Morphisms in Context

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Abstract. Morphisms constitute a general tool for modelling complex relationships between mathematical objects in a disciplined fashion. In Formal Concept Analysis (FCA), morphisms can be used for the study of structural properties of knowledge represented in formal contexts, with applications to data transformation and merging. In this paper we present a comprehensive treatment of some of the most important morphisms in FCA and their relationships, including dual bonds, scale measures, infomorphisms, and their respective relations to Galois connections. We summarize our results in a concept lattice that cumulates the relationships among the considered morphisms. The purpose of this work is to lay a foundation for applications of FCA in ontology research and similar areas, where morphisms help formalize the interplay among distributed knowledge bases.

1 Introduction

Formal Concept Analysis (FCA) [1] provides a fundamental mathematical methodology for the creation, analysis, and manipulation of data and knowledge. Its field of application ranges from social and natural sciences to most prominently computer science. The automated processing of knowledge necessitates an understanding of its structural properties in order to develop sound transformation algorithms, ontology merging procedures, and other operations needed for practical applications. FCA is ideally suited for such an understanding due to its sound mathematical and philosophical base, rooted in algebra and logic.

Fundamental structural properties can be captured by category-theoretical treatments [2], the heart of which are *morphisms* as structure-preserving mappings. In turn, morphisms provide abstract means for the modelling of data translation, communication, and distributed reasoning, to give a few examples. Thus the theory and application of morphisms between formal contexts have recently become a focal point in FCA.

Institution theory [3], developed in the 80's, uses formal contexts and appropriate morphisms to represent a broad class of logics. The resulting mathematical theory has been applied as a basis for various programming languages. More recently, similar ideas

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have been used as a foundation of a theory of *information flow* [4], recently considered in the context of *ontology research* [5, 6]. While the focus of the above is more on communication and transport of information, research on *Chu spaces* [7] (a special case of which are formal contexts) considers similar morphisms from a categorical viewpoint in order to obtain categories with certain specific properties.

Although morphisms between contexts have been investigated in all of the above research areas, they mostly study the same kind of morphisms, which today are typically called *infomorphisms*. These, however, are only a special choice for morphisms in FCA, and it is unclear whether they are in general preferable to other possible notions. At least two other kinds of morphisms known in FCA deserve particular attention. One is the so-called *(dual) bond*, a specific kind of relation between contexts which is of importance due to its close relationship to Galois connections. The other is *scale measure*, characterized by certain functional continuity properties.

In order to develop concrete applications to knowledge processing from structural analysis based on category theory, it is of fundamental importance to understand the properties of and relationships between different notions of morphisms. So far, only a few order- and category-theoretic treatments of morphisms in FCA are available, either studying one kind of morphism in isolation or focusing on further specific types of morphisms (an overview is given at the end of the following section). The purpose of this paper is thus to present a comprehensive study of the relations among the three types of major morphisms in FCA mentioned above. We explicate the rich interrelationships and dependencies as a step stone for further developments.

The paper is structured as follows. After explaining some preliminaries in Section 2, we study dual bonds and their relationships to direct products of formal contexts and Galois connections in Section 3. In Section 4, dual bonds featuring certain continuity properties are identified as an important subclass. Section 5 deals with the relationship between scale measures, functional types of dual bonds, and Galois connections, while Section 6 is devoted to infomorphisms. In Section 7, we summarize some of our results in the form of a concept lattice of context-morphisms, which we obtain by attribute exploration. We conclude our results by discussing various possible directions for future research in Section 8.

2 Preliminaries

Our notation basically follows [1], with a few exceptions to enhance readability for our purposes. Especially, we avoid the use of the symbol $'$ to denote the operations that are induced by a context. This will both clarify the exposition and allow us to use $'$ to enrich our pool of possible entity names (like in "$g, g' \in G$").We shortly review the main terminology using our notation, but we assume that the reader is familiar with the notation and terminology from [1]. Our treatment also requires some basic knowledge of (*antitone*) *Galois connections* and their *monotone* variant (a.k.a. *residuated maps*), which can also be found in [1].

A *(formal) context* $\mathbb{K}$ is a triple (G, M, I) where G is a set of *objects*, M is a set of *attributes*, and $I \subseteq G \times M$ is an *incidence relation*. Given $O \subseteq G$ and $A \subseteq M$, we define:

$$O^I := \{m \in M \mid g\ I\ m \text{ for all } g \in O\}, \qquad I(O) := \{m \in M \mid g\ I\ m \text{ for some } g \in O\},$$
$$A^I := \{g \in G \mid g\ I\ m \text{ for all } m \in A\}, \qquad I^{-1}(A) := \{g \in G \mid g\ I\ m \text{ for some } m \in A\}.$$

For singleton sets we use the common abbreviations $g^I := \{g\}^I$, $I(g) := I(\{g\})$, etc. The notation X^I can be ambiguous if it is not clear whether X is considered a set of objects or a set of attributes, so we will be careful to avoid such situations. We refer to $I(O)$ as the *image* of O and to $I^{-1}(A)$ as the *preimage* of A with respect to I. We use these notations for arbitrary binary relations.

A subset $O \subseteq G$ is an *extent* of $\mathbb{K}$ whenever $O = O^{II}$. O is an *attribute extent* (*object extent*) if there is some attribute n (object g) such that $O = n^I$ ($O = g^{II}$). *Intents*, *object intents* and *attribute intents* are defined dually. A *concept* of $\mathbb{K}$ is an extent-intent pair (O, A) such that $O = A^I$ (or, equivalently, $A = O^I$).

Since the extent and intent of a concept determine each other uniquely, we will usually prefer to consider only one of them. Our use of the terms *object extent* and *attribute intent* constitutes a slight deviation from standard terminology.

The central result of FCA is that contexts can be used to represent complete lattices.

Theorem 1 ([1, Theorem 3]). *For any context $\mathbb{K} = (G, M, I)$, the mapping $(\cdot)^{II} : 2^G \to 2^G$ constitutes a closure operator on the powerset 2^G. The corresponding closure system (in the sense of [1]) is the set $\mathsf{B}_o(\mathbb{K}) := \{O \subseteq G \mid O = O^{II}\}$ of all extents of $\mathbb{K}$.*

Similar statements are true for the mapping $(\cdot)^{II} : 2^M \to 2^M$, which induces a closure system $\mathsf{B}_a(\mathbb{K})$. Under set inclusion, $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K})$ are dually order-isomorphic, with $(\cdot)^I : 2^G \to 2^A$ and $(\cdot)^I : 2^A \to 2^G$ as the according isomorphisms.

We refer to $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K})$ ordered by set inclusion as the *object-* and *attribute-concept lattices*.

An important aspect of FCA is that contexts can be dualized and complemented to obtain new structures. These operations turn out to be vital for our subsequent studies. Given a context $\mathbb{K} = (G, M, I)$, the context *dual* to $\mathbb{K}$ is $\mathbb{K}^{\mathsf{d}} := (M, G, I^{-1})$. It is easy to see that dualizing a context merely changes the roles of extent and intent. Thus, with respect to the order of the concept lattices we have $\mathsf{B}_o(\mathbb{K}^{\mathsf{d}}) = \mathsf{B}_a(\mathbb{K})$ and $\mathsf{B}_a(\mathbb{K}^{\mathsf{d}}) = \mathsf{B}_o(\mathbb{K})$. The situation for complement, defined as $\mathbb{K}^{\mathsf{c}} = (G, M, \not{I})$ with $\not{I} := (G \times M) \setminus I$, is more involved since the concept lattices of $\mathbb{K}$ and $\mathbb{K}^{\mathsf{c}}$ are in general not (dually) isomorphic to each other. We can observe immediately that dualization and complementation commute: $\mathbb{K}^{\mathsf{cd}} = \mathbb{K}^{\mathsf{dc}}$. Furthermore, the following lemma will be helpful.

Lemma 1. *Given a context $\mathbb{K} = (G, M, I)$ with objects $g, h \in G$, we find that $g \in h^{II}$ if and only if $h \in g^{\not{I}\not{I}}$.*

Proof. If $g \in h^{II}$ then $g\ I\ m$ for all $m \in h^I$. Thus $h\ I\ m$ implies $g\ I\ m$. Contrapositively, $g \not{I} m$ entails $h \not{I} m$, which shows $h \in g^{\not{I}\not{I}}$. □

Definitions of the relevant context-morphisms will be introduced in the subsequent sections. An overview of the existing results on morphisms in FCA is given in [1, Chapter 7], which incorporates much information from [8], though the latter contains further details from a more category-theoretic viewpoint. Bonds and infomorphisms, as well as several other kinds of morphisms that we shall not consider in this paper, have been studied in greater detail in [9]. Some newer results on dual bonds and *relational Galois* *connections* between contexts can be found in [10]. Further related investigations can be found in [11], where infomorphisms are studied in conjunction with monotone Galois connections, *complete homomorphisms*, and the so-called *concept lattice morphisms*. Morphisms relating FCA, domain theory, and logic have been studied in [12].

3 Dual Bonds and Direct Product

The construction of concept lattices exploits the fact that the derivation operators $(\cdot)^I$ form an antitone Galois connection. Naturally, Galois connections are also of interest when one looks for suitable morphisms for concept lattices. To represent Galois connections on the level of contexts, functions between the sets of attributes or objects turn out to be too specific. Instead, one uses certain relations called *dual bonds* which we study in this section. Most of the materials before Lemma 3 can be found in [1, 9, 10].[3]

Definition 1. *A* dual bond *between formal contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ is a relation $R \subseteq G \times H$ for which the following hold:*

- *for every object $g \in G$, g^R (which is equal to $R(g)$) is an extent of $\mathbb{L}$ and*
- *for every object $h \in H$, h^R (which is equal to $R^{-1}(h)$) is an extent of $\mathbb{K}$.*

This definition is motivated by the following result:

Theorem 2 ([1, Theorem 53]). *Consider a dual bond R between contexts $\mathbb{K}$ and $\mathbb{L}$ as above. The mappings*

$$\vec{\phi}_R : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto X^R \quad \text{and} \quad \overleftarrow{\phi}_R : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto Y^R$$

form an antitone Galois connection between the (object) concept lattices of $\mathbb{K}$ and $\mathbb{L}$.

Conversely, given such an antitone Galois connection $(\vec{\phi}, \overleftarrow{\phi})$, the relation $R_{(\vec{\phi},\overleftarrow{\phi})} = \left\{(g, h) \mid h \in \vec{\phi}(g^{II})\right\} = \left\{(g, h) \mid g \in \overleftarrow{\phi}(h^{JJ})\right\}$ is a dual bond, and these constructions are mutually inverse in the following sense:

$$\vec{\phi} = \vec{\phi}_{R_{(\vec{\phi},\overleftarrow{\phi})}} \qquad \overleftarrow{\phi} = \overleftarrow{\phi}_{R_{(\vec{\phi},\overleftarrow{\phi})}} \qquad R = R_{\vec{\phi}_R,\overleftarrow{\phi}_R}$$

Hence, formal contexts with dual bonds are "equivalent" to complete lattices with antitone Galois connections. Referring to dual bonds as morphisms might be somewhat misleading, since they do not immediately satisfy the necessary axioms for category theoretic morphisms. However, we will adhere to this terminology since it is indeed possible to use dual bonds in a categorical fashion, provided that objects, homsets and composition are chosen appropriately (see [13] for details).

Before proceeding, let us note the following consequence of Lemma 1.

Lemma 2. *Consider a dual bond R between contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. Then $R(g^{\not{I}\not{I}}) = R(g)$ and $R^{-1}(h^{\not{J}\not{J}}) = R^{-1}(h)$ holds for any $g \in G$, $h \in H$. Especially, $R(g^{\not{I}\not{I}})$ and $R^{-1}(h^{\not{J}\not{J}})$ are extents.*

[3] Note that one could as well work with monotone Galois connections without affecting any result. We do not feel any need to deviate from the traditional formulation here.

Proof. The inclusion $R(g) \subseteq R(g^{\not{I}\not{I}})$ is obvious for any relation R, since $g \in g^{\not{I}\not{I}}$. For the converse, assume that $h \in R(g^{\not{I}\not{I}})$, i.e. there is some $g' \in g^{\not{I}\not{I}}$ such that $g' \; R \; h$. By Lemma 1 we conclude $g \in g'^{II}$ which is a subset of $R^{-1}(h)$ since the latter is an extent. This shows $h \in R(g)$ as required. The statement for R^{-1} follows by a similar reasoning. □

Now we ask how the dual bonds between two contexts can be represented. Since extents are closed under intersections, the same is true for the set of all dual bonds between two contexts. Thus the dual bonds form a closure system and one might ask for a way to cast this into a formal context which has dual bonds as concepts. An immediate candidate for this purpose is the direct product of the contexts.

Definition 2. *Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, the* direct product *of $\mathbb{K}$ and $\mathbb{L}$ is the context $\mathbb{K} \times \mathbb{L} = (G \times H, M \times N, \nabla)$, where $(g, h) \nabla (m, n)$ iff $g \; I \; m$ or $h \; J \; n$.*

Proposition 1 ([10]). *Extents of a direct product $\mathbb{K} \times \mathbb{L}$ are dual bonds from $\mathbb{K}$ to $\mathbb{L}$.*

Proof. It suffices to show that attribute extents are dual bonds, because any extent is an intersection of attribute extents and intersections of dual bonds are still dual bonds. Thus consider $(m, n) \in M \times N$ and define $R = (m, n)^{\nabla}$. We find that $R = (m^I \times H) \cup (G \times n^J)$. Thus, for any $g \in G$, $g^R = H$ or $g^R = n^J$, both of which are extents in $\mathbb{L}$. Likewise, for $h \in H$, $h^R = m^I$ or $h^R = H$, such that R is indeed a dual bond. □

However, it is known that the converse of this result is false, i.e. there are dual bonds which are not extents of the direct product. We give the following counterexample:

Counterexample 1. Consider the context $\mathbb{K} = (\{1, 2, 3\}, \{a, b, c\}, I)$ with incidence relation I given as follows:

I	a	b	c
1	×		
2		×	
3			×

Obviously, the relation $R = \{(1, 1), (2, 2), (3, 3)\}$ is a dual bond from $\mathbb{K}$ to itself, since all singleton sets are extents. However, we find $R^{\nabla} = \emptyset$ in $\mathbb{K} \times \mathbb{K}$. Thus $R \neq R^{\nabla\nabla} = \{1, 2, 3\} \times \{1, 2, 3\}$ is not an extent of the direct product.

As a consequence, the direct product does only represent a distinguished subset of all dual bonds. In order to find additional characterizations for these relations, we use the following result.

Lemma 3. *Given a binary relation R between objects, let R^{∇} denote the intent associated with R when viewed as a set of objects of the direct product. Consider the contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and a relation $R \subseteq G \times H$. For any attribute $m \in M$, the following sets are equal:*

- $X_1 := R^{\nabla}(m) = \{n \in N \mid (m, n) \in R^{\nabla}\}$
- $X_2 := R(m^{\not{I}})^J = \{h \in H \mid \text{there is } g \in G \text{ with } g \not{I} \; m \text{ and } (g, h) \in R\}^J$

– $X_3 := \bigcap_{g \in m^{\not I}} R(g)^J$

Furthermore, for any object $g \in G$, we find that $R^{\nabla\nabla}(g) = R^{\nabla}(g^{\not I})^J = \bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ}$.

Proof. We first show the equality between X_1 and X_2. If $(m, n) \in R^{\nabla}$ then $(g, h)\nabla(m, n)$ holds for all $(g, h) \in R$. Thus, if $g \not I m$ for some $(g, h) \in R$, one certainly has $h\ J\ n$. Hence $n \in X_2$ and we obtain $X_1 \subseteq X_2$. For the other direction consider some $n \in X_2$. Then for all $(g, h) \in R$, $g \not I m$ implies $h\ J\ n$. Hence $(m, n) \in R^{\nabla}$ and $X_2 \subseteq X_1$ as required.

Next observe that X_2 clearly can be expressed as $\left(\bigcup_{g \in m^{\not I}} R(g)\right)^J$. The fact that this is equal to X_3 is a basic result of formal concept analysis (see e.g. [1, Proposition 11]).

For the rest of the proof, note that R^{∇} is a relation between the sets of objects of the dual contexts $\mathbb{K}^{\mathsf{d}}$ and $\mathbb{L}^{\mathsf{d}}$. Thus we can apply the first part of the lemma on R^{∇} to obtain the equality

$$R^{\nabla\nabla}(g) = R^{\nabla}(g^{\not I})^J = \bigcap_{m \in g^{\not I}} R^{\nabla}(m)^J.$$

Another application of the above results shows that $R^{\nabla}(m) = R(m^{\not I})^J$ and we obtain $\bigcap_{m \in g^{\not I}} R^{\nabla}(m)^J = \bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ}$ as required. □

Now we can state a characterization theorem for dual bonds in the direct product.

Theorem 3. *Consider the contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and a relation $R \subseteq G \times H$. The following are equivalent:*

(i) R is an extent of the direct product $\mathbb{K} \times \mathbb{L}$.
(ii) For all $g \in G$, $R(g) = R^{\nabla}(g^{\not I})^J \ \left(= \bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ}\right)$.
(iii) R is a dual bond and, for all $g \in G$, $\bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ} = R(g^{\not I \not I})$

Proof. The equivalence of (i) and (ii) follows immediately from Lemma 3 where we established that $R^{\nabla}(g^{\not I})^J = \bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ} = R^{\nabla\nabla}(g)$. Using Lemma 2 on condition (iii) yields $\bigcap_{m \in g^{\not I}} R(m^{\not I})^{JJ} = R(g)$, which is just condition (ii). □

Another feature of dual bonds in the direct product allows for the construction of Galois connections other than those considered in Theorem 2. Given a dual bond R in $\mathbb{K} \times \mathbb{L}$, its intent R^{∇} is a dual bond from $\mathbb{K}^{\mathsf{d}}$ to $\mathbb{L}^{\mathsf{d}}$, which induces another antitone Galois connection between the dual concept lattices. This Galois connection appears to have no simple further relationship to the antitone Galois connection derived from R.

Corollary 1. *Consider the contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and an extent R of the direct product $\mathbb{K} \times \mathbb{L}$. There are two distinguished Galois connections $\phi_R : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ and $\phi_{R^{\nabla}} : \mathsf{B}_o(\mathbb{K})^{\mathsf{op}} \to \mathsf{B}_o(\mathbb{L})^{\mathsf{op}}$ and each of R, R^{∇}, ϕ_R and $\phi_{R^{\nabla}}$ uniquely determines the others (using $(\cdot)^{\mathsf{op}}$ to denote the order duals of the respective concept lattices).*

Proof. Just use Theorem 2 on R and R^{∇}. □

Of course any antitone Galois connection between two posets contravariantly induces another antitone Galois connection, obtained by exchanging both adjoints. But there appears to be no general way to construct an additional antitone Galois connection between the order duals of the original posets. Some of our results, like Proposition 3 and Theorem 7 below, can be extended to account for this second Galois connection, but we will usually prefer to save space and refrain from stating this explicitly.

4 Continuity for Dual Bonds

Continuity is a central concept in many branches of mathematics. It is also of importance for formal concept analysis. However, we will generally not be dealing with functions but with relations such as dual bonds, so the notion of continuity will be lifted accordingly (the following is partially taken from [1]).

Definition 3. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. A relation $R \subseteq G \times H$ is* extensionally continuous *if it* reflects *extents of $\mathbb{L}$, i.e. if for every extent O of $\mathbb{L}$ the preimage $R^{-1}(O)$ is an extent of $\mathbb{K}$.*

R is extensionally object-continuous *(*attribute-continuous*) if it reflects all object extents (attribute extents) of $\mathbb{L}$, i.e. if for every object extent $O = h^{JJ}$ (attribute extent $O = n^J$) the preimage $R^{-1}(O)$ is an extent of $\mathbb{K}$ (but not necessarily an object extent).*

A relation is extensionally closed *from $\mathbb{K}$ to $\mathbb{L}$ if it* preserves *extents of $\mathbb{K}$, i.e. if its inverse is extensionally continuous from $\mathbb{L}$ to $\mathbb{K}$. Extensional object- and attribute-closure are defined accordingly.*

The dual definitions give rise to intensional *continuity and closure properties.*

Lemma 2 earlier shows that extensional object-continuity and -closure are properties of any dual bond when considered as a relation between one context and the complement of the other. We thus focus on extensional attribute-continuity and -closure in the present section. The other notions will however become important later on in Section 5.

Whenever it is clear whether we are dealing with a relation on attributes or on objects, we will tend to omit the additional qualifications "extensionally" and "intensionally." We also remark that neither object- nor attribute-continuity is sufficient to obtain full continuity in the general case, as can be seen from R^{∇} in Counterexample 2.

Now we can investigate the interaction between continuity and the representation of dual bonds.

Theorem 4. *Consider a dual bond R from $\mathbb{K} = (G, M, I)$ to $\mathbb{L} = (H, N, J)$. If R is extensionally attribute-continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$, then R is an extent of $\mathbb{K} \times \mathbb{L}$ and R^{∇} is intensionally object-closed from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$.*

Proof. We will first show that $R(g)^J = R^{\nabla}(g^{\not I})$ holds for arbitrary $g \in G$ $(*)$. Clearly, $R^{\nabla}(g^{\not I}) \subseteq R(g)^J$, since $n \in R(g)^J$ for any $(m, n) \in R^{\nabla}$ for which $g \not I m$.

For the other direction, assume that there is $n \in R(g)^J$, i.e. all objects which are R-related to g satisfy n. Thus g relates to no objects that do not satisfy n, i.e. $g \notin R^{-1}(n^{\not J})$. Due to attribute-continuity of R, the latter is closed in $\mathbb{K}$ and thus there must be some element $m \in R^{-1}(n^{\not J})^I$ such that $g \not I m$. We want to show that $(m, n) \in R^{\nabla}$ which follows if any pair in R is ∇-related to (m, n). We only need to consider pairs which have a first component g' such that $g' \not I m$. But then $g' \notin R^{-1}(n^{\not J})^{II} = R^{-1}(n^{\not J})$ and we find that $n \in R(g')^J$. Hence all pairs $(g', h') \in R$ satisfy (m, n) and we conclude that $(m, n) \in R^{\nabla}$. Together with the above information that $g \not I m$, this finishes the proof of $(*)$.

Now it is immediate that R is an extent of the direct product. Indeed, by property $(*)$, we obtain $R(g)^{JJ} = R^{\nabla}(g^{\not I})^J$. Now since $R(g) = R(g)^{JJ}$, this yields condition (ii) of Theorem 3 which establishes the claim.

I	a	b
1	×	
2		×

J	c	d	e
3	×		
4		×	

I	a	b	c
1	×		
2		×	
3			

Fig. 1. Formal contexts for Counterexamples 2 (left) and 3 (right).

Finally, note that $(*)$ also shows that the set $R^{\nabla}(g^{\not{I}})$ is an intent of $\mathbb{L}$, such that R^{∇} is indeed object-closed. □

Of course, analogous results can be obtained for closure by exchanging the roles of $\mathbb{K}$ and $\mathbb{L}$. One may wonder whether similar statements can be proven for dual bonds which are fully continuous and/or closed. However, this is not the case:

Counterexample 2. Consider the contexts $\mathbb{K} = (\{1,2\}, \{a,b\}, I)$ and $\mathbb{L} = (\{3,4\}, \{c,d,e\}, J)$ depicted in Fig. 1 (left). Define $R = \{(1,3),(2,4)\}$. All subsets of $\{1,2\}$ are extents of both $\mathbb{K}$ and $\mathbb{K}^{c}$. Likewise, all subsets of $\{3,4\}$ are extents of $\mathbb{L}$ and $\mathbb{L}^{c}$. Thus R is trivially closed and continuous in every sense. However, we find that $R^{\nabla} = \{(a,d),(b,c)\}$ is not closed from $\mathbb{K}^{c}$ to $\mathbb{L}$. Indeed, $\{a,b\}$ is an intent of $\mathbb{K}^{c}$ but $R^{\nabla}(\{a,b\}) = \{c,d\}$ is not an intent of $\mathbb{L}$, since $\{c,d\}^{JJ} = \{c,d,e\}$.

Other easy counterexamples for this claim can be obtained by exploiting the fact that for any relation the image and preimage of the empty set is necessarily empty. By adding appropriate attributes, one can always assure that the empty set is not an intent in order to find cases where no relation can be intentionally continuous even if numerous extensionally closed and continuous dual bonds exist.

Another false assumption one might have is that the conditions given in Theorem 4 for being an extent of the direct product are not just sufficient but also necessary. However, neither closure nor continuity is needed for a dual bond to be represented in the direct product.

Counterexample 3. Consider the context $\mathbb{K} = (\{1,2,3\}, \{a,b,c\}, I)$ depicted in Fig. 1 (right). Define $R = \{(1,1),(2,2)\}$. We find that $R^{\nabla} = \{(a,b),(b,a)\}$. Thus $R = R^{\nabla\nabla}$ and R is a dual bond which is an extent of the direct product $\mathbb{K} \times \mathbb{K}$. However, R is not even attribute-continuous from $\mathbb{K}$ to $\mathbb{K}^{c}$, since $R^{-1}(c^{\not{I}}) = R^{-1}(\{1,2,3\}) = \{1,2\}$ is not closed in $\mathbb{K}$. On the other hand, using that $R = R^{-1}$, we find that R is not attribute-closed from $\mathbb{K}^{c}$ to $\mathbb{K}$ either.

Although this shows that continuity is not a characteristic feature of all dual bonds in the direct product, we still find that there are many situations where there is a wealth of continuous dual bonds. This is the content of the following theorem.

Theorem 5. *Consider the contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. If*

$$\emptyset \text{ is an extent of } \mathbb{K} \qquad \text{or} \qquad \emptyset \text{ is not an extent of } \mathbb{L}^{c}$$

then the set of all dual bonds which are continuous from $\mathbb{K}$ to $\mathbb{L}^{c}$ is $\bigcap$-dense in $\mathsf{B}_o(\mathbb{K}\times\mathbb{L})$ and thus forms a basis for the closure system of all dual bonds in the direct product.

If the assumptions also hold with $\mathbb{K}$ and $\mathbb{L}$ exchanged, then the set of all dual bonds which are both continuous from $\mathbb{K}$ to $\mathbb{L}^{c}$ and closed from $\mathbb{K}^{c}$ to $\mathbb{L}$ is $\bigcap$-dense as well.

Proof. From Theorem 4 we know that the above sets of dual bonds are subsets of the extents of the direct product. For density, we recall that the set of all attribute extents $(m,n)^\nabla$ is $\bigcap$-dense in the lattice of extents. For every $(m,n) \in M \times N$, we find that $(m,n)^\nabla = m^I \times H \cup G \times n^J$. Therefore, for arbitrary extents $O \subseteq H$ we calculate

$$(m,n)^{\nabla-1}(O) = \begin{cases} \emptyset & \text{if } O = \emptyset, \\ G \cup m^I = G & \text{if } n^J \cap O \neq \emptyset, \\ m^I & \text{otherwise.} \end{cases}$$

In each case $(m,n)^{\nabla-1}(O)$ is an extent of $\mathbb{K}$, where we use the initial assumption that $\emptyset$ is an extent of $\mathbb{K}$ if $O = \emptyset$ is an extent of $\mathbb{L}^c$. Thus any $(m,n)^\nabla$ is continuous from $\mathbb{K}$ to $\mathbb{L}^c$ and the attribute extents must form a subset of the set of continuous dual bonds. This shows the required density property.

Using the additional assumptions for the last part of the theorem, this shows that the dual bonds $(m,n)^\nabla$ are also closed from $\mathbb{K}^c$ to $\mathbb{L}$. Hence the continuous and closed dual bonds form a $\bigcap$-dense set as required. □

Note that the previous theorem could of course also be stated using closure in place of continuity. Furthermore it is evident that dual bonds of the form $(m,n)^\nabla$ are such that the (pre)image of almost any set is an extent. The only exception is the empty set, which is why we needed to add the given preconditions. We remark that these conditions are indeed very weak. By removing or adding full rows, any context can be modified in such a way that the empty set either is an extent or not. Since the concept lattices of the context and its complement are not affected by this procedure, one can enforce the necessary conditions without loosing generality.

5 Functional Bonds and Scale Measures

In FCA, (extensionally) continuous functions have been studied under the name *scale measures*, the importance of which stems from the fact that they can be regarded as a model for concept scaling and data abstraction. Topology provides additional interpretations for continuous functions in the context of knowledge representation and reasoning, but we will not give further details here.[4] We merely remark that continuity between topological spaces coincides with continuity between appropriate contexts.

Continuity for functions constitutes a special case of continuity in the relational case as defined above.

Definition 4. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. A function $f : G \to H$ is* extensionally continuous *whenever its graph $\{(x, f(x)) \mid x \in G\}$ is an extensionally continuous relation, i.e. if $f^{-1}(O)$ is an extent of $\mathbb{K}$ for any extent O of $\mathbb{L}$.*

Extensional attribute- *and* object-continuity*, as well as the according intensional properties and closures are defined similarly based on the graph of the function.*

[4] Roughly speaking, the potential of topology for our purposes resides in its well-known connections to FCA (data representation), formal logic (reasoning), and domain theory (computation/approximation), all of which are based on essentially the same mechanisms of *Stone duality* (see [13] for further details).

This definition agrees with [1, Definition 89], where extensionally continuous maps have also been called *scale measures*. Extensional attribute-continuity (and thus intensional object-continuity) is of course redundant, as the following lemma shows.

Lemma 4. *Given contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, a function $f : G \to H$ is extensionally continuous iff it is extensionally attribute-continuous.*

Proof. The forward implication is trivial, so assume that f is attribute-continuous. Consider an extent B^J of $\mathbb{L}$. According to the *Basic Theorem on Concept Lattices* [1] one has that $B^J = \bigcap_{n \in B} n^J$. We find that $f^{-1}\left(\bigcap_{n \in B} n^J\right) = \bigcap_{n \in B} f^{-1}(n^J)$. By attribute-continuity, the latter is an intersection of concepts of $\mathbb{K}$, and thus a concept. □

This statement relies on the fact that attribute extents are $\bigcap$-dense in the object concept lattice and that preimage commutes with intersection. On the one hand, this is not true for images of functions, and hence extensional attribute-closure does not yield full closure. On the other hand, though object extents are supremum-dense, the respective suprema are not the set-theoretical unions. Hence extensional object-continuity and -closure are reasonable notions as well.

The link from functions to our earlier studies of dual bonds is established through a specific class of dual bonds which can be represented by functions.

Definition 5. *Consider a dual bond R between contexts (G, M, I) and (H, N, J). Then R is* functional *whenever, for any $g \in G$, the extent $R(g)$ is generated by a unique object $f_R(g) \in H$:*

$$R(g) = f_R(g)^{JJ}.$$

In this case R is said to induce *the corresponding function $f_R : G \to H$.*

It is obvious that functional dual bonds are uniquely determined by the function they induce. In fact, it is easy to see that R is the least dual bond that contains the graph of the function f_R. However, not for every function will this construction yield a dual bond that is functional. The next result characterizes the functions that are of the form f_R for some functional dual bond R.

Proposition 2. *Consider a context $\mathbb{K} = (G, M, I)$ and a context $\mathbb{L} = (H, N, J)$ for which the map $h \mapsto h^J$ is injective. There is a bijective correspondence between*

- *the set of all functional dual bonds from $\mathbb{K}$ to $\mathbb{L}$ and*
- *the set of all extensionally object-continuous functions from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$.*

The required bijections consist of the functions

- *$R \mapsto f_R$ mapping each functional dual bond to the induced function and*
- *$f \mapsto R_f$ mapping each object-continuous function to the least dual bond which contains its graph $\{(g, f(g)) \mid g \in G\}$.*

Proof. Consider a functional dual bond R from $\mathbb{K}$ to $\mathbb{L}$ and the induced mapping $f = f_R$. For some object $h \in H$, we find that $R^{-1}(h) = f^{-1}(h^{\mathcal{J}\mathcal{J}})$ follows from the defining property of f and Lemma 1. Since R is a dual bond, $R^{-1}(h)$ must be an extent and hence f is extensionally object-continuous in the required sense.

Conversely, if $f : G \to H$ is an object-continuous function from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$, then a relation $R \subseteq G \times H$ is defined by setting $R(g) = f(g)^{JJ}$ for any $g \in G$. Clearly R maps objects of $\mathbb{K}$ to extents of $\mathbb{L}$. For the converse, consider $h \in H$. As before we find that $R^{-1}(h) = f^{-1}(h^{\not{J}\not{J}})$ which is an extent of $\mathbb{K}$ by object-continuity. Thus R is a dual bond. Moreover, it is easy to see that R is the least dual bond that contains the graph of f. Due to the assumptions on $\mathbb{L}$, we have that R is functional inducing the function f and we obtain the required bijection. □

Object-continuity of the functions f_R is not too much of a surprise in the light of Lemma 2. The fact that this property suffices for the above result demonstrates how specific functional dual bonds really are. In contrast, the properties established in Lemma 2 are generally not sufficient for a relation to be a dual bond.

Also note that the additional requirements for $\mathbb{L}$, which guarantee that no two functions induce the same dual bond, are again rather weak. Indeed, they are implied by the common assumption that the contexts under consideration are clarified.

We can now go further and characterize the antitone Galois connections obtained from functional dual bonds.

Proposition 3. *Consider a context $\mathbb{K} = (G, M, I)$ and a context $\mathbb{L} = (H, N, J)$ for which the map $h \mapsto h^J$ is injective. The bijection between dual bonds and antitone Galois connections given in Theorem 2 restricts to a bijective correspondence between*

- *the set of all functional dual bonds from $\mathbb{K}$ to $\mathbb{L}$ and*
- *the set of all antitone Galois connections from $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L})$ which map object extents of $\mathbb{K}$ to object extents of $\mathbb{L}$.*

Proof. Consider a functional dual bond R from $\mathbb{K}$ to $\mathbb{L}$ and the antitone Galois connection $(\vec{\phi}_R, \overleftarrow{\phi}_R)$ as constructed in Theorem 2. We claim that $\vec{\phi}_R$ maps object extents to object extents. Thus consider $\vec{\phi}_R(g^{II})$ for some $g \in G$ and let f_R be the function induced by R. The set $R^{-1}(f_R(g))$ contains g and is an extent since R is a dual bond. Consequently $g^{II} \subseteq R^{-1}(f_R(g))$. But this shows that $f_R(g) \in \vec{\phi}_R(g^{II})$ since the latter is equal to $\bigcap\{R(x) \mid x \in g^{II}\}$. Therefore we have $f_R(g)^{JJ} \subseteq \vec{\phi}_R(g^{II})$. The opposite inclusion follows, since $\vec{\phi}_R(g^{II})$ is an intersection of a collection of sets which includes $f_R(g)^{JJ} = R(g)$. Thus $\vec{\phi}_R(g^{II}) = f_R(g)^{JJ}$, which is an object extent of $\mathbb{L}$ as required.

Now let $(\vec{\phi}, \overleftarrow{\phi})$ be a Galois connection such that $\vec{\phi}$ maps object extents to object extents. There is a unique function $f : G \to H$ for which $\vec{\phi}(g^{II}) = f(g)^{JJ}$ hold for arbitrary $g \in G$. Let $R = R_{(\vec{\phi}, \overleftarrow{\phi})}$ be the dual bond induced by $(\vec{\phi}, \overleftarrow{\phi})$ as in Theorem 2. But then $R(g) = \vec{\phi}(g^{II}) = f(g)^{JJ}$, for arbitrary $g \in G$, such that R is indeed functional. □

In the light of the previous proposition we give a definition for the corresponding property of Galois connections.

Definition 6. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and a (monotone or antitone) Galois connection $\phi = (\vec{\phi}, \overleftarrow{\phi})$ between $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_o(\mathbb{L})$.*

Then ϕ is functional *(from $\mathbb{K}$ to $\mathbb{L}$) if $\vec{\phi}$ maps object extents to object extents and, for any $g \in G$ there is a unique object $f_{\vec{\phi}}(g)$ such that*

$$\vec{\phi}(g^{II}) = f_{\vec{\phi}}(g)^{JJ}.$$

In this case, ϕ is said to induce *the function $f_{\vec{\phi}} : G \to H$.*

Proposition 3 shows rather natural classes of dual bonds and Galois connections, respectively. However, functional dual bonds do not generally arise as extents of the direct product. Moreover, the corresponding class of extensionally object-continuous functions as described in Proposition 2 appears to be unidentified. As Theorem 6 below shows, the more common class of extensionally continuous functions still allows for a nice characterization in terms of dual bonds. It will be helpful to first state the following lemma.

Lemma 5. *Consider contexts* $\mathbb{K} = (G, M, I)$ *and* $\mathbb{L} = (H, N, J)$*. If* R *is a functional dual bond from* $\mathbb{K}$ *to* $\mathbb{L}$ *then we find that for any extent* O *of* $\mathbb{L}^{\mathsf{c}}$

$$R^{-1}(O) = f_R^{-1}(O).$$

Proof. Let O be an arbitrary extent of $\mathbb{L}^{\mathsf{c}}$. The inclusion $R^{-1}(O) \supseteq f_R^{-1}(O)$ is obvious, since R contains the graph of f_R.

For the converse note that $R^{-1}(O)$ is just the union of the sets $R^{-1}(h)$ for all $h \in O$. As noted in the proof of Proposition 2, we have $R^{-1}(h) = f_R^{-1}(h^{\not{J}\not{J}})$ for arbitrary $h \in H$. But since O is an extent of $\mathbb{L}^{\mathsf{c}}$, $f_R^{-1}(h^{\not{J}\not{J}}) \subseteq f_R^{-1}(O)$ for all $h \in O$. Hence we obtain $R^{-1}(O) \subseteq f_R^{-1}(O)$ as required. □

Theorem 6. *Consider a context* $\mathbb{K} = (G, M, I)$ *and a context* $\mathbb{L} = (H, N, J)$ *for which the map* $h \mapsto h^J$ *is injective. The bijection given in Proposition 2 restricts to a bijective correspondence between*

- *the set of all extensionally continuous functions from* $\mathbb{K}$ *to* $\mathbb{L}^{\mathsf{c}}$ *and*
- *the set of all functional dual bonds from* $\mathbb{K}$ *to* $\mathbb{L}$ *that are continuous from* $\mathbb{K}$ *to* $\mathbb{L}^{\mathsf{c}}$*.*

Especially, every dual bond R_f *induced by a continuous function from* $\mathbb{K}$ *to* $\mathbb{L}^{\mathsf{c}}$ *is an extent of the direct product* $\mathbb{K} \times \mathbb{L}$*.*

Proof. Given a function f which is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$, we must show that the dual bond R_f as specified in Proposition 2 is also continuous. From the same proposition we know that $f = f_{R_f}$ and so we can apply Lemma 5 to show that $R_f^{-1}(O) = f^{-1}(O)$ for any extent O of $\mathbb{L}^{\mathsf{c}}$. Continuity of R_f then follows from continuity of f.

Conversely, consider the function f_R for any functional dual bond R that is continuous in the above sense. Using Lemma 5 again, we find that $R^{-1}(O) = f_R^{-1}(O)$ for every extent O of $\mathbb{L}^{\mathsf{c}}$ and hence obtain continuity of f_R.

Finally, to show that R_f is an extent of the direct product, one can apply Theorem 4 and continuity of R_f. □

Thus we find that extensionally continuous functions, or scale measures, are a rather specific kind of dual bonds. Again we must be careful: It is certainly not the case that all functional dual bonds which are extents in the direct product are continuous. Just consider the context $\mathbb{K} = (\{g\}, \{m\}, \{(g, m)\})$. The relation $R = \{(g, g)\}$ is an extent of the direct product $\mathbb{K} \times \mathbb{K}$ and it is functional with f_R being the identity. However, the preimage of the empty set (which is closed in $\mathbb{K}^{\mathsf{c}}$) is not an extent of $\mathbb{K}$.

As a dual bond, every continuous function naturally induces an antitone Galois connection – Propositions 2 and 3 discussed the according constructions for object-continuous functions. Due to their special structure, continuous functions can additionally be used to derive another monotone Galois connection. It should not come as a surprise that these entities determine each other uniquely under some mild assumptions.

Theorem 7. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and a function $f : G \to H$ which is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$.*

(i) An antitone Galois connection $\phi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ is given by the mappings

$$\vec{\phi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto \bigcap\{f(x)^{JJ} \mid x \in X\} \quad \textit{and}$$
$$\overleftarrow{\phi}_f : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \bigcap\{f^{-1}(y^{\not{J}\not{J}}) \mid y \in Y\}.$$

(ii) A monotone Galois connection $\psi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ is given by the mappings

$$\vec{\psi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) : X \mapsto f(X)^{\not{J}\not{J}} \quad \textit{and}$$
$$\overleftarrow{\psi}_f : \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto f^{-1}(Y).$$

Moreover, if $\mathbb{L}$ is such that $h \mapsto h^J$ is injective, the above mappings provide bijective correspondences between

- *the set of all extensionally continuous functions from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$,*
- *the set of all antitone Galois connections $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L})$ that are functional (from $\mathbb{K}$ to $\mathbb{L}$) and for which the induced function is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$,*
- *the set of all monotone Galois connections $\mathsf{B}_o(\mathbb{K})$ to $\mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ that are functional (from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$).*

Proof. We observe that $\vec{\phi}_f(X) = X^{R_f}$ and $\overleftarrow{\phi}_f(Y) = Y^{R_f}$ such that (i) is an immediate consequence of Theorem 2 and Proposition 2. The according bijection follows from Proposition 2 and Proposition 3.

For part (ii), we repeat the proof given in [1, Propositions 118 and 119]. Due to continuity $\overleftarrow{\psi}_f = f^{-1}$ is a function between the specified object concept lattices. Like the preimage of any function, it preserves all intersections,which are exactly the infima in the given lattices. Thus $\overleftarrow{\psi}_f$ is the upper adjoint of some monotone Galois connection. The lower adjoint of $\overleftarrow{\psi}_f$ then is defined to be the function

$$X \mapsto \bigcap\left\{Y^{\not{J}\not{J}} \mid X \subseteq f^{-1}(Y^{\not{J}\not{J}})\right\} = \bigcap\left\{Y^{\not{J}\not{J}} \mid f(X) \subseteq Y^{\not{J}\not{J}}\right\} = f(X)^{\not{J}\not{J}} = \vec{\psi}_f(X).$$

Consequently $\vec{\psi}_f$ is adjoint to $\overleftarrow{\psi}_f$ as required.

To show that $\vec{\psi}_f$ maps object extents of $\mathbb{K}$ to object extents of $\mathbb{L}^{\mathsf{c}}$ consider some arbitrary $g \in G$. $f^{-1}(f(g)^{\not{J}\not{J}})$ is an extent of $\mathbb{K}$ which contains g and hence g^{II}. Thus $f(g^{II}) \subseteq f(g)^{\not{J}\not{J}}$ and therefore $\vec{\psi}_f(g^{II}) \subseteq f(g)^{\not{J}\not{J}}$. But since $f(g) \in f(g^{II})$ this shows $\vec{\psi}_f(g^{II}) = f(g)^{\not{J}\not{J}}$ as required. Now it is easy to see that if $h \mapsto h^J$ is injective, then so are $h \mapsto h^{JJ}$, $h \mapsto h^{\not{J}}$, and $h \mapsto h^{\not{J}\not{J}}$. Injectivity of $h \mapsto h^{\not{J}\not{J}}$ entails that $(\vec{\psi}_f, \overleftarrow{\psi}_f)$ is functional.

For the converse of the claimed bijection, consider any monotone Galois connection $(\vec{\psi}, \overleftarrow{\psi}) : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ which is functional in the above sense, and let f be the induced function. Given some extent X of $\mathbb{K}$ we calculate

$$\begin{aligned}\vec{\psi}(X) &= \vec{\psi}\left(\bigvee\{x^{II} \mid x \in X\}\right) = \bigvee\{\vec{\psi}(x^{II}) \mid x \in X\} \\ &= \bigvee\{f(x)^{\not{J}\not{J}} \mid x \in X\} = \left(\bigcup\{f(x)^{\not{J}\not{J}} \mid x \in X\}\right)^{\not{J}\not{J}} = f(X)^{\not{J}\not{J}},\end{aligned}$$

where we used that $\vec{\psi}$ preserves suprema and that f represents the value of $\vec{\psi}$ on object extents. But this shows that $\vec{\psi}$ is indeed the mapping $\vec{\psi}_f$ induced by f as above.

As an extension to the proof from [1], we also show explicitly that the function f is continuous from $\mathbb{K}$ to $\mathbb{L}^{\mathsf{c}}$, which does not seem to be entirely obvious. Thus consider some extent Y of $\mathbb{L}^{\mathsf{c}}$ and observe that

$$\begin{aligned}\vec{\psi}\left(f^{-1}(Y)^{II}\right) &= \vec{\psi}\left(\bigvee\{g^{II} \mid g \in f^{-1}(Y)\}\right) \\ &= \bigvee\{\vec{\psi}(g^{II}) \mid g \in f^{-1}(Y)\} \;=\; \bigvee\{f(g)^{\not J \not J} \mid g \in f^{-1}(Y)\},\end{aligned}$$

which is clearly a subset of the extent Y. Now for every $g' \in f^{-1}(Y)^{II}$, we find $f(g') \in \vec{\psi}(f^{-1}(Y)^{II})$ and hence $f(g') \in Y$ as required. □

Part (ii) of the theorem and the corresponding bijections are known (see [1, Propositions 118 and 119]). Note that the two Galois connections from the preceding result are not obtained from each other by some simple dualizing. This is also evident when comparing the different side conditions in both cases: functional monotone Galois connections always relate to continuous functions, while continuity has to be required explicitly for functional antitone Galois connections. To further explain the situation, we can dualize $\mathbb{L}$ to obtain the following result:

Corollary 2. *Given contexts* $\mathbb{K} = (G, M, I)$ *and* $\mathbb{L} = (H, N, J)$*, there is a bijection between*

- *the set of antitone Galois connections* $\mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ *which map object extents to attribute extents and*
- *the set of functions* $f : G \to N$ *which are extensionally continuous from* $\mathbb{K}$ *to* $\mathbb{L}^{\mathsf{d}}$.

6 Infomorphisms

Infomorphisms are a special kind of morphism between formal contexts that have been considered quite independently in rather different research disciplines. The name "infomorphism" we use here has been coined in the context of *information flow theory* [4]. Literature on Chu spaces means the same when speaking about "Chu mappings"; *institution theory* [3] refers the corresponding definition as the "Satisfaction condition" without naming the emerging morphisms at all. In FCA, the antitone version of these morphisms has been studied under the name *(context-)Galois connection* [9, 10].

Probably the most decisive feature of informorphisms is self-duality, an immediate consequence of their symmetry. Some of the relationships between infomorphisms and Galois connections are known, but our results in earlier sections reveal a more complete picture.

Definition 7. *Given contexts* $\mathbb{K} = (G, M, I)$ *and* $\mathbb{L} = (H, N, J)$*, an* infomorphism *from* $\mathbb{K}$ *to* $\mathbb{L}$ *is a pair of mappings* $\vec{f} : G \to H$ *and* $\overleftarrow{f} : N \to M$ *such that*

$$g \; I \; \overleftarrow{f}(n) \qquad \text{if and only if} \qquad \vec{f}(g) \; J \; n$$

holds for arbitrary $g \in G$, $n \in N$.

We first establish the following basic facts.

Lemma 6. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$. The infomorphisms from $\mathbb{K}$ to $\mathbb{L}$ are exactly the infomorphisms from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$.*

Given such an infomorphism $(\vec{f}, \overleftarrow{f})$ and sets $O \subseteq G$, $A \subseteq N$, we find that

$$\vec{f}^{-1}(A^J) = \overleftarrow{f}(A)^I, \quad \vec{f}^{-1}(A^{\not J}) = \overleftarrow{f}(A)^{\not I}, \quad \overleftarrow{f}^{-1}(O^I) = \vec{f}(O)^J \quad \text{and} \quad \overleftarrow{f}^{-1}(O^{\not I}) = \vec{f}(O)^{\not J}.$$

Especially, $\vec{f}$ is extensionally continuous from $\mathbb{K}^{(\mathsf{c})}$ to $\mathbb{L}^{(\mathsf{c})}$ and $\overleftarrow{f}$ is intensionally continuous from $\mathbb{L}^{(\mathsf{c})}$ to $\mathbb{K}^{(\mathsf{c})}$.

Proof. The first statement is immediate from the definition of infomorphisms. Now for some $n \in N$ we find that $g \in \vec{f}^{-1}(n^J)$ iff $\vec{f}(g)\ J\ n$ iff $g\ I\ \overleftarrow{f}(n)$ iff $g \in \overleftarrow{f}(n)^I$. This shows that $\vec{f}^{-1}(n^J) = \overleftarrow{f}(n)^I$. Now for arbitrary sets $A \subseteq N$, $A^J = \bigcap_{n \in A} n^J$ and we can calculate

$$\begin{aligned} \vec{f}^{-1}(A^J) &= \vec{f}^{-1}\left(\textstyle\bigcap_{n\in A} n^J\right) = \textstyle\bigcap_{n \in A} \vec{f}^{-1}(n^J) \\ &= \textstyle\bigcap_{n \in A} \overleftarrow{f}(n)^I \quad = \left(\textstyle\bigcup_{n \in A} \overleftarrow{f}(n)\right)^I = \overleftarrow{f}(A)^I \end{aligned}$$

The other cases follow by dualization and/or complementation of this reasoning. □

Using these continuity properties, we can already specify a number of possible Galois connections constructed from infomorphisms. We remark that continuity between two contexts is in general not equivalent to continuity between the respective complements, such that Theorem 7 can be applied to one part of an infomorphism in two different ways, whereas this is not possible for arbitrary continuous functions.

From Theorem 6, we know that we can obtain continuous dual bonds from both $\vec{f}$ and $\overleftarrow{f}$. Since these relations are extents and intents, respectively, in the direct product, one may ask whether they belong to the same concepts or not. The following proposition shows the expected result.

Proposition 4. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and an infomorphism $(\vec{f}, \overleftarrow{f})$ from $\mathbb{K}$ to $\mathbb{L}$. Define relations $R \subseteq G \times H$ and $S \subseteq M \times N$ by setting*

$$R(g) = \vec{f}(g)^{JJ} \quad \text{and} \quad S^{-1}(n) = \overleftarrow{f}(n)^{\not I \not I}.$$

Then R is a dual bond from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$ which is an extent of $\mathbb{K}^{\mathsf{c}} \times \mathbb{L}$ with $R^{\nabla} = S$.

Furthermore, R is extensionally continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ and S^{-1} is intensionally continuous from $\mathbb{L}^{\mathsf{c}}$ to $\mathbb{K}$.

Proof. Since $\vec{f}$ is continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ (Lemma 6), the fact that R is an extent of $\mathbb{K}^{\mathsf{c}} \times \mathbb{L}$ and continuous in the required sense follows from Theorem 6. S^{-1} is obtained accordingly from $\overleftarrow{f}$ and thus is a dual bond from $\mathbb{L}^{\mathsf{d}}$ to $\mathbb{K}^{\mathsf{cd}}$ which is continuous as required.

As already observed in the proof of Proposition 2, the definition of S^{-1} yields that $S(m) = \overleftarrow{f}^{-1}(m^{II})$ for arbitrary $m \in M$. Thus $S(m) = \bigcup_{g \in m^I} \overleftarrow{f}^{-1}(g^I)$ which is equal to $\bigcup_{g \in m^I} \vec{f}(g)^J$ by Lemma 6. Due to S^{-1} being a dual bond from $\mathbb{L}^{\mathsf{d}}$ to $\mathbb{K}^{\mathsf{cd}}$, $S(m)$ is an intent of $\mathbb{L}$. Hence the above union is equal to $\left(\bigcup_{g \in m^I} \vec{f}(g)^{JJ}\right)^J$ which is just $R(m^I)^J$. By Lemma 3, $R(m^I)^J = R^{\nabla}(m)$ such that we find $S(m) = R^{\nabla}(m)$ and thus $S = R^{\nabla}$. □

Observe that the above construction of R (and S) relies only on the continuity of $\overrightarrow{f}$ from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$ (and the corresponding continuity of $\overleftarrow{f}$). One can also construct a dual bond based on the continuity properties of these functions between the non-complemented contexts. However, Proposition 4 does not imply any relationship between these two dual bonds beyond the obvious fact that they induce the same infomorphism.

We already know that the dual bonds induced by (one part of) an infomorphism have rather specific properties. The next result shows that these features are sufficient for characterizing the respective dual bonds.

Proposition 5. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ and let R be a dual bond from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$ such that both R and $R^{\nabla-1}$ are functional. If R is extensionally continuous then the functions induced by R and $R^{\nabla-1}$ constitute an infomorphism from $\mathbb{K}$ to $\mathbb{L}$.*

Proof. Denote the functions induced by R and $R^{\nabla-1}$ by $\overrightarrow{f}$ and $\overleftarrow{f}$, respectively, and consider some $n \in N$. We calculate

$$\overleftarrow{f}(n)^{\not I} = R^{\nabla-1}(n)^{\not I} = R^{-1}(n^{\not J})^{\not I\not I} = R^{-1}(n^{\not J}),$$

where the first and second equalities follow from Proposition 4 and Lemma 3, respectively, and the last equality uses continuity of R. Clearly $\overrightarrow{f}^{-1}(n^{\not J}) \subseteq R^{-1}(n^{\not J})$. For the other direction, assume that $g \in R^{-1}(n^{\not J})$. Then there is some $h \not J n$ with $g \; R \; h$, i.e. $h \in \overrightarrow{f}(g)^{JJ}$. But then $h^J \supseteq \overrightarrow{f}(g)^J$ and therefore $\overrightarrow{f}(g) \not J n$. This shows $g \in \overrightarrow{f}^{-1}(n^{\not J})$ such that the latter is equal to $R^{-1}(n^{\not J})$. In summary, we thus obtain $\overleftarrow{f}(n)^{\not I} = \overrightarrow{f}^{-1}(n^{\not J})$ which is equivalent to the statement

$$g \not I \overleftarrow{f}(n) \quad \text{iff} \quad \overrightarrow{f}(g) \not J n,$$

which states that $(\overrightarrow{f}, \overleftarrow{f})$ is an infomorphism as claimed. □

Note that, according to Lemma 4, extensional continuity of a functional dual bond R is equivalent to extensional attribute-continuity. This in turn implies intensional object-closure of R^{∇} (Theorem 4) which, since $R^{\nabla-1}$ is also functional, implies the closure of R^{∇}. Thus our assumptions are perfectly symmetrical. Furthermore, Propositions 4 and 5 induce a bijection between infomorphisms and the described class of dual bonds.

Having understood how infomorphisms are characterized in terms of dual bonds, we can specify their relationship with Galois connections.

Theorem 8. *Consider contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$, and let $f = (\overrightarrow{f}, \overleftarrow{f})$ be an infomorphism from $\mathbb{K}$ to $\mathbb{L}$.*

- *An antitone Galois connection $\phi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ is given by the mappings*

  $$\overrightarrow{\phi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) : X \mapsto \bigcap\{\overrightarrow{f}(x)^{\not J\not J} \mid x \in X\} \;=\; \bigcap\{\overleftarrow{f}^{-1}(x^{\not I})^{\not J} \mid x \in X\} \quad \text{and}$$

  $$\overleftarrow{\phi}_f : \mathsf{B}_o(\mathbb{L}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \bigcap\{\overrightarrow{f}^{-1}(y^{JJ}) \mid y \in Y\} \;=\; \bigcap\{\overleftarrow{f}(y^J)^I \mid y \in Y\}.$$

  Further, three antitone Galois connections $\phi_f^{\mathsf{c}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{L})$, $\phi_f^{\mathsf{d}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{d}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{cd}})$ and $\phi_f^{\mathsf{cd}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{cd}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{d}})$ are defined similarly, using the complemented incidence relations (for $(\cdot)^{\mathsf{c}}$) and exchanging $\overrightarrow{f}$ and $\overleftarrow{f}$ (for $(\cdot)^{\mathsf{d}}$), respectively.

– *A monotone Galois connection* $\psi_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L})$ *is given by the mappings*

$$\vec{\psi}_f : \mathsf{B}_o(\mathbb{K}) \to \mathsf{B}_o(\mathbb{L}) : X \mapsto \vec{f}(X)^{JJ} \;=\; \overleftarrow{f}^{-1}(X^I)^J \quad \textit{and}$$

$$\overleftarrow{\psi}_f : \mathsf{B}_o(\mathbb{L}) \to \mathsf{B}_o(\mathbb{K}) : Y \mapsto \vec{f}^{-1}(Y) \;=\; \overleftarrow{f}(Y^J)^I.$$

Another monotone Galois connection $\psi_f^{\mathsf{c}} : \mathsf{B}_o(\mathbb{K}^{\mathsf{c}}) \to \mathsf{B}_o(\mathbb{L}^{\mathsf{c}})$ *is defined similarly, but with all incidence relations complemented.*

Proof. The fact that the above mappings consitute Galois connections between the given concept lattices is an immediate consequence from Theorem 7 together with the continuity properties of infomorphisms as established in Proposition 4.

We have to show that the claimed equalities hold. For ϕ_f the equalities are obtained by applying Lemma 6 to the sets of objects $\{x\}$ $(x \in X)$ and y^J $(y \in Y)$, respectively. Likewise, the equalities within the definition of ψ_f follow by using Lemma 6 on X and Y^J. □

Note that Proposition 4 shows that the antitone Galois connections ϕ_f^{d} and ϕ_f^{cd} can also be constructed as in Corollary 1 from the two dual bonds induced by the function $\vec{f}$. Especially, Corollary 1 does not yield any further Galois connections.

7 A Concept Lattice of Morphisms

The above considerations show that scale measures and infomorphisms can be identified with special types of dual bonds, and thus that part of this work can also be regarded as a study of various attributes of dual bonds and of the implications between them. The resulting concept lattice of context-morphisms is represented by the *nested line diagram*[5] in Fig. 2.

To see that the information represented in this concept lattice is indeed correct, one can compute the induced set of implications between its attributes to obtain the following collection of inference rules:

attr.-continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	extent of $\mathbb{K} \times \mathbb{L}$	Theorem 3
continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	attr.-continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	Definition 3
infomorphism $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$, functional $\mathbb{K} \to \mathbb{L}$	Proposition 5
functional $\mathbb{K} \to \mathbb{L}$, attr.-cont. $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	$\Rightarrow$	continuous $\mathbb{K} \to \mathbb{L}^{\mathsf{c}}$	Lemma 4
attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	extent of $\mathbb{K} \times \mathbb{L}$	Theorem 3
closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	Definition 3
infomorphism $\mathbb{L} \to \mathbb{K}^{\mathsf{c}}$	$\Rightarrow$	closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$, functional $\mathbb{L} \to \mathbb{K}$	Proposition 5
functional $\mathbb{L} \to \mathbb{K}$, attr.-closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	$\Rightarrow$	closed $\mathbb{K}^{\mathsf{c}} \to \mathbb{L}$	Lemma 4

As usual, collections of attributes on either side of the implications are comprehended as conjunctions. As the last column documents, each of these implications has indeed already been established within this document.

[5] The concept lattice represented by a nested line diagram consists of the boldfaced nodes, where connections between boxes represent parallel connections between boldfaced nodes at corresponding positions wrt. the background structure. See [1, pp. 75].

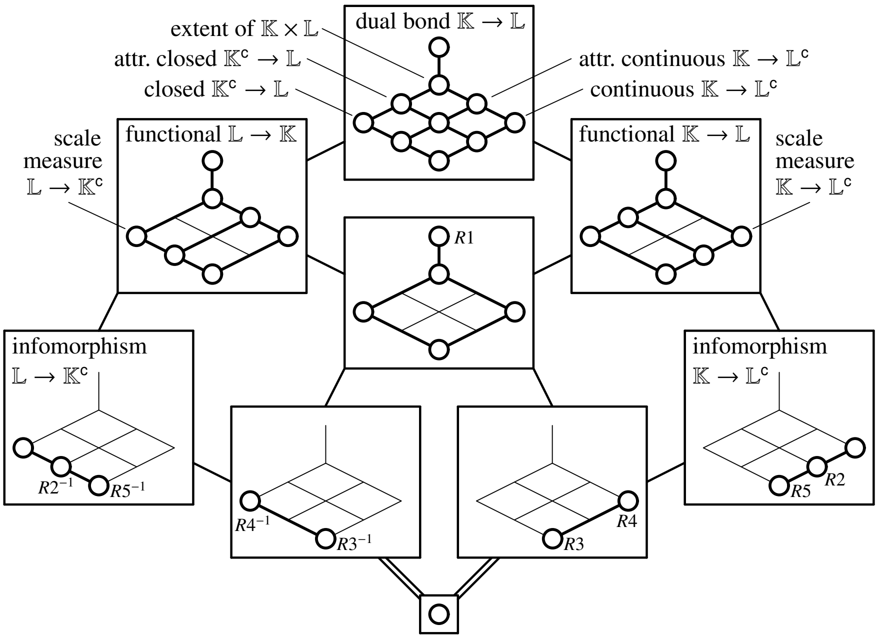

Fig. 2. The concept lattice of the discussed properties of dual bonds, displayed as a nested line diagram. The included attributes are defined in Definition 1 (dual bond), 2 ($\mathbb{K}\times\mathbb{L}$), 3 (continuity and closure), and 5 (functionality). The attributes "scale measure" and "infomorphism" refer to the dual bonds described in Theorem 6 and Proposition 5, respectively, and thus imply functionality. The labels $R1$ to $R5$ and $R2^{-1}$ to $R5^{-1}$ denote the objects of the formal context in Fig. 3.

Conversely, we claim that no further implications between conjunctions of attributes hold for the considered properties. To substantiate this claim, we conducted an *attribute exploration* (see [1, pp. 85]) for the attributes used in Fig. 2 – a task that was greatly simplified through the use of the free software *ConExp*.[6] After reducing the resulting collection of objects, we obtained the dual bonds and formal context displayed in Fig. 3. To check that each of the given objects indeed has the specified attributes, first note that the attributes of $R2^{-1}$ to $R5^{-1}$ are determined by the properties of their inverted variants. Thus it remains to verify the attributes for $R1$ to $R5$. Considering the fact that the above implications have already been shown, this task reduces to a small number of straightforward computations, which we will not include here.

Finally, we want to remark that the conjunctive implications considered in FCA cannot describe all possible relationships between the attributes of a context. In particular, it could still occur that some properties are just disjunctions of others, i.e. that some suprema in the concept lattice are computed as simple set-unions. Counterexample 3 demonstrates the reasoning that is necessary to exclude such cases explicitly. We refrain from giving similar counterexamples for each of the 40 concepts in Fig. 2, since it is rather evident that all of them are indeed object-concepts of appropriate dual bonds.

[6] *Concept Explorer*: `http://sourceforge.net/projects/conexp`

	extent of $\mathbb{K} \times \mathbb{L}$	functional $\mathbb{K} \to \mathbb{L}$	infomorph. $\mathbb{K} \to \mathbb{L}^c$	functional $\mathbb{L} \to \mathbb{K}$	infomorph. $\mathbb{L} \to \mathbb{K}^c$	attr.-cont. $\mathbb{K} \to \mathbb{L}^c$	continuous $\mathbb{K} \to \mathbb{L}^c$	attr.-closed $\mathbb{K}^c \to \mathbb{L}$	closed $\mathbb{K}^c \to \mathbb{L}$
$R1$		×		×					
$R2$	×	×	×			×	×	×	
$R3$	×	×	×	×		×	×	×	×
$R4$	×	×	×	×		×	×		
$R5$	×	×	×			×	×	×	×
$R2^{-1}$	×			×	×	×		×	×
$R3^{-1}$	×	×		×	×	×	×	×	×
$R4^{-1}$	×	×		×	×			×	×
$R5^{-1}$	×			×	×	×	×	×	×

$\mathbb{K}_1$	a	b
1	×	
2		×
3		

$\mathbb{K}_2$	a	b	c	d
1	×		×	
2		×	×	
3			×	×

$\mathbb{K}_3$	a	b	c	d	e
1		×		×	
2	×			×	
3	×	×			×

$\mathbb{K}_4$	a	b
1	×	
2		×

$\mathbb{K}_5$	a	b	c	d
1	×		×	
2		×	×	

$R1$: R from Counterexample 1
$R2 : \mathbb{K}_4 \to \mathbb{K}_1 \quad R2 = \{(1,1),(2,2)\}$
$R3 : \mathbb{K}_5 \to \mathbb{K}_4 \quad R3 = \{(1,1),(2,2)\}$
$R4 : \mathbb{K}_3 \to \mathbb{K}_2 \quad R4 = \{(1,1),(2,2),(3,3)\}$
$R5 : \mathbb{K}_5 \to \mathbb{K}_4 \quad R5 = \{(1,1),(2,1)\}$

Fig. 3. A formal context for the concept lattice from Fig. 2 and the definition of the dual bonds that consitute its set of objects.

8 Conclusion and Future Work

In spite of the rather complete picture of the mutual relationships between dual bonds, scale measures and infomorphisms obtained in our considerations, there are many other aspects of the theory of morphisms in FCA which could not be considered within this article; they are left as possible directions for future research. As mentioned in the introduction, the use of morphisms to model knowledge transfer and information sharing may employ methods from category theory (see e.g. [14, 15]). But not all of the above morphisms immediately yield categories of contexts, especially since antitone Galois connections cannot be composed in an obvious way. As a solution, one can dualize one context and consider *bonds* which yield monotone Galois connections that can be composed easily [1]. One can also restrict to special classes of dual bonds: scale measures, infomorphisms, and dual bonds that are both closed and continuous all allow for rather obvious composition mechanisms.

The next step after identifying possible categories is to investigate the properties of these structures. What are their natural interpretations in terms of knowledge representation? Do they support all of the constructions that one may be interested in? How are they related to other known categories, e.g. from formal logic, order theory, or topology? This does also involve comparisons to the usage of context-morphisms in institution theory and information flow, where a relaxation of the rather strict definition of infomorphisms may yield advantages for certain applications.

In institution theory, many specific collections of formal contexts have been introduced in order to handle given logics, basically by considering the consequence relation between the models and the formulae of a logic as a formal context. In this setting, dual bonds allow for a proof theoretic interpretation as *consequence relations* and may have special properties due to the additional (logical) restrictions on contexts. For example, *compactness*[7] of classical propositional logic yields additional continuity and closure properties of dual bonds between (appropriate complements of) the respective contexts. Furthermore, extensionally continuous functions between such contexts are continuous in the usual topological sense with respect to the associated Stone spaces (see [13]).

Besides the mentioned (onto-)logical and categorical investigations, there are also further questions related to lattice theory. We already gave characterizations for the Galois connections that are induced by certain types of dual bonds, especially in the functional case (Proposition 3, Theorems 7 and 8). For many other types of dual bonds, the corresponding descriptions are missing. Likewise, although dual bonds are closed under intersections, we are aware of no (non-canonical) context that has all dual bonds as extents.

In FCA, the concept lattice of the direct product $\mathbb{K} \times \mathbb{L}$ is known as the *tensor product* of the lattices $\mathsf{B}_o(\mathbb{K})$ and $\mathsf{B}_o(\mathbb{L})$. Theorem 5 showed that the study of dual bonds can also yield additional results on the tensor product, but further relationships between both subjects have not been investigated yet. As shown in [9, Satz 15], infomorphisms can be represented by a concept lattice as well, but the role of this structure in the light of our present investigations still needs to be explored.

Finally, many other results from [1, 9–11] could not be discussed here due to space limitations. It would be a useful endeavor to compile the available knowledge from these publications in a systematic way and to investigate what additional insights are obtained in the sum.

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[7] This basically amounts to saying that the induced concept lattice is *(co-)algebraic*, see [12].

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