2. $h \geq k \quad \Leftrightarrow \quad h\sigma c^+ \geq k\sigma c^+$

In Proposition 11, the last hedge is independent of the others. Conversely, when being the first hedge, it does not affect the meaning of the others.

Proposition 13. *Given a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then, $\forall h \in H : \sigma_1c^+ > \sigma_2c^+ \Leftrightarrow \sigma_1hc^+ > \sigma_2hc^+$.*

Proof. Let $\sigma_1 = h_p \dots h_1$, $\sigma_2 = k_q \dots k_1$. Because $\sigma_1c^+ > \sigma_2c^+$, we have $\sigma_1 \neq \sigma_2$. Thus, there exists $n \geq 0$ such that $h_n \dots h_1 = k_n \dots k_1$ and $h_{n+1} \neq k_{n+1}$. We have

$$\begin{aligned}
& h_p \dots h_1c^+ > k_q \dots k_1c^+ \\
\Leftrightarrow \quad & h_{n+1} \dots h_1c^+ > k_{n+1} \dots k_1c^+ \\
\Leftrightarrow \quad & h_{n+1} >_h k_{n+1} \\
\Leftrightarrow \quad & h_{n+1}h_n \dots h_1hc^+ > k_{n+1}k_n \dots k_1hc^+ \\
\Leftrightarrow \quad & h_p \dots h_1hc^+ > k_q \dots k_1hc^+
\end{aligned}$$

Hence, $\sigma_1 c^+ > \sigma_2 c^+ \Leftrightarrow \sigma_1 h c^+ > \sigma_2 h c^+$. $\square$

In the general case, when the generator is either c^+ or c^-, a similar property holds.

Proposition 14. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. We have $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 h c_1 > \sigma_2 h c_2$, for $c_1, c_2 \in \{c^+, c^-\}$.*

Proof.

($\Rightarrow$) Let us prove by case analysis. There are 3 cases as follows.

+ $c_1 = c_2 = c^+$: it is proved by Proposition 13.

+ $c_1 = c_2 = c^-$: we have the following equivalent transformation.

$$\begin{array}{lcl} & \sigma_1 c^- > \sigma_2 c^- & \\ \Leftrightarrow & \sigma_1 c^+ < \sigma_2 c^+ & \\ \Leftrightarrow & \sigma_1 h c^+ < \sigma_2 h c^+ & \text{(Proposition 13)} \\ \Leftrightarrow & \sigma_1 h c^- > \sigma_2 h c^- & \end{array}$$

+ $c_1 = c^+, c_2 = c^-$: we always have $\sigma_1 h c_1 = \sigma_1 h c^+ > \sigma_2 h c^- = \sigma_2 h c_2$.

($\Leftarrow$) It is proved analogously by case analysis.

$\square$

The following corollary follows immediately.

Corollary 15. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then, $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 \delta c_1 > \sigma_2 \delta c_2$.*

Therefore, in monotonic HAs, a hedge is not only independent of other hedges in the hedge chain but also independent of the generators, if it is the first or the last hedge in the chain. This property will help us to define the concept of inverse hedges in the next subsection.

2.3 Inverse Mappings of Hedges

In daily life, people often use words in relative assessments, e.g., "it is quite true that Robert is very old". As discussed in [9], assessments like that can be considered as a composition of an individual, e.g., Robert, a fuzzy predicate, e.g., VeryOld, and a truth value, e.g., QuiteTrue. In the context of fuzzy DLs, the above assessment is typically represented by

$$(VeryOld)^{\mathcal{I}}(Robert^{\mathcal{I}}) = QuiteTrue.$$

In a fuzzy linguistic logic [9], the following two assessments are equivalent: "it is true that Robert is very old" and "it is very true that Robert is old". It means that somehow the modifier from the truth value can be moved to the fuzzy predicate and vice versa. This idea is formalized in [4] by the two following rules represented in a DL representation:

$$RT1: \quad (hC)^{\mathcal{I}}(a) = \sigma c \rightarrow C^{\mathcal{I}}(a) = \sigma hc$$
$$RT2: \quad C^{\mathcal{I}}(a) = \sigma hc \rightarrow (hC)^{\mathcal{I}}(a) = \sigma c$$

in which C is a concept, hC a concept C modified by a hedge h, a an individual, $\mathcal{I}$ an interpretation, σ a hedge chain, and c a generator of the truth domain.

However, the rules are not complete. E.g., if the truth-degree of "John is Young" is VeryTrue and we want to compute the truth-degree of "John is MoreYoung", then no rules are applicable. This problem motivates the definition of a so-called *inverse mapping of a hedge*, and based on this definition, a generalized version of rule (RT2).

Suppose that for each hedge $h \in H$, there exists a mapping $h^- : X \rightarrow X$ such that $h^-(\sigma hc) = \sigma c$ for all $\sigma \in H^*, c \in G$. Then the rule $(RT2)$ is generalized as follows:

$$GRT2 : C^{\mathcal{I}}(a) = \delta c \rightarrow (hC)^{\mathcal{I}}(a) = h^-(\delta c).$$

Note that $GRT2$ becomes $RT2$ when $\sigma h = \delta$.

In the following, we define h^- formally by axiomization. Given a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$ and a hedge $h \in H$. Suppose $h^- : X \rightarrow X$ is a mapping such that

$$h^-(\sigma hc) = \sigma c, \qquad \text{for } c \in \{c^+, c^-\} \tag{2}$$

According to Propostion 14, $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 hc_1 > \sigma_2 hc_2$. By (2), $h^-(\sigma_1 hc_1) = \sigma_1 c_1$, $h^-(\sigma_2 hc_2) = \sigma_2 c_2$. Hence, $h^-(\sigma_1 hc_1) > h^-(\sigma_2 hc_2) \Leftrightarrow \sigma_1 hc_1 > \sigma_2 hc_2$. Generalizing this idea, h^- should satisfy:

$$\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow h^-(\sigma_1 c_1) > h^-(\sigma_2 c_2) \tag{3}$$

Definition 16. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$ and a hedge $h \in H$. A mapping $h^- : X \rightarrow X$ is called an* inverse mapping *of h iff it satisfies* (2) *and* (3)*.*

A question concerning the existence of such mappings can be raised. Let us consider the following example.

Example 17. *Consider the HA given in Example 3. For $H(True)$, the* *inverse mappings are defined as follows*

$$V^-(\sigma T) = \begin{cases} \delta T & \textit{if } \sigma = \delta V, \\ \sigma MolMolT & \textit{otherwise.} \end{cases}$$

$$M^-(\sigma T) = \begin{cases} \sigma VVT & \textit{if } \sigma = \delta V, \\ \delta T & \textit{if } \sigma = \delta M, \\ \sigma MolMolT & \textit{otherwise.} \end{cases}$$

$$P^-(\sigma T) = \begin{cases} \sigma MolMolT & \textit{if } \sigma = \delta Mol, \\ \delta T & \textit{if } \sigma = \delta P, \\ \sigma VVT & \textit{otherwise.} \end{cases}$$

$$Mol^-(\sigma T) = \begin{cases} \delta T & \textit{if } \sigma = \delta Mol, \\ \sigma VVT & \textit{otherwise.} \end{cases}$$

in which T, V, M, and P stand for True, Very, More, and Probably resp.

For $H(False)$, the mappings are defined as $h^-(\sigma False) = -h^-(\sigma True)$ for each $h \in H$.

It is easily verified that these mappings satisfy (2) *and* (3)*.*

Given hedges $h_1, \ldots, h_p$, one may need to construct an inverse mapping $(h_p \ldots h_1)^- : X \to X$ of a hedge chain $h_p \ldots h_1$. On the one hand, we expect $(h_p \ldots h_1)^-(\sigma h_p \ldots h_1 c) = \sigma c$. On the other hand, we have

$$\begin{aligned} & {h_p}^-(\ldots({h_2}^-({h_1}^-(\sigma h_p \ldots h_2 h_1 c)))\ldots) \\ = \; & {h_p}^-(\ldots({h_2}^-(\sigma h_p \ldots h_2 c))\ldots) \\ & \ldots \\ = \; & {h_p}^-(\sigma h_p c) = \sigma c \end{aligned}$$

Therefore, $(h_p \ldots h_1)^-$ can be defined as

$$(h_p \ldots h_1)^-(\sigma c) = {h_p}^-(\ldots({h_1}^-(\sigma c))\ldots) \tag{4}$$

We have the following property which is the general case of Corollary 15.

Proposition 18. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$, a hedge chain δ and its inverse mapping δ^-. Then, $\sigma_1 c_1 > \sigma_2 c_2$ iff $\delta^-(\sigma_1 c_1) > \delta^-(\sigma_2 c_2)$.*

Proof. Let $\delta = h_p \ldots h_1$. According to (4), $\delta^-(\sigma c) = {h_p}^-(\ldots({h_1}^-(\sigma c))\ldots)$. According to (3), we have $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow {h_1}^-(\sigma_1 c_1) > {h_1}^-(\sigma_2 c_2) \Leftrightarrow \ldots \Leftrightarrow {h_p}^-(\ldots({h_1}^-(\sigma_1 c_1))\ldots) > {h_p}^-(\ldots({h_1}^-(\sigma_2 c_2))\ldots)$. □

3 $\mathcal{ALC}_{\mathcal{FL}}$

This section discusses the *fuzzy linguistic description logic* $\mathcal{ALC}_{\mathcal{FL}}$, i.e., a DL in which the truth domain of interpretations is represented by a hedge algebra.

The syntax of $\mathcal{ALC}_{\mathcal{FL}}$ is similar to that of $\mathcal{L}$–$\mathcal{ALC}$ except that $\mathcal{ALC}_{\mathcal{FL}}$ allows concept modifiers. Hence, $\mathcal{ALC}_{\mathcal{FL}}$-concepts are defined by

$$A \,|\, \top \,|\, \bot \,|\, \neg C \,|\, C \sqcap D \,|\, C \sqcup D \,|\, \delta C \,|\, \exists R.C \,|\, \forall R.C,$$

where A denotes primitive concepts, R roles, C and D concepts, and δ modifiers.

The semantics is based on the notion of interpretations. Given a monotonic HA $AX = (X, \{True, False\}, H, >)$, an interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ in which $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}}$ is a mapping, which maps different individuals to different elements in $\Delta^{\mathcal{I}}$, concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to X$ and role R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to X$. The extension of $\mathcal{I}$ for complex concepts is

$$\begin{array}{rcl}
\top^{\mathcal{I}}(d) &=& \sup(True) \text{ for all } d \in \Delta^{\mathcal{I}}, \\
\bot^{\mathcal{I}}(d) &=& \inf(False) \text{ for all } d \in \Delta^{\mathcal{I}}, \\
(\neg C)^{\mathcal{I}}(d) &=& -C^{\mathcal{I}}(d), \\
(C \sqcap D)^{\mathcal{I}}(d) &=& C^{\mathcal{I}}(d) \wedge D^{\mathcal{I}}(d), \\
(C \sqcup D)^{\mathcal{I}}(d) &=& C^{\mathcal{I}}(d) \vee D^{\mathcal{I}}(d), \\
(\delta C)^{\mathcal{I}}(d) &=& \delta^{-}\left(C^{\mathcal{I}}(d)\right), \\
(\forall R.C)^{\mathcal{I}}(d) &=& \bigwedge_{d' \in \Delta^{\mathcal{I}}}(-R^{\mathcal{I}}(d,d') \vee C^{\mathcal{I}}(d')), \\
(\exists R.C)^{\mathcal{I}}(d) &=& \bigvee_{d' \in \Delta^{\mathcal{I}}}(R^{\mathcal{I}}(d,d') \wedge C^{\mathcal{I}}(d')),
\end{array}$$

where $\wedge, \vee$ are the meet and join operations resp., $-x$ is the contradictory element of x, and δ^{-} is the inverse of the hedge chain δ.

Fuzzy assertions are expressions of the forms $\langle \alpha \circ x \rangle$ where $\circ \in \{>, \geq, \leq, <\}$, α is of type $a : C$ or $(a, b) : R$, and $x \in X$. *Fuzzy terminological axioms*, the semantics of fuzzy assertions and terminological axioms are defined similarly to those in [2, 3, 7, 8].

Semantically, two concepts C, D are said to be equivalent, denoted by $C \equiv D$, iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all $\mathcal{I}$. For example, $\top \equiv \neg\bot$ or $C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$. Some equivalences concerning the hedge modification are showed in the following proposition.

Proposition 19. *We have the following semantical equivalence:*

$$\begin{array}{c}
\delta(C \sqcap D) \equiv \delta(C) \sqcap \delta(D) \\
\delta(C \sqcup D) \equiv \delta(C) \sqcup \delta(D) \\
\delta_1(\delta_2 C) \equiv (\delta_1\delta_2)C
\end{array}$$

Proof. Consider an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, and $x \in \Delta^{\mathcal{I}}$.

- Let $C^{\mathcal{I}}(x) = \sigma_1 c_1, D^{\mathcal{I}}(x) = \sigma_2 c_2$. Because the hedge algebra AX is linear, we have either $\sigma_1 c_1 \geq \sigma_2 c_2$ or $\sigma_1 c_1 \leq \sigma_2 c_2$. Wlog, suppose $\sigma_1 c_1 \geq \sigma_2 c_2$. Therefore, $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \wedge D^{\mathcal{I}}(x) = \sigma_2 c_2$. And thus, $(\delta(C \sqcap D))^{\mathcal{I}}(x) = \delta^-(\sigma_2 c_2)$.

  We have $(\delta(C))^{\mathcal{I}}(x) = \delta^-(\sigma_1 c_1), (\delta(D))^{\mathcal{I}}(x) = \delta^-(\sigma_2 c_2)$. According to Proposition 18, because $\sigma_1 c_1 \geq \sigma_2 c_2$ we have $\delta^-(\sigma_1 c_1) \geq \delta^-(\sigma_2 c_2)$. Therefore, $(\delta(C) \sqcap \delta(D))^{\mathcal{I}}(x) = (\delta(C))^{\mathcal{I}}(x) \wedge (\delta(D))^{\mathcal{I}}(x) = \delta^-(\sigma_1 c_1) \wedge \delta^-(\sigma_2 c_2) = \delta^-(\sigma_2 c_2)$.

  Hence, $(\delta(C \sqcap D))^{\mathcal{I}}(x) = (\delta(C) \sqcap \delta(D))^{\mathcal{I}}(x)$ for all interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$. Therefore, $\delta(C \sqcap D) \equiv \delta(C) \sqcap \delta(D)$.

- The proof of $\delta(C \sqcup D) \equiv \delta(C) \sqcup \delta(D)$ is similar.

- Let $C^{\mathcal{I}}(x) = \sigma c$. We have $(\delta_1(\delta_2 C))^{\mathcal{I}}(x) = \delta_1^-(\delta_2^-(\sigma c)), ((\delta_1 \delta_2)\, C)^{\mathcal{I}}(x) = (\delta_1 \delta_2)^-(\sigma c)$.

  Let $\delta_1 = h_p \ldots h_1, \delta_2 = k_q \ldots k_1$. According to (4) we have:

  $$\delta_1^-(\delta_2^-(\sigma c)) = h_p^-(\ldots(h_1^-(k_q^-(\ldots(k_1^-(\sigma_c))))))$$
  $$(\delta_1 \delta_2)^-(\sigma c) = h_p^-(\ldots(h_1^-(k_q^-(\ldots(k_1^-(\sigma_c))))))$$

  Therefore, $\delta_1^-(\delta_2^-(\sigma c)) = (\delta_1 \delta_2)^-(\sigma c)$, and thus $(\delta_1(\delta_2 C))^{\mathcal{I}}(x) = ((\delta_1 \delta_2)\, C)^{\mathcal{I}}(x)$. Because it holds for all interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$, we have $\delta_1(\delta_2 C) \equiv (\delta_1 \delta_2) C$.

□

For instance, $Very(MolC) \equiv (VeryMol)C$. Therefore the drawback of $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$ specified in Section 1 is solved in $\mathcal{ALC}_{\mathcal{FL}}$.

4 The Satisfiability Problem

Similarly to fuzzy $\mathcal{ALC}$, $\mathcal{ALC}_{\mathcal{FH}}$, $\mathcal{ALC}_{\mathcal{FLH}}$, and $\mathcal{L}$–$\mathcal{ALC}$, in $\mathcal{ALC}_{\mathcal{FL}}$ the entailment problem can be converted to the satisfiability problem, which can be solved by a tableau algorithm. As usual, starting from a set S of fuzzy constraints, the propagation rules are applied step by step to add "simpler" constraints preserving the satisfiability. This process terminates and gives a *completion set* to which no rules are applicable. If there is no clash in the completion set, we can construct a model for S, otherwise, S is unsatisfiable.

A set S of fuzzy constraints contains a *clash* iff it contains either one of the unsatisfiable constraints $\langle a : \bot \geq x \rangle$ where $x > 0$, $\langle a : \top \leq x \rangle$ where $x < 1$, $\langle \psi < 0 \rangle, \langle \psi > 1 \rangle, \langle a : \bot > x \rangle$, or $\langle a : \top < x \rangle$, or S contains a conjugated pair of fuzzy constraints as in Table 1.

Our calculus for solving the unsatisfiability problem in $\mathcal{ALC}_{\mathcal{FL}}$ consists of transformation rule RT1 and a set of constraint propagation rules. The set of propagation rules is given in Table 2. Note that all rules except the ones that handle concept modifiers are similar to those for fuzzy $\mathcal{ALC}$, but not as complicated as those for $\mathcal{L}$–$\mathcal{ALC}$. The reason is that in a general lattice, for incomparable elements u and v we have $u \wedge v, u \vee v \notin \{u, v\}$, but in $\mathcal{ALC}_{\mathcal{FL}}$ we always have $u \wedge v, u \vee v \in \{u, v\}$ because the HA in consideration is linear. Therefore, rules handling $\neg, \sqcap, \sqcup, \forall$, and $\exists$ constructors are obtained by replacing n by σc, $1 - n$ by $\sigma \bar{c}$, where $\bar{c} = -c$, in those rules for fuzzy $\mathcal{ALC}$. To handle concept modifiers, we add four new rules $(\delta_>), (\delta_\geq), (\delta_\leq)$ and $(\delta_<)$. The reading of these rules is the same as that in [2, 3, 7, 8].

Proposition 20. *A finite set S of fuzzy constraints is satisfiable iff there exists a clash-free completion of S.*

Proof. Because the HA representing the truth domain is linear, the argument is similar to the ones for fuzzy $\mathcal{ALC}$ in [7] except for the rules handling hedge modifiers. Hence, we focus on these rules only in order to save space.

$(\Rightarrow)$ By case analysis, it is easily verified that the rules are sound, i.e., if we apply a rule to a satisfiable set S_1 of constrants, the result S_2 is also satisfiable, and thus, clash-free. Let us consider the rule $(\delta_\geq)$, for rules $(\delta_>), (\delta_<)$ and $(\delta_\leq)$, similar arguments can be used.

$(\delta_\geq)$ Assume that $(\delta_\geq)$ is applicable, i.e., S_1 contains $\langle w : \delta C \geq \sigma c \rangle$ for some $w, \sigma c$, and modified concept δC. Since S_1 is satisfiable, there exist an interpretation $\mathcal{I}$ that satisfies $\langle w : \delta C \geq \sigma c \rangle$, i.e., $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) \geq \sigma c$. Let $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0 \geq \sigma c$, $\delta = h_p \ldots h_1$ where $p \geq 0, h_i \in H \forall i = 1 \ldots p$. Applying RT1 p times, we have $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma h_p \ldots h_1 c_0 = \gamma \delta c_0$. There are two cases: $\gamma c_0 = \sigma c$ or $\gamma c_0 > \sigma c$. In the first case, $\gamma c_0 = \sigma c$ implies $\gamma \delta c_0 = \sigma \delta c$. In the second case, according to Corollary 15, $\gamma c_0 > \sigma c \Leftrightarrow \gamma \delta c_0 > \sigma \delta c$. Therefore, we always have $\gamma \delta c_0 \geq \sigma \delta c$. Hence, $C^{\mathcal{I}}(w^{\mathcal{I}}) \geq \sigma \delta c$, i.e., $\mathcal{I}$ satisfies $\langle w : C \geq \sigma \delta c \rangle$ and S_2 as well.

$(\Leftarrow)$ Assume S' is a clash-free completion of S. Let us construct a model for the fuzzy constraints in S' that contains only primitive concepts or roles, and prove that it is a model of S', and S as well.

Since S' is clash-free, for each concept A and element w that appear in S' in the form $\langle w : A \circ x \rangle$, there exists a non-empty set $\tau(A, w) = \{x_0 \in X \mid \forall \langle w : A \circ x \rangle \in S'.x_0 \circ x\}$ in which $\circ \in \{<, \leq, \geq, >\}$. Similarly, for each role R

	$\langle \alpha \leq y \rangle$	$\langle \alpha < y \rangle$
$\langle \alpha \geq x \rangle$	$x > y$	$x \geq y$
$\langle \alpha > x \rangle$	$x \geq y$	$x \geq y$

Table 1: Conjugated pairs

$$\langle w : \neg C \geq \sigma c\rangle \rightarrow \langle w : C \leq \sigma \bar{c}\rangle \qquad (\neg_{\geq})$$
$$\langle w : \neg C \leq \sigma c\rangle \rightarrow \langle w : C \geq \sigma \bar{c}\rangle \qquad (\neg_{\leq})$$

$$\langle w : C \sqcap D \geq \sigma c\rangle \rightarrow \langle w : C \geq \sigma c\rangle, \langle w : D \geq \sigma c\rangle \qquad (\sqcap_{\geq})$$
$$\langle w : C \sqcup D \leq \sigma c\rangle \rightarrow \langle w : C \leq \sigma c\rangle, \langle w : D \leq \sigma c\rangle \qquad (\sqcup_{\leq})$$

$$\langle w : C \sqcup D \geq \sigma c\rangle \rightarrow \langle w : C \geq \sigma c\rangle \mid \langle w : D \geq \sigma c\rangle \qquad (\sqcup_{\geq})$$
$$\langle w : C \sqcap D \leq \sigma c\rangle \rightarrow \langle w : C \leq \sigma c\rangle \mid \langle w : D \leq \sigma c\rangle \qquad (\sqcap_{\leq})$$

$$\langle w_1 : \forall R.C \geq \sigma c\rangle, \psi \rightarrow \langle w_2 : C \geq \sigma c\rangle \quad \text{if } \psi \text{ is conjugated to } \langle (w_1, w_2) : R \leq \sigma \bar{c}\rangle \qquad (\forall_{\geq})$$

$$\langle w_1 : \exists R.C \leq \sigma c\rangle, \psi \rightarrow \langle w_2 : C \leq \sigma c\rangle \quad \text{if } \psi \text{ is conjugated to } \langle (w_1, w_2) : R \leq \sigma c\rangle \qquad (\exists_{\leq})$$

$$\langle w : \exists R.C \geq \sigma c\rangle \rightarrow \langle (w, x) : R \geq \sigma c\rangle, \langle x : C \geq \sigma c\rangle \qquad (\exists_{\geq})$$

if x is a new variable and there is no w' such that both $\langle (w, w') : R \geq \sigma c\rangle$ and$\langle w' : C \geq \sigma c\rangle$ are already in the constraint set

$$\langle w : \forall R.C \leq \sigma c\rangle \rightarrow \langle (w, x) : R \geq \sigma \bar{c}\rangle, \langle x : C \leq \sigma c\rangle \qquad (\forall_{\leq})$$

if x is a new variable and there is no w' such that both $\langle (w, w') : R \geq \sigma \bar{c}\rangle$ and $\langle w' : C \leq \sigma c\rangle$ are already in the constraint set

$$\langle w : \delta C \geq \sigma c\rangle \rightarrow \langle w : C \geq \sigma \delta c\rangle \qquad (\delta_{\geq})$$
$$\langle w : \delta C \leq \sigma c\rangle \rightarrow \langle w : C \leq \sigma \delta c\rangle \qquad (\delta_{\leq})$$

Table 2: The rules of the decision procedure. In addition to the presented rules there are rules $(\neg_{>})$, $(\neg_{<})$, $(\sqcap_{>}) \ldots (\delta_{<})$ for the strict relations. These can easily be obtained from the rules above by replacing $\geq$ by $>$ and $\leq$ by $<$. Note that $\bar{c} = -c$

and pair (w_1, w_2) that appear in S' in the form $\langle (w_1, w_2) : R \circ x \rangle$, there exists a non-empty set $\tau(R, w_1, w_2) = \{x_0 \in X \mid \forall \langle (w_1, w_2) : R \circ x \rangle \in S'.x_0 \circ x\}$ in which $\circ \in \{<, \leq, \geq, >\}$.

Consider an interpretation $\mathcal{I}$ such that the domain $\Delta^{\mathcal{I}}$ is the set of objects appearing in S', $\forall w \in \Delta^{\mathcal{I}}.w^{\mathcal{I}} = w$, and $A^{\mathcal{I}}(w^{\mathcal{I}}) \in \tau(A, w)$, $R^{\mathcal{I}}({w_1}^{\mathcal{I}}, {w_2}^{\mathcal{I}}) \in \tau(R, w_1, w_2)$.

It is easily verified that this interpretation satifies all constraints for primitive concepts and roles in S' if S' is clash-free. The satisfaction of the other fuzzy constraints in S' are shown by induction on the structure of the $\mathcal{ALC}_{\mathcal{FL}}$ -formula in the constraints. Once again, let us just represent one case for space reasons.

Case $\langle w : \delta C > \sigma c \rangle$ Because S' is complete, $\langle w : C > \sigma\delta c \rangle$ is in S' and is satisfied by $\mathcal{I}$ by induction assumption, i.e., $C^{\mathcal{I}}(w^{\mathcal{I}}) > \sigma\delta c$. Let $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0 > \sigma\delta c$. According to Proposition 18, we have $\gamma c_0 > \sigma\delta c \Rightarrow \delta^-(\gamma c_0) > \delta^-(\sigma\delta c)$. Since $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0$, we have$(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \delta^-(\gamma c_0)$. Besides, $\delta^-(\sigma\delta c) = \sigma c$. Hence, $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \delta^-(\gamma c_0) > \delta^-(\sigma\delta c) = \sigma c$. Therefore, $\mathcal{I}$ satisfies $\langle w : \delta C > \sigma c \rangle$. □

Let us close the section by an example to demonstrate how the calculus works.

Example 21. *Consider a knowledge base* Σ*:*

"A car is a sport car if it is very likely that it can run very very fast" holds to a degree at least $True$*. In particular, for Audi_TT cars:*

$$\langle tt : \neg(\exists speed.VVFast) \sqcup Sport \geq True \rangle \tag{5}$$

"An Audi_TT car can run at 250km/h" holds to a degree more than more-or-less True:

$$\langle (tt, 250) : speed \geq MolTrue \rangle \tag{6}$$

"250km/h is fast" holds to a degree at least More True:

$$\langle 250 : Fast \geq MTrue \rangle \tag{7}$$

We want to prove that Σ *entails that "Audi_TT cars are sport cars" to a degree more than Probably True. That is,* Σ *together with (8) is unsatisfiable.*

$$\langle tt : Sport < PTrue \rangle \tag{8}$$

The rule $(\sqcup_{\geq})$ *gives two choices:*

$$\langle tt : \neg(\exists speed.VVFast) \geq True \rangle \tag{9}$$

$$\langle tt : Sport \geq True \rangle \tag{10}$$

The latter immediately yields a clash with (8). The application of the rule $(\neg_{\geq})$ *on the former one gives:*

$$\langle tt : (\exists speed.VVFast) \leq False\rangle \tag{11}$$

Since (6) is conjucated to $\langle (tt, 250) : speed \leq False\rangle$*, rule* $(\exists_{\leq})$ *yields:*

$$\langle 250 : VVFast \leq False\rangle \tag{12}$$

Rule $(\delta_{\leq})$ *applying on (12) yields*

$$\langle 250 : Fast \leq VVFalse\rangle \tag{13}$$

which clashes with (7).

Hence, there is no clash-free completion of $\Sigma \cup \{(8)\}$*, i.e.,* $\Sigma \cup \{(8)\}$ *is unsatisfiable. Therefore,* $\Sigma \models \langle tt : Sport \geq PTrue\rangle$*.*

5 Conclusions

In the paper, we have presented the fuzzy linguistic DL $\mathcal{ALC}_{\mathcal{FL}}$, where the truth domain is represented by a monotonic HA. The main feature is that an element a belongs to a concept C with a degree specified by a linguistic value, which itself is part of a lattice represented by a HA. Besides, $\mathcal{ALC}_{\mathcal{FL}}$ allows the modification by hedges. Furthermore, the ambiguity of the hedge application in $\mathcal{ALC}_{\mathcal{FH}}$ and $\mathcal{ALC}_{\mathcal{FLH}}$ is solved in $\mathcal{ALC}_{\mathcal{FL}}$. To the best of our knowledge, no DL that not only allows the modification by hedges but also computes directly with words has been proposed.

A sound and complete decision procedure for the satisfiability problem in $\mathcal{ALC}_{\mathcal{FL}}$ has been also presented. In the future, we plan to consider the subsumption problem in $\mathcal{ALC}_{\mathcal{FL}}$.

Note that in this paper, we restrict ourselves to linear HAs in order to define inverse mappings of hedges with ease. In the general case of HAs [6, 4], where there are some incomparable hedges in H^+ or H^-, e.g., *Possibly* with *Approximately*, the domain becomes a lattice instead of linear. Thus, our further work is to extend our logic to the case where the truth domain is not linear.

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References

[1] F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi, and P. F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, 2003.

[2] S. Hölldobler, H.-N. Nguyen, and D.-K. Tran. The fuzzy description logic ALC_{FLH}. In *Proc. 9th IASTED International Conference on Artificial Intelligence and Soft Computing*, pages 99–104, 2005.

[3] S. Hölldobler, H.-P. Störr, D.-K. Tran, and H.-N. Nguyen. The subsumption problem in the fuzzy description logic $\mathcal{ALC}_{\mathcal{FH}}$. In *Proc. Tenth International Conference IPMU 2004: Information Processing and Managment of Uncertainty in Knowledge-Based Systems*, pages 243–250, 2004.

[4] C.-H. Nguyen, D.-K. Tran, V.-N. Huynh, and H.-C. Nguyen. Linguistic-valued logic and their application to fuzzy reasoning. *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, 7(4):347–361, 1999.

[5] C.-H. Nguyen and W. Wechler. Hegde algebras: An algebraic approach to structure of sets of linguistic truth value. *Fuzzy Sets and Systems*, 35:281–293, 1990.

[6] C.-H. Nguyen and W. Wechler. Extended hegde algebras and their application to fuzzy logic. *Fuzzy Sets and Systems*, 52:259–281, 1992.

[7] U. Straccia. Reasoning within fuzzy description logics. *Journal of Artificial Intelligence Research*, 14:137–166, 2001.

[8] U. Straccia. Description logics over lattices. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2005. To appear.

[9] L. A. Zadeh. The concept of a linguistic variable and its application in approximate reasoning. *Information Sciences*, 1975. Part I - 8:199–249, Part II - 8:301–357, Part III - 9:43–80.