MELL in the Calculus of Structures

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Abstract

Gentzen's sequent calculus is a tool to study properties of logics independently of semantics. The main property is cut-elimination which entails the subformula property and consistency. It is also the basis of methods of automated deduction. Although the sequent calculus is very appropriate for classical logic, it raises some concerns in dealing with more refined logics like linear logic. An example is the global behaviour of the promotion rule. The *calculus of structures* is a recent development that is able to overcome those difficulties without losing the ability of performing a cut-elimination proof. Moreover, the cut rule can be reduced to its atomic form in the same way as the identity axiom can. In this paper I will carry out the exercise of describing the multiplicative exponential fragment of linear logic in the calculus of structures. We get the following advantages over the sequent calculus representation: no non-deterministic splitting of the context in the times rule, a local rule for promotion, a modular proof for the cut-elimination theorem, and a decomposition theorem for derivations and proofs.

1 Introduction

The sequent calculus [2],[3] has been the main tool for proof theorists to specify their systems and to prove cut-elimination. It has been remarkably successful in making the study of logical systems independent of their semantics, which is important if semantics is missing, incomplete or under development, as it is often the case in computer science. This success of the sequent calculus is based on the following two facts: First, a proof in the sequent calculus is a tree where branching occurs when inference rules with more than one premise are used, and we have a proof of the conclusion if we have a proof of each premise. Second, the main connective plays a central rôle in the application of an inference rule because the rule gives a meaning to the main conective in the conclusion by saying that the conclusion is provable if certain subformulae obtained by removing the connective are provable.

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However, these two facts make the sequent calculus unnecessarily rigid. More precisely, if we consider the sequent calculus representation of the multiplicative exponential fragment of linear logic (MELL), we can make the following observations:

- Consider the *times* rule

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad .$$

  From the point of view of proof search, this rule has a serious problem. We have to decide how to split the context of the formula $A \otimes B$ at the moment the rule is applied. For n formulas in Φ, Ψ, there are 2^n possibilities. Although there are methods, like lazy evaluation, that can circumvent this problem inside an implementation, there still remains the question whether this problem can be solved inside a calculus.

- The *promotion* rule has the following shape:

$$! \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \quad .$$

  If we want to apply this rule, we have to check each formula in the context of A, whether it has the form $?B$. This global behaviour raises the question whether it is possible to design a system where all rules are local.

- In the sequent calculus of linear logic, the cut rule cannot be reduced to atomic form, whereas the identity axiom can. This asymmetry is caused by the fact that proofs are trees, and therefore asymmetric objects in the top-down perspective.

The *calculus of structures*, introduced in [6], is a recent development that is able to overcome those difficulties [7]. Because in the calculus of structures, it is possible to relax the branching of derivations and the decomposition of the conclusion around one main connective in one formula, all three problems mentioned above disappear.

Structures are a uniform notation for formulae and sequents. In the calculus of structures all inference rules are of the shape $\rho \, \dfrac{S\{T\}}{S\{R\}}$, i.e. all rules have only one premise. Premise and conclusion are structures. The structure $S\{R\}$ consists of the structural context $S\{\ \}$ and the structure R, which fills the hole of $S\{\ \}$. The rule ρ above simply says that if a structure matches the premise $S\{T\}$, then it can be rewritten as $S\{R\}$, where the context $S\{\ \}$ does not change. This means that the rule ρ corresponds to the implication $T \Rightarrow R$, where $\Rightarrow$ stands for the implication that is modelled in the system. For instance, the non-deterministic splitting of the context in the times rule of linear logic is avoided by using the linear implication $A \otimes (B \mathbin{⅋} C) \multimap (A \otimes B) \mathbin{⅋} C$ in a rule. And the implication $!(A \mathbin{⅋} B) \multimap !A \mathbin{⅋} ?B$ gives us a local promotion rule.

Observe that there is a danger involved here, because any axiom $T \Rightarrow R$ of a Hilbert system could be used in a rule, with the consequence that there would not be any structural relation between T and R. And so all good proof theoretical properties, like cut-elimination, would be lost. Therefore, the challenge is to design inference rules that, on the one hand, are liberal enough to overcome the strictness of the sequent calculus and, on the other hand, are conservative enough to allow a proof of cut-elimination and a subformula property.

In the calculus of structures, derivations are chains of instances of inference rules. This means that they have (contrarily to what happens in the sequent calculus whose derivations are trees) a top-down symmetry. As a consequence we obtain the ability to reduce the cut-rule to its atomic form, in the same way as this is possible for the identity axiom. Furthermore, new manipulations to derivations become possible. For instance, we can negate a derivation and flip it upside down, and it remains a valid derivation. Moreover, inside a derivation, inference rules can not only be permuted up over other rules, but also permuted down under other rules.

Let me now sketch the outline of this paper. Because I will give a specification of MELL in the calculus of structures, the next section contains a short introduction to MELL. In Section 3, I will introduce the calculus of structures. In Section 4, I will present a set of rules, called system SELS (symmetric, or self-dual multiplicative exponential linear logic in the calculus of structures), that exhibits the abilities of the calculus of structures. I will also show the equivalence of SELS to MELL. Section 7 is devoted to the permutation of rules. The result will be a decomposition theorem, which exhibits the top-down symmetry of derivations in the calculus of structures. It is also crucial for the cut-elimination proof of Section 8. This proof will be very different from all known cut-elimination proofs because of its modularity.

2 The Multiplicative Exponential Fragment of Linear Logic

The calculus of structures itself is not tied to any particular logic. It can be used to represent many different logical systems, in the same way as the sequent calculus has been used for various systems, for instance classical and intuitionistic logic [2], the Lambek-calculus [8] or linear logic [4]. In this paper, I will restrict myself to the multiplicative exponential fragment of linear logic.

2.1 Definitionhe multiplicative exponential fragment of linear logic (MELL) is defined as follows:

- *Formulae*, denoted with A, B and C, are built over atoms according to the following syntax:
$$A ::= a \mid 1 \mid \bot \mid A \parr A \mid A \otimes A \mid !A \mid ?A \mid A^\perp \quad ,$$
where the binary connectives $\parr$ and $\otimes$ are called *par* and *times*, respectively, the unary connectives ! and ? are called *of course* and *why not*, respectively, and $A^\perp$ is the *negation* of A. When necessary, parentheses are used to disambiguate expressions. Negation obeys the De Morgan laws:
$$\begin{array}{rcl} (A \parr B)^\perp &=& A^\perp \otimes B^\perp \quad , \\ (A \otimes B)^\perp &=& A^\perp \parr B^\perp \quad , \\ (!A)^\perp &=& ?A^\perp \quad , \\ (?A)^\perp &=& !A^\perp \quad , \\ 1^\perp &=& \bot \quad , \\ \bot^\perp &=& 1 \quad , \\ A^{\perp\perp} &=& A \quad . \end{array}$$

$$\mathsf{id}\,\frac{}{\vdash A, A^{\perp}} \qquad \mathsf{cut}\,\frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$

$$\parr\,\frac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi} \qquad \otimes\,\frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \perp\,\frac{\vdash \Phi}{\vdash \perp, \Phi} \qquad \mathsf{1}\,\frac{}{\vdash \mathsf{1}}$$

$$\mathsf{dr}\,\frac{\vdash A, \Phi}{\vdash ?A, \Phi} \qquad \mathsf{ct}\,\frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \qquad \mathsf{wk}\,\frac{\vdash \Phi}{\vdash ?A, \Phi} \qquad !\,\frac{\vdash A, ?B_1, \dots, ?B_n}{\vdash !A, ?B_1, \dots, ?B_n} \quad (\text{for } n \geqslant 0)$$

Figure 1: System MELL in the sequent calculus

Formulae are considered equivalent modulo the smallest congruence satisfying the equations above.

- *Sequents*, denoted with Σ, are expressions of the kind

$$\vdash A_1, \dots, A_h \quad ,$$

where $h \geqslant 0$ and the comma between the formulae $A_1, \dots, A_h$ stands for multiset union. Multisets of formulae are denoted with Φ and Ψ.

- *Derivations*, denoted with Δ, are trees where the nodes are sequents to which a finite number (possibly zero) of instances of the inference rules shown in Figure 1 are applied. The sequents in the leaves are called *premises*, and the sequent in the root is the *conclusion*. A derivation with no premises is a *proof*, denoted with Π.

3 The Calculus of Structures

In the sequent calculus, rules apply to sequents which in turm are built from formulae. In the caclulus of structures, rules apply to structures, which are kind of an intermediate expressions between formulae and sequents.

3.1 Definition There are countably many *atoms*, denoted with a, b, c, ... Then, *structures*, denoted with P, Q, R, S, ..., are generated by

$$R ::= a \mid \perp \mid \mathsf{1} \mid [\underbrace{R, \dots, R}_{>0}] \mid (\underbrace{R, \dots, R}_{>0}) \mid !R \mid ?R \mid \bar{R} \quad ,$$

where $[R_1, \dots, R_h]$ is called a *par structure*, $(R_1, \dots, R_h)$ is called a *times structure*, $!R$ is called an *of-course structure*, and $?R$ is called a *why-not structure*; $\bar{R}$ is the *negation* of the structure R. Structures are considered to be equivalent modulo the relation $=$, which is the smallest congruence relation induced by the equations shown in Figure 2, where $\vec{R}$ and $\vec{T}$ stand for finite, non-empty sequences of structures, i.e. for all structures $R, R', R_1, R'_1, \dots, R_h, R'_h$ and $h > 0$,

Associativity

$$[\vec{R},[\vec{T}]] = [\vec{R},\vec{T}]$$
$$(\vec{R},(\vec{T})) = (\vec{R},\vec{T})$$

Commutativity

$$[\vec{R},\vec{T}] = [\vec{T},\vec{R}]$$
$$(\vec{R},\vec{T}) = (\vec{T},\vec{R})$$

Units

$$[\bot,\vec{R}] = [\vec{R}]$$
$$(1,\vec{R}) = (\vec{R})$$

Singleton

$$[R] = R = (R)$$

Exponentials

$$?\bot = \bot$$
$$!1 = 1$$
$$??R = ?R$$
$$!!R = !R$$

Negation

$$\overline{\bot} = 1$$
$$\overline{1} = \bot$$
$$\overline{[R_1,\dots,R_h]} = (\bar{R}_1,\dots,\bar{R}_h)$$
$$\overline{(R_1,\dots,R_h)} = [\bar{R}_1,\dots,\bar{R}_h]$$
$$\overline{?R} = !\bar{R}$$
$$\overline{!R} = ?\bar{R}$$
$$\bar{\bar{R}} = R$$

Figure 2: Syntactic congruence =

if $R = R'$ then $!R = !R'$ and $?R = ?R'$; and if $R_1 = R'_1, \dots, R_h = R'_h$ then $[R_1,\dots,R_h] = [R'_1,\dots,R'_h]$ and $(R_1,\dots,R_h) = (R'_1,\dots,R'_h)$.

3.2 Definition In the same setting, we can define *structure contexts*, which are structures with a hole. Formally, they are generated by

$$S ::= \{\ \} \mid [\underbrace{R,\dots,R}_{\geqslant 0},S,\underbrace{R,\dots,R}_{\geqslant 0}] \mid (\underbrace{R,\dots,R}_{\geqslant 0},S,\underbrace{R,\dots,R}_{\geqslant 0}) \mid !S \mid ?S \quad .$$

Because of the de Morgan laws there is no need to include the negation into the definition of the context, which means that the structure that is plugged into the hole of a context will always be positive. Structure contexts will be denoted with $R\{\ \}, S\{\ \}, T\{\ \}, \dots$. Then, $S\{R\}$ denotes the structure that is obtained by replacing the hole $\{\ \}$ in the context $S\{\ \}$ by the structure R. The structure R is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. For a better readability, I will omit the context braces if no ambiguity is possible, e.g. I will write $S[R,T]$ instead of $S\{[R,T]\}$.

It is also possible to define contexts with more than one hole, for example, the structure $S[R,T]\{!V\}$ is obtained from the context $S\{\ \}\{\ \}$ (with two holes) by putting the structure $[R,T]$ into the first and $!V$ into the second hole. I will use this notation only if no ambiguity is possible.

3.3 Definition In the calculus of structures, an *inference rule* is a scheme of the kind

$$\rho\,\frac{T}{R} \quad ,$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*. An inference rule is called an *axiom* if its premise is empty, i.e. the rule is of the shape

$$\rho\,\frac{}{R}\quad .$$

A typical rule has shape $\rho\,\dfrac{S\{T\}}{S\{R\}}$ and specifies a step of rewriting, by the implication $T \Rightarrow R$, inside a generic context $S\{\ \}$. Rules with empty contexts correspond to the case of the sequent calculus.

3.4 Definition A (*formal*) *system* $\mathscr{S}$ is a finite set of inference rules.

3.5 Definition A *derivation* Δ in a certain formal system is a finite or infinite chain of instances of inference rules in the system:

$$\begin{array}{r@{}c} & \vdots \\ \rho'' & \overline{} \\ & T \\ \rho' & \overline{} \\ & R \\ \rho & \overline{} \\ & \vdots \end{array}\quad .$$

A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the *premise* of the derivation, and the bottommost structure, if present, is called its *conclusion*. A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathscr{S}$ will be indicated with $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$. A *proof* Π in the calculus of structures is a finite derivation whose topmost inference rule is an axiom. It will be denoted by $\begin{array}{c}\Pi\Vert\mathscr{S}\\ R\end{array}$.

3.6 Definition A rule ρ is *strongly admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every application of $\rho\,\dfrac{T}{R}$ there is a derivation $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$. A rule ρ is *(weakly) admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\begin{array}{c}\Pi\Vert\mathscr{S}\cup\{\rho\}\\ R\end{array}$ there is a proof $\begin{array}{c}\Pi'\Vert\mathscr{S}\\ R\end{array}$.

3.7 Definition Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$ there is a derivation $\begin{array}{c}T\\ \Delta'\Vert\mathscr{S}'\\ R\end{array}$, and vice versa. Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *(weakly) equivalent* if for every proof $\begin{array}{c}\Pi\Vert\mathscr{S}\\ R\end{array}$ there is a proof $\begin{array}{c}\Pi'\Vert\mathscr{S}'\\ R\end{array}$, and vice versa.

3.8 Definition The function $\underline{\cdot}_{\mathsf{s}}$ defines the obvious translation from MELL formulae into struc-

$$\mathsf{id}'\,\frac{}{[R_A,\bar{R}_A]}\qquad \mathsf{id}''\,\frac{S}{(S,[R_A,\bar{R}_A])}\qquad \mathsf{cut}'\,\frac{(S,[R_A,P_\Phi],[\bar{R}_A,Q_\Psi])}{(S,[P_\Phi,Q_\Psi])}$$

$$\mathbin{\rotatebox[origin=c]{180}{$\&$}}'\,\frac{(S,[R_A,T_B,P_\Phi])}{(S,[[R_A,T_B],P_\Phi])}\qquad \otimes'\,\frac{(S,[R_A,P_\Phi],[T_B,Q_\Psi])}{(S,[(R_A,T_B),P_\Phi,Q_\Psi])}\qquad \bot'\,\frac{(S,P_\Phi)}{(S,[\bot,P_\Phi])}\qquad \mathsf{1}'\,\frac{}{\mathsf{1}}\qquad \mathsf{1}''\,\frac{S}{(S,\mathsf{1})}$$

$$\mathsf{dr}'\,\frac{(S,[R_A,P_\Phi])}{(S,[?R_A,P_\Phi])}\qquad \mathsf{ct}'\,\frac{(S,[?R_A,?R_A,P_\Phi])}{(S,[?R_A,P_\Phi])}\qquad \mathsf{wk}'\,\frac{(S,P_\Phi)}{(S,[?R_A,P_\Phi])}\qquad !'\,\frac{(S,[R_A,?T_{B_1},\dots,?T_{B_n}])}{(S,[!R_A,?T_{B_1},\dots,?T_{B_n}])}$$

Figure 3: System MELL' in the calculus of structures

tures:

$$\begin{aligned}
\underline{a}_{\mathsf{s}} &= a \quad ,\\
\underline{\bot}_{\mathsf{s}} &= \bot \quad ,\\
\underline{\mathsf{1}}_{\mathsf{s}} &= \mathsf{1} \quad ,\\
\underline{A \mathbin{\rotatebox[origin=c]{180}{$\&$}} B}_{\mathsf{s}} &= [\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}}] \quad ,\\
\underline{A \otimes B}_{\mathsf{s}} &= (\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}}) \quad ,\\
\underline{?A}_{\mathsf{s}} &= ?\underline{A}_{\mathsf{s}} \quad ,\\
\underline{!A}_{\mathsf{s}} &= !\underline{A}_{\mathsf{s}} \quad ,\\
\underline{A^\bot}_{\mathsf{s}} &= \overline{\underline{A}_{\mathsf{s}}} \quad .
\end{aligned}$$

The domain of $\underline{\cdot}_{\mathsf{s}}$ is extended to sequents by

$$\begin{aligned}
\underline{\vdash}_{\mathsf{s}} &= \bot \quad \text{and}\\
\underline{\vdash A_1,\dots,A_h}_{\mathsf{s}} &= [\underline{A_1}_{\mathsf{s}},\dots,\underline{A_h}_{\mathsf{s}}] \quad ,\ \text{for } h \geqslant 0 \quad .
\end{aligned}$$

The translation $\underline{\cdot}_{\mathsf{s}}$ induces trivially a set of rules for the calculus of structures that are able to mimic the derivations in MELL. These rules are shown in Figure 3. (The rules $\mathbin{\rotatebox[origin=c]{180}{$\&$}}'$, $\bot'$, and $\mathsf{1}''$ are vacuous.) They are a one-to-one translation of the rules of the sequent calculus shown in Figure 1. Of course, they are simply the rules of Figure 1 written in an awkward way. It is easy to see that for every derivation in MELL there is a corresponding derivation in the calculus of structures using that set of rules, and vice versa. This shows that the calculus of structures is at least as powerful as the calculus of sequents, but it hardly justifies the use of the calculus of structures. However, in the next section, I will build a system that is equivalent to the one in Figure 3 and that exhibits the extraordinary abilities of the calculus of structures. Its rules will be much simpler and will not have the flaws mentioned in the introduction.

3.9 Definition The translation from structures into MELL formulae is given by the function $\underline{\cdot}_{\mathsf{L}}$:

$$
\begin{array}{rcl}
\underline{a}_{\mathsf{L}} & = & a \quad , \\
\underline{\bot}_{\mathsf{L}} & = & \bot \quad , \\
\underline{1}_{\mathsf{L}} & = & 1 \quad , \\
\underline{[R_1, \dots, R_h]}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \parr \cdots \parr \underline{R_h}_{\mathsf{L}} \quad , \\
\underline{(R_1, \dots, R_h)}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \otimes \cdots \otimes \underline{R_h}_{\mathsf{L}} \quad , \\
\underline{?R}_{\mathsf{L}} & = & ?\, \underline{R}_{\mathsf{L}} \quad , \\
\underline{!R}_{\mathsf{L}} & = & !\, \underline{R}_{\mathsf{L}} \quad , \\
\underline{\bar{R}}_{\mathsf{L}} & = & (\underline{R}_{\mathsf{L}})^{\bot} \quad .
\end{array}
$$

4 A Symmetric Set of Rules

4.1 In [6], system BV is introduced, which is essentially the multiplicative fragment of linear logic (MLL) extended by a self-dual non-commutative connective. A crucial ingredient of system BV is the *switch* rule:

$$\mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \quad ,$$

which totally captures the behavior of the *par* and the *times* with respect to each other. In this paper, it will also play a central rôle. The system ELS that I propose in this paper will be an extension of flat system BV (i.e. system BV without non-commutativity), and therefore, the switch rule will be an essential part of ELS.

4.2 If we want to end up with a system that is equivalent to MELL, we have to capture the behavior of *of-course* and *why-not* with respect to *par* and *times*. The following two rules

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{and} \qquad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}} \quad ,$$

called *promotion* and *co-promotion*, respectively, are sufficient. The rule $\mathsf{p}{\downarrow}$ captures the behavior of ! and ? with respect to the *par* and the rule $\mathsf{p}{\uparrow}$ the behavior of ! and ? with respect to the *times*. Observe that $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$ are dual to each other, whereas the switch is dual to itself.

4.3 The next step is to capture weakening and contraction. This is done via the rules

$$\mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \text{and} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \quad ,$$

called *weakening* and *absorption*, respectively. For the sake of symmetry, I also include their dual rules *co-weakening* and *co-absorption*:

$$\mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{1\}} \qquad \text{and} \qquad \mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R,R)} \quad .$$

4.4 Up to now we totally captured the behavior of the logical connectives, what remains are the identity rules. Following the work outlined in [6], the obvious thing to do is to introduce the rules

$$\mathsf{i}\!\downarrow \frac{S\{1\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\bot\}} \quad ,$$

which are called *interaction* and *co-interaction* (or *cut*). The former is the same as the identity axiom of linear logic and the latter is a general formulation of the cut rule.

Now consider the rules

$$\mathsf{a}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{and} \qquad \mathsf{a}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}} \quad ,$$

called *atomic interaction* and *atomic co-interaction* (or *atomic cut*), respectively. They are obviously instances of the two rules above. However, we can replace the general interaction rules by the atomic rules.

4.5 Proposition *The rule* $\mathsf{i}\!\downarrow$ *is strongly admissible for the rules* $\{\mathsf{a}\!\downarrow, \mathsf{s}, \mathsf{p}\!\downarrow\}$. *Dually, the rule* $\mathsf{i}\!\uparrow$ *is strongly admissible for* $\{\mathsf{a}\!\uparrow, \mathsf{s}, \mathsf{p}\!\uparrow\}$.

Proof: For a given application of $\mathsf{i}\!\downarrow \frac{S\{1\}}{S[R,\bar{R}]}$, we will by structural induction on R construct an equivalent derivation that contains only $\mathsf{a}\!\downarrow$, s and $\mathsf{p}\!\downarrow$.

- $R = \bot$ or $R = 1$: In this case $S[R,\bar{R}] = S\{1\}$.
- R is an atom: Then the given instance of $\mathsf{i}\!\downarrow$ is an instance of $\mathsf{a}\!\downarrow$.
- $R = [P,Q]$, where $P \neq \bot \neq Q$: Apply the induction hypothesis on

$$\mathsf{s}\,\frac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow \dfrac{\mathsf{i}\!\downarrow \dfrac{S\{1\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[Q,([P,\bar{P}],\bar{Q})]}}{S[P,Q,(\bar{P},\bar{Q})]} \quad .$$

- $R = (P,Q)$, where $P \neq 1 \neq Q$: Similar to the previous case.
- $R = ?P$, where $P \neq \bot$: Apply the induction hypothesis on

$$\mathsf{p}\!\downarrow \frac{\mathsf{i}\!\downarrow \dfrac{S\{1\}}{S\{![P,\bar{P}]\}}}{S[?P,!\bar{P}]} \quad .$$

  (Note that $S\{1\} = S\{!1\}$.)
- $R = !P$, where $P \neq 1$: Similar to the previous case.

The second statement is dual to the first. For the sake of convenience let me show the two interesting derivations:

$$\mathsf{i}\!\uparrow \frac{\mathsf{i}\!\uparrow \dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(P,Q,[\bar{P},\bar{Q}])}{S(Q,[(P,\bar{P}),\bar{Q}])}}{S[(P,\bar{P}),(Q,\bar{Q})]}}{S(Q,\bar{Q})}}{S\{\bot\}} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{\mathsf{p}\!\uparrow \dfrac{S(!P,?\bar{P})}{S\{?(P,\bar{P})\}}}{S\{\bot\}} \quad .$$

$$
\mathsf{a}{\downarrow}\,\frac{S\{1\}}{S[a,\bar a]} \qquad \mathsf{a}{\uparrow}\,\frac{S(a,\bar a)}{S\{\bot\}}
$$

$$
\mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]}
$$

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}}
$$

$$
\mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{1\}}
$$

$$
\mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \qquad \mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R,R)}
$$

Figure 4: System SELS

□

4.6 I will call the system $\{\mathsf{a}{\downarrow},\mathsf{a}{\uparrow},\mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow},\mathsf{w}{\downarrow},\mathsf{w}{\uparrow},\mathsf{b}{\downarrow},\mathsf{b}{\uparrow}\}$, shown in Figure 4, *symmetric (or self-dual) multiplicative exponential linear logic in the calculus of structures*, or system SELS.

4.7 There is another strong admissibility result involved here, that has already been observed in [6]. If the rules $\mathsf{i}{\downarrow}$, $\mathsf{i}{\uparrow}$ and s are in a system, then for each rule ρ, its *co-rule* ρ', i.e. the rule obtained from ρ by exchanging and negating premise and conclusion, is strongly admissible. Let $\rho\,\frac{S\{P\}}{S\{Q\}}$ be given. Then any instance of $\rho'\,\frac{S\{\bar Q\}}{S\{\bar P\}}$ can be replaced by the following derivation:

$$
\mathsf{i}{\uparrow}\,\frac{\rho\,\frac{\mathsf{s}\,\frac{\mathsf{i}{\downarrow}\,\frac{S\{\bar Q\}}{S(\bar Q,[P,\bar P])}}{S[(\bar Q,P),\bar P]}}{S[(\bar Q,Q),\bar P]}}{S\{\bar P\}} \quad .
$$

4.8 Proposition *Every rule* $\mathsf{x}{\uparrow}$ *in system* SELS *is strongly admissible for* $\{\mathsf{i}{\downarrow},\mathsf{i}{\uparrow},\mathsf{s},\mathsf{x}{\downarrow}\}$.

4.9 Propositions 4.5 and 4.8 together say, that the general cut-rule is as powerful as the whole up-fragment of the system, and vice versa.

4.10 Observe that in Proposition 4.5 only the rules s, $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$ are used to decompose the general interaction and the general cut into their atomic form, whereas the rules $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$ and b↑ are not used. This suggests the following definition: In system SELS, the rules s, p↓ and p↑ are called *core* part, whereas the rules w↓, w↑, b↓ and b↑ are *non-core*.

4.11 So far we are only able to describe derivations. In order to formulate proofs, we need an axiom. I will take the rule

$$\mathsf{e}{\downarrow}\,\frac{}{\mathsf{1}} \quad ,$$

called *empty*. In the language of linear logic it simply says that $\vdash \mathsf{1}$ is provable. Observe that in every proof, the rule e↓ occurs exactly once, namely as the topmost rule of the proof.

4.12 Theorem *If a given structure R is provable in system* $\mathsf{SELS} \cup \{\mathsf{e}{\downarrow}\}$*, then its translation* $\vdash \underline{R}_{\mathsf{L}}$ *is provable in* MELL*.*

Proof: Suppose, we have a proof Π of R in system $\mathsf{SELS} \cup \{\mathsf{e}{\downarrow}\}$. By induction on Π, let us build a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{R}_{\mathsf{L}}$ in MELL.

Base case: Π is $\mathsf{e}{\downarrow}\,\frac{}{\mathsf{1}}$: Let $\underline{\Pi}_{\mathsf{L}}$ be the proof $\mathsf{1}\,\frac{}{\vdash \mathsf{1}}$.

Inductive case: Suppose Π is $\begin{array}{c} \Pi' \Big\| \mathsf{SELS}\cup\{\mathsf{e}{\downarrow}\} \\ \rho\,\dfrac{S\{R\}}{S\{T\}} \end{array}$, i.e. $\rho\,\dfrac{S\{R\}}{S\{T\}}$ is the last rule to be applied in Π. The following MELL-proofs show that $\vdash (\underline{R}_{\mathsf{L}})^{\perp}, \underline{T}_{\mathsf{L}}$ is provable in MELL for every rule $\rho\,\dfrac{S\{R\}}{S\{T\}}$ in SELS, i.e. $\underline{R}_{\mathsf{L}} \multimap \underline{T}_{\mathsf{L}}$ is a theorem in MELL:

$$\bot\,\dfrac{⅋\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, a^{\perp}}}{\vdash a ⅋ a^{\perp}}}{\vdash \bot, a ⅋ a^{\perp}} \quad , \qquad \mathsf{wk}\,\dfrac{\mathsf{1}\,\dfrac{}{\vdash \mathsf{1}}}{\vdash \mathsf{1}, ?R} \quad , \qquad \mathsf{ct}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash !R^{\perp}, ?R} \qquad \mathsf{dr}\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R}}{\vdash R^{\perp}, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R} \quad ,$$

$$⅋\,\dfrac{⅋\,\dfrac{\otimes\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash U^{\perp}, U}}{\vdash R^{\perp}, U^{\perp}, R \otimes U} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, R \otimes U, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, (R \otimes U) ⅋ T}}{\vdash (R^{\perp} \otimes T^{\perp}) ⅋ U^{\perp}, (R \otimes U) ⅋ T} \quad \text{and} \quad ⅋\,\dfrac{!\,\dfrac{\mathsf{dr}\,\dfrac{\mathsf{dr}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, !T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R ⅋ !T} \quad .$$

Since linear implication is closed under positive context, we also have that $\underline{S\{R\}}_{\mathsf{L}} \multimap \underline{S\{T\}}_{\mathsf{L}}$ is a theorem in MELL, i.e. $\vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}$ is provable in MELL. By induction hypothesis we have a proof $\underline{\Pi'}_{\mathsf{L}}$ of $\vdash \underline{S\{R\}}_{\mathsf{L}}$ in MELL. Now we can get a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{S\{T\}}_{\mathsf{L}}$ by applying the cut-rule:

$$\mathsf{cut}\,\dfrac{\vdash \underline{S\{R\}}_{\mathsf{L}} \qquad \vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}}{\vdash \underline{S\{T\}}_{\mathsf{L}}} \quad .$$

□

4.13 Theorem *If a given sequent* $\vdash \Phi$ *is provable in* MELL*, then the structure* $\underline{\vdash \Phi}_{\mathsf{s}}$ *is provable in system* $\mathsf{SELS} \cup \{\mathsf{e}{\downarrow}\}$*.*

Proof: Let Π be the proof of $\vdash \Phi$ in MELL. By structural induction on Π, we will construct a proof $\underline{\Pi}_{\mathsf{s}}$ of $\underline{\vdash \Phi}_{\mathsf{s}}$ in system $\mathsf{SELS} \cup \{\mathsf{e}{\downarrow}\}$.

- If Π is $\mathsf{id}\,\dfrac{}{\vdash A, A^{\perp}}$ for some formula A, let $\underline{\Pi}_{\mathsf{s}}$ be the proof obtained from $\mathsf{i}{\downarrow}\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{}{1}}{[\underline{A}_{\mathsf{s}}, \bar{\underline{A}}_{\mathsf{s}}]}$ via Proposition 4.5.

- If $\mathsf{cut}\,\dfrac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Vert \mathsf{SELS} \\ [\overline{\underline{A}}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof obtained from

$$\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\begin{array}{c} \mathsf{e}{\downarrow}\,\dfrac{}{1} \\ \Delta_1' \Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \\ \Delta_2' \Vert \mathsf{SELS} \\ ([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\overline{\underline{A}}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]) \end{array}}{[([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], \overline{\underline{A}}_{\mathsf{s}}), \underline{\Psi}_{\mathsf{s}}]}}{[\underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}, (\underline{A}_{\mathsf{s}}, \overline{\underline{A}}_{\mathsf{s}})]}}{[\underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]}$$

  via Proposition 4.5.

- If $\mathbin{\bindnasrepma}\,\dfrac{\vdash A, B, \Phi}{\vdash A \mathbin{\bindnasrepma} B, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $[\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]$ that exists by induction hypothesis.

- If $\otimes\,\dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Vert \mathsf{SELS} \\ [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\begin{array}{c} \mathsf{e}{\downarrow}\,\dfrac{}{1} \\ \Delta_1' \Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \\ \Delta_2' \Vert \mathsf{SELS} \\ ([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]) \end{array}}{[([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], \underline{B}_{\mathsf{s}}), \underline{\Psi}_{\mathsf{s}}]}}{[(\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}), \underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]} \;.$$

- If $\perp\,\dfrac{\vdash \Phi}{\vdash \perp, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $\underline{\vdash \Phi}_{\mathsf{s}}$ that exists by induction hypothesis.

- If Π is $1\,\dfrac{}{\vdash 1}$, let $\underline{\Pi}_{\mathsf{s}}$ be $\mathsf{e}{\downarrow}\,\dfrac{}{1}$.

- If $\mathsf{dr}\,\dfrac{\vdash A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} \Big\Vert \mathsf{SELS}\cup\{\mathsf{e}\!\downarrow\} \\ \mathsf{w}\!\downarrow \dfrac{[\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{\mathsf{b}\!\downarrow \dfrac{[?\underline{A}_{\mathsf{s}}, \underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}} \end{array} .$$

- If $\mathsf{ct}\,\dfrac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} \Big\Vert \mathsf{SELS}\cup\{\mathsf{e}\!\downarrow\} \\ \mathsf{b}\!\downarrow \dfrac{[??\underline{A}_{\mathsf{s}}, ?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{[??\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \end{array} .$$

(Note that $??\underline{A}_{\mathsf{s}} = ?\underline{A}_{\mathsf{s}}$.)

- If $\mathsf{wk}\,\dfrac{\vdash \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} \Big\Vert \mathsf{SELS}\cup\{\mathsf{e}\!\downarrow\} \\ \mathsf{w}\!\downarrow \dfrac{\underline{\Phi}_{\mathsf{s}}}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \end{array} .$$

- If $!\,\dfrac{\vdash A, ?B_1, \dots, ?B_n}{\vdash\, !A, ?B_1, \dots, ?B_n}$ is the last rule applied in Π, then there is by induction hypothesis a derivation

$$\begin{array}{c} 1 \\ \Delta \Big\Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \dots, ?\underline{B_n}_{\mathsf{s}}] \end{array}$$

. Now let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} \mathsf{e}\!\downarrow \dfrac{}{!1} \\ \Delta' \Big\Vert \mathsf{SELS} \\ \mathsf{p}\!\downarrow \dfrac{![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \dots, ?\underline{B_n}_{\mathsf{s}}]}{\vdots} \\ \mathsf{p}\!\downarrow \dfrac{}{\mathsf{p}\!\downarrow \dfrac{[![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}], ??\underline{B_2}_{\mathsf{s}}, \dots, ??\underline{B_n}_{\mathsf{s}}]}{[!\underline{A}_{\mathsf{s}}, ??\underline{B_1}_{\mathsf{s}}, ??\underline{B_2}_{\mathsf{s}}, \dots, ??\underline{B_n}_{\mathsf{s}}]}} \end{array} .$$

□

5 Permutability of Rules

The top-down symmetry of derivations in the calculus of structures enables us to study the permutability of rules with respect to each other in a very natural way. This allows to formulate a decomposition theorem (Theorem 7.1) which is not possible in the sequent calculus.

5.1 Definition A rule ρ *permutes up over* a rule π (or π *permutes down under* ρ) if for every derivation $\pi\,\dfrac{Q}{\rho\,\dfrac{U}{P}}$ there is a derivation $\rho\,\dfrac{Q}{\pi\,\dfrac{V}{P}}$ for some structure V.

In order to study the permutation properties of rules, some more definitions are needed. The inference rules of SELS, as it is presented in Figure 4, are all of the kind $\rho\,\dfrac{S\{W\}}{S\{Z\}}$: the structure Z is called the *redex* and W the *contractum* of the rule's instance. A sub-structure that occurs both in the redex and in the contractum of a rule without changing is called *passive*, and all the sub-structures of redexes and contracta, that are not passive, (i.e. that change, dissappear or are duplicated) are called *active*. Consider for example the rules

$$\mathsf{p}{\downarrow}\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{and} \qquad \mathsf{b}{\downarrow}\frac{S[?R,R]}{S\{?R\}} \quad .$$

In $\mathsf{p}{\downarrow}$, the redex is $[!R,?T]$ and the contractum is $![R,T]$; the structures R and T are passive; the structures $[!R,?T]$, $!R$ and $?T$ active in the redex; and the structures $![R,T]$ and $[R,T]$ are active in the contractum. In $\mathsf{b}{\downarrow}$ there are no passive structures; in the redex the structures $?R$ and R are active and in the contractum $[?R,R], ?R, R$ and R are active (i.e. both occurances of the structure R are active).

5.2 Definition An application of a rule $\rho\,\frac{T}{S}$ will be called *trivial* if $S=T$.

5.3 In order to find out whether a rule ρ permutes up over a rule π, we have to consider all possibilities of interference of the redex of π and the contractum of ρ in a situation $\begin{array}{c}\pi\,\frac{Q}{U}\\ \rho\,\overline{P}\end{array}$.
Similarly as in the study of critical pairs in term rewriting systems, it can happen that one is inside the other, that they overlap or that they are independent. Although the situation is symmetric with respect to ρ and π, in almost all proofs of this paper, the situation to be considered will be of the shape $\begin{array}{c}\pi\,\frac{Q}{S\{W\}}\\ \rho\,\overline{S\{Z\}}\end{array}$, where Z is the redex of ρ and W the contractum of ρ.
Then the following six cases exhaust all possibilities. Figure 5 shows an example for each case.

(1) The redex of π is inside the context $S\{\ \}$ of ρ.

(2) The contractum of ρ is inside a passive structure of the redex of π.

(3) The redex of π is inside a passive structure of the contractum W of ρ.

(4) The redex of π is inside an active structure of the contractum W of ρ but not inside a passive one.

(5) The contractum W of ρ is inside an active structure of the redex of π but not inside a passive one.

(6) The contractum W of ρ and the redex of π overlap.

In the first two cases, we have that $Q=S'\{W\}$ for some context $S'\{\ \}$, and we can obtain a derivation $\begin{array}{c}\rho\,\frac{S'\{W\}}{S'\{Z\}}\\ \pi\,\overline{S\{Z\}}\end{array}$. In the third case, we have that $Z=Z'\{R\}$ and $W=W'\{R\}$ and $Q=S\{W'\{R'\}\}$ for some structures R (which is passive for ρ) and R' and some contexts $Z'\{\ \}$ and $W'\{\ \}$. So, we can obtain a derivation $\begin{array}{c}\rho\,\frac{S\{W'\{R'\}\}}{S\{Z'\{R'\}\}}\\ \pi\,\overline{S\{Z'\{R\}\}}\end{array}$. This means that in a proof of a

(1) $$\mathsf{s}\,\dfrac{\mathsf{a}{\downarrow}\,\dfrac{(d,[a,c],b)}{([b,\bar{b}],d,[a,c],b)}}{([b,\bar{b}],d,[(a,b),c])}$$
(2) $$\mathsf{a}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{(!(a,c),[\bar{a},d])}{[\bar{a},(!(a,c),d)]}}{[\bar{a},(!(a,[b,\bar{b}],c),d)]}$$
(3) $$\mathsf{s}\,\dfrac{\mathsf{a}{\downarrow}\,\dfrac{([a,c],b)}{([(a,[b,\bar{b}]),c],b)}}{[(a,[b,\bar{b}],b),c]}$$

(4) $$\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{(a,![b,(c,d)])}{(a,[!b,?(c,d)])}}{[(a,!b),?(c,d)]}$$
(5) $$\mathsf{a}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{[a,b]}{[a,b,?[(c,\bar{c}),\bar{a}]]}}{[a,b,?\bar{a}]}$$
(6) $$\mathsf{b}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{[?[a,b],a,([b,c],d)]}{[?[a,b],a,b,(c,d)]}}{[?[a,b],(c,d)]}$$

Figure 5: Examples, how redex and contractum of two rules can interfere

permutation result the case (1)–(3) are always trivial, whereas for the remaining cases (4)–(6), more elaboration will be necessary.

In every proof concerning a permutation result I will follow this schema.

5.4 Lemma *The rule* $\mathsf{w}{\downarrow}$ *permutes up over the rules* $\mathsf{a}{\downarrow}, \mathsf{a}{\uparrow}, \mathsf{s}, \mathsf{p}{\downarrow}$ *and* $\mathsf{w}{\uparrow}$*.*

Proof: Consider a derivation $\mathsf{w}{\downarrow}\,\dfrac{\pi\,\dfrac{Q}{S\{\bot\}}}{S\{?R\}}$, where $\pi \in \{\mathsf{a}{\downarrow}, \mathsf{a}{\uparrow}, \mathsf{s}, \mathsf{p}{\downarrow}, \mathsf{w}{\uparrow}\}$. Without loss of generality, assume that the application of π is not trivial. According to 5.3, the following cases exhaust all possibilities.

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum $\bot$ of $\mathsf{w}{\downarrow}$. Not possible because there are no passive structures.

(4) The redex of π is inside the contractum $\bot$. Not possible because the application of π is not trivial.

(5) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ is inside an active structure of the redex of π but not inside a passive one. Not possible.

(6) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ and the redex of π overlap. Not possible.

□

5.5 Lemma *The rule* $\mathsf{w}{\uparrow}$ *permutes down under the rules* $\mathsf{a}{\downarrow}, \mathsf{a}{\uparrow}, \mathsf{s}, \mathsf{p}{\downarrow}$ *and* $\mathsf{w}{\downarrow}$*.*

Proof: The statement is dual to the previous lemma. □

5.6 Lemma *The rule* $\mathsf{a}{\downarrow}$ *permutes up over the rules* $\mathsf{a}{\uparrow}, \mathsf{s}, \mathsf{p}{\uparrow}$ *and* $\mathsf{w}{\uparrow}$*.*

Proof: Consider a derivation $\mathsf{a}{\downarrow}\,\dfrac{\pi\,\dfrac{Q}{S\{1\}}}{S[a,\bar{a}]}$, where $\pi \in \{\mathsf{a}{\uparrow}, \mathsf{s}, \mathsf{p}{\uparrow}\}$. Without loss of generality, assume that the application of π is not trivial. Again, follow 5.3.

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum 1 of $\mathsf{a}{\downarrow}$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum 1 of $\mathsf{a}\!\downarrow$. Not possible because there are no passive structures.

(4) The redex of π is inside the contractum 1. Not possible because the application of π is not trivial.

(5) The contractum 1 of $\mathsf{a}\!\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. Not possible.

(6) The contractum 1 of $\mathsf{a}\!\downarrow$ and the redex of π overlap. Not possible.

□

5.7 Lemma *The rule* $\mathsf{a}\!\uparrow$ *permutes down under the rules* $\mathsf{a}\!\downarrow, \mathsf{s}, \mathsf{p}\!\downarrow$ *and* $\mathsf{w}\!\downarrow$*.*

Proof: Dual to Lemma 5.6. □

Observe that the rule $\mathsf{w}\!\downarrow$ does not permute up over $\mathsf{p}\!\uparrow$. This is easy to see from the derivation

$$\mathsf{w}\!\downarrow \frac{\mathsf{p}\!\uparrow \dfrac{S(?U, !V)}{S\{?[(U,V),\bot]\}}}{S\{?[(U,V),?R]\}} \quad .$$

However, with the help of the switch rule we can get

$$\mathsf{s}\frac{\mathsf{p}\!\uparrow \dfrac{\mathsf{w}\!\downarrow \dfrac{S(?U, !V)}{S(?U, ![V, ?R])}}{S\{?(U,[V,?R])\}}}{S\{?[(U,V),?R]\}} \quad .$$

For the rules $\mathsf{a}\!\downarrow$ and $\mathsf{p}\!\downarrow$ the situation is similar. Furthermore, the rule $\mathsf{a}\!\downarrow$ does not permute over $\mathsf{w}\!\downarrow$. For example, in the derivation

$$\mathsf{a}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S\{\bot\}}{S\{?(a,b)\}}}{S\{?(a,[c,\bar{c}],b)\}} \quad ,$$

we cannot permute $\mathsf{a}\!\downarrow$ up, but we could replace the whole derivation by a single application of $\mathsf{w}\!\downarrow$:

$$\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?(a,[c,\bar{c}],b)\}} \quad .$$

This leads to the following definition.

5.8 Definition A rule ρ *permutes up over* a rule π *by* a rule σ if for every derivation $\rho \dfrac{\pi \dfrac{Q}{U}}{P}$ there is either a derivation $\pi \dfrac{\rho \dfrac{Q}{V}}{P}$ for some structure V or a derivation $\sigma \dfrac{\pi \dfrac{\rho \dfrac{Q}{V}}{V'}}{P}$ for some structures V and

V' or a derivation $\pi \dfrac{Q}{P}$ or a derivation $\rho \dfrac{Q}{P}$ or a derivation $\sigma \dfrac{Q}{P}$. Dually, a rule π *permutes down under* a rule ρ *by* a rule σ if for every derivation $\begin{array}{l} \pi \dfrac{Q}{U} \\ \rho \dfrac{}{P} \end{array}$ there is either a derivation $\begin{array}{l} \rho \dfrac{Q}{V} \\ \pi \dfrac{}{P} \end{array}$ for some structure V or a derivation $\begin{array}{l} \sigma \dfrac{Q}{V} \\ \rho \dfrac{}{V'} \\ \pi \dfrac{}{P} \end{array}$ for some structures V and V' or a derivation $\pi \dfrac{Q}{P}$ or a derivation $\rho \dfrac{Q}{P}$ or a derivation $\sigma \dfrac{Q}{P}$.

5.9 Lemma *(a) The rule* $\mathsf{w}\!\downarrow$ *permutes up over* $\mathsf{p}\!\uparrow$ *by* s*. (b) The rule* $\mathsf{w}\!\uparrow$ *permutes down under* $\mathsf{p}\!\downarrow$ *by* s*. (c) The rule* $\mathsf{a}\!\downarrow$ *permutes up over* $\mathsf{p}\!\downarrow$ *and* $\mathsf{w}\!\downarrow$ *by* s*. (d) The rule* $\mathsf{a}\!\uparrow$ *permutes down under* $\mathsf{p}\!\uparrow$ *and* $\mathsf{w}\!\uparrow$ *by* s*.*

Proof: (a) Consider a derivation $\begin{array}{l} \mathsf{p}\!\uparrow \dfrac{Q}{S\{\bot\}} \\ \mathsf{w}\!\downarrow \dfrac{}{S\{?R\}} \end{array}$, where $\mathsf{p}\!\uparrow$ is not trivial. Then the cases (1)-(4) and (6) are as in the proof of Lemma 5.4. The only nontrivial case is:

(5) The contractum $\bot$ of $\mathsf{w}\!\downarrow$ is inside an active structure of the redex of $\mathsf{p}\!\uparrow$ but not inside a passive one. Then $S\{\bot\} = S'\{?[(U,V),\bot]\}$. We have

$$\begin{array}{l} \mathsf{p}\!\uparrow \dfrac{S'(?U,!V)}{S'\{?[(U,V),\bot]\}} \\ \mathsf{w}\!\downarrow \dfrac{}{S'\{?[(U,V),?R]\}} \end{array} \quad \text{yields} \quad \begin{array}{l} \mathsf{w}\!\downarrow \dfrac{S'(?U,!V)}{S'(?U,![V,?R])} \\ \mathsf{p}\!\uparrow \dfrac{}{S'\{?(U,[V,?R])\}} \\ \mathsf{s} \dfrac{}{S'\{?[(U,V),?R]\}} \end{array} \quad .$$

(b) Dual to (a).

(c) Consider a derivation $\begin{array}{l} \pi \dfrac{Q}{S\{1\}} \\ \mathsf{a}\!\downarrow \dfrac{}{S[a,\bar{a}]} \end{array}$, where $\pi \in \{\mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow\}$ is not trivial. The cases (1)-(4) and (6) are as in the proof of Lemma 5.6. The only nontrivial case is:

(5) The contractum 1 of $\mathsf{a}\!\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. There are three subcases.

(i) $\pi = \mathsf{p}\!\downarrow$ and $S\{1\} = S'[(!R,1),?T]$. Then

$$\begin{array}{l} \mathsf{p}\!\downarrow \dfrac{S'\{![R,T]\}}{S'[(!R,1),?T]} \\ \mathsf{a}\!\downarrow \dfrac{}{S'[(!R,[a,\bar{a}]),?T]} \end{array} \quad \text{yields} \quad \begin{array}{l} \mathsf{a}\!\downarrow \dfrac{S'\{![R,T]\}}{S'(![R,T],[a,\bar{a}])} \\ \mathsf{p}\!\downarrow \dfrac{}{S'([!R,?T],[a,\bar{a}])} \\ \mathsf{s} \dfrac{}{S'[(!R,[a,\bar{a}]),?T]} \end{array} \quad .$$

(ii) $\pi = \mathsf{p}\!\downarrow$ and $S\{1\} = S'[!R,(?T,1)]$. Similar to (i).

(iii) $\pi = \mathsf{w}\!\downarrow$ and $S\{1\} = S'\{?S''\{1\}\}$. Then

$$\mathsf{a}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S'\{\bot\}}{S'\{?S''\{1\}\}}}{S'\{?S''[a,\bar{a}]\}} \qquad \text{yields} \qquad \mathsf{w}\!\downarrow \frac{S'\{\bot\}}{S'\{?S''[a,\bar{a}]\}} \quad .$$

(d) Dual to (c). □

5.10 This is sufficient to show that in any derivation that does not contain the rules $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$, we can permute all instances of $\mathsf{w}\!\downarrow$ and $\mathsf{a}\!\downarrow$ to the top of the derivation and all instances of $\mathsf{w}\!\uparrow$ and $\mathsf{a}\!\uparrow$ to the bottom. For the full decomposition theorem it is necessary to handle the rules $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$. However, this is not possible with a trivial permutation argument because they neither permute up over nor down under any other rule.

6 Circles in Derivations

This section is devoted to a very important property of derivation, which is crucial for the decomposition theorem in the next section as well as for cut elimination. However, the proof of it is very difficult and technical. In the first reading of the paper, this section might be skipped entirely.

6.1 Definition A !-*link* is any of-course structure $!R$ that occurs as substructure of a structure S inside a derivation Δ.

6.2 In order to avoid ambiguity, I will always mark !-links with a $!^\bullet$. For example, the derivation

$$\mathsf{p}\!\uparrow \frac{\mathsf{s}\dfrac{\mathsf{p}\!\downarrow \dfrac{(!^\bullet[(b,!a),\bar{a}],!c)}{([!^\bullet(b,!a),?\bar{a}],!c)}}{([!(b,!^\bullet a),(?\bar{a},!c)])}}{([!(b,!a),?(\bar{a},c)])}$$

contains many !-links, but only three of them are marked.

6.3 Definition Two !-links $!^\bullet R$ and $!^\bullet R'$ inside a derivation Δ are *connected* if they occur in two consecutive structures, i.e. Δ is of the shape

$$\begin{array}{c} P \\ \| \\ \rho\,\dfrac{S'\{!^\bullet R'\}}{S\{!^\bullet R\}} \\ \| \\ Q \end{array}$$

, such that one of the following cases holds (see Figure 6):

(1) The link $!^\bullet R$ is inside the context of ρ, i.e. $R = R'$ and $S\{!^\bullet R\} = S''\{!^\bullet R\}\{Z\}$ and $S'\{!^\bullet R'\} = S''\{!^\bullet R\}\{W\}$, where Z and W are redex and contractum of ρ.

(2) The link $!^\bullet R$ is inside a passive structure of the redex of ρ, i.e. $R = R'$ and $S\{!^\bullet R\} = S''\{Z\{!^\bullet R\}\}$ and $S'\{!^\bullet R'\} = S''\{W\{!^\bullet R\}\}$, where $Z\{!^\bullet R\}$ and $W\{!^\bullet R\}$ are redex and contractum of ρ.

(3) The redex of ρ is inside R, i.e. $S\{\ \} = S'\{\ \}$ and $S\{!^\bullet R\} = S\{!^\bullet R''\{Z\}\}$ and $S'\{!^\bullet R'\} = S\{!^\bullet R''\{W\}\}$, where Z and W are redex and contractum of ρ.

(4) The link $!^\bullet R$ is inside an active structure of the redex of ρ, but not inside a passive one. Then six subcases are possible:

(i) $\rho = \mathsf{p}{\downarrow}$, $S\{!^\bullet R\} = S'[!^\bullet R, ?T]$ and $S'\{!^\bullet R'\} = S'\{!^\bullet [R, T]\}$, i.e. $R' = [R, T]$ for some structure T.

(ii) $\rho = \mathsf{b}{\downarrow}$, $R = R'$, $S\{!^\bullet R\} = S''\{?T\{!^\bullet R\}\}$ and $S'\{!^\bullet R'\} = S''[?T\{!^\bullet R\}, T\{!R\}]$.

(iii) $\rho = \mathsf{b}{\downarrow}$, $R = R'$, $S\{!^\bullet R\} = S''\{?T\{!^\bullet R\}\}$ and $S'\{!^\bullet R'\} = S''[?T\{!R\}, T\{!^\bullet R\}]$.

(iv) $\rho = \mathsf{b}{\uparrow}$, $R = R'$, $S\{!^\bullet R\} = S'(!^\bullet R, R)$ and $S'\{!^\bullet R'\} = S'\{!^\bullet R\}$.

(v) $\rho = \mathsf{b}{\uparrow}$, $R = R'$, $S\{!^\bullet R\} = S''(!V\{!^\bullet R\}, V\{!R\})$ and $S'\{!^\bullet R'\} = S''\{!V\{!^\bullet R\}\}$.

(vi) $\rho = \mathsf{b}{\uparrow}$, $R = R'$, $S\{!^\bullet R\} = S''(!V\{!R\}, V\{!^\bullet R\})$ and $S'\{!^\bullet R'\} = S''\{!V\{!^\bullet R\}\}$.

$$(1)\ \rho\,\frac{S''\{!^\bullet R\}\{W\}}{S''\{!^\bullet R\}\{Z\}} \qquad (2)\ \rho\,\frac{S''\{W\{!^\bullet R\}\}}{S''\{Z\{!^\bullet R\}\}} \qquad (3)\ \rho\,\frac{S\{!^\bullet R''\{W\}\}}{S\{!^\bullet R''\{Z\}\}}$$

$$(4.\mathrm{i})\ \mathsf{p}{\downarrow}\,\frac{S'\{!^\bullet [R,T]\}}{S'[!^\bullet R, ?T]} \qquad (4.\mathrm{ii})\ \mathsf{b}{\downarrow}\,\frac{S''[?T\{!^\bullet R\}, T\{!R\}]}{S''\{?T\{!^\bullet R\}\}} \qquad (4.\mathrm{iii})\ \mathsf{b}{\downarrow}\,\frac{S''[?T\{!R\}, T\{!^\bullet R\}]}{S''\{?T\{!^\bullet R\}\}}$$

$$(4.\mathrm{iv})\ \mathsf{b}{\uparrow}\,\frac{S'\{!^\bullet R\}}{S'(!^\bullet R, R)} \qquad (4.\mathrm{v})\ \mathsf{b}{\uparrow}\,\frac{S''\{!V\{!^\bullet R\}\}}{S''(!V\{!^\bullet R\}, V\{!R\})} \qquad (4.\mathrm{vi})\ \mathsf{b}{\uparrow}\,\frac{S''\{!V\{!^\bullet R\}\}}{S''(!V\{!R\}, V\{!^\bullet R\})}$$

Figure 6: Connection of !-links

For example, in the derivation shown in 6.2, the two topmost marked !-links are connected, whereas the two bottommost marked !-links are not connected.

6.4 Definition A *!-chain* χ inside a derivation Δ is a sequence of connected !-links. The bottommost !-link of χ is called its *tail* and the topmost !-link of χ is called its *head*.

Throughout this paper, I will visualize !-chains by giving the derivation and marking all !-links of the chain by $!^\bullet$. For example the lefthand-side derivation in Figure 7 shows a !-chain with tail $!^\bullet(b, ?a)$ and head $!^\bullet b$.

6.5 Definition The notion of *?-link* is defined in the same way as the one of !-link. The notion of *?-chain* is defined dually to !-chain, in particular, the *tail* of a ?-chain is its topmost ?-link and its *head* is its bottommost ?-link.

The righthand-side derivation in Figure 7 shows an example for a ?-chain with tail $?^\bullet a$ and head $?^\bullet(a, c)$.

6.6 Definition An *upper link* is any structure of the shape $[!R, ?T]$ that occurs as substructure of a structure S inside a derivation Δ. Dually, a *lower link* is any structure of the shape $(?T, !R)$ that occurs as substructure of a structure S inside a derivation Δ.

As !-links and ?-links, I will always mark upper links as $[!^\bullet R, ?^\bullet T]$ and lower links as $(?^\bullet T, !^\bullet R)$.

$$
\begin{array}{rc}
 & (!^\bullet b, !c) \\ \hline
\mathsf{b}\!\uparrow & (!^\bullet b, !c, b) \\ \hline
\mathsf{a}\!\downarrow & (!^\bullet(b, [a, \bar{a}]), !c, b) \\ \hline
\mathsf{w}\!\downarrow & (!^\bullet(b, [?a, a, \bar{a}]), !c, b) \\ \hline
\mathsf{s} & (!^\bullet[(b, [?a, a]), \bar{a}], !c, b) \\ \hline
\mathsf{b}\!\downarrow & (!^\bullet[(b, ?a), \bar{a}], !c, b) \\ \hline
\mathsf{p}\!\downarrow & ([!^\bullet(b, ?a), ?\bar{a}], !c, b) \\ \hline
\mathsf{s} & ([!^\bullet(b, ?a), (?\bar{a}, !c)], b) \\ \hline
\mathsf{p}\!\uparrow & ([!^\bullet(b, ?a), ?(\bar{a}, c)], b)
\end{array}
\qquad
\begin{array}{rc}
 & ![!c, (?^\bullet a, !c), 1] \\ \hline
\mathsf{p}\!\downarrow & [?!c, ![(?^\bullet a, !c), 1]] \\ \hline
\mathsf{w}\!\downarrow & [?!([?a, 1], !c), ![(?^\bullet a, !c), 1]] \\ \hline
\mathsf{s} & [?![(?a, !c), 1], ![(?^\bullet a, !c), 1]] \\ \hline
\mathsf{b}\!\downarrow & ?![(?^\bullet a, !c), 1] \\ \hline
\mathsf{a}\!\downarrow & ?![(?^\bullet a, !c), b, \bar{b}] \\ \hline
\mathsf{p}\!\downarrow & ?[![(?^\bullet a, !c), b], ?\bar{b}] \\ \hline
\mathsf{b}\!\uparrow & ?[(![(?^\bullet a, !c), b], [(?a, !c), b]), ?\bar{b}] \\ \hline
\mathsf{p}\!\uparrow & ?[(![?^\bullet(a, c), b], [(?a, !c), b]), ?\bar{b}]
\end{array}
$$

Figure 7: A !-chain and a ?-chain

6.7 Definition Let Δ be a derivation. The set $X(\Delta)$ of *chains* in Δ is defined inductively as follows:

(1) For every !-chain χ, in Δ, we have $\chi \in X(\Delta)$.

(2) For every ?-chain χ, in Δ, we have $\chi \in X(\Delta)$.

(3) If Δ contains two chains χ_1 and χ_2 and an upper link $[!^\bullet R, ?^\bullet T]$ such that $!^\bullet R$ is the head of χ_1 and $?^\bullet T$ is the tail of χ_2, then the union of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(4) If Δ contains two chains χ_1 and χ_2 and a lower link $(?^\bullet T, !^\bullet R)$ such that $?^\bullet T$ is the head of χ_1 and $!^\bullet R$ is the tail of χ_2, then the union of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

6.8 Definition Let Δ be a derivation. A chain $\chi \in X(\Delta)$ is called a *circle* if Δ contains an upper link $[!^\bullet R, ?^\bullet T]$ such that $!^\bullet R$ is the head of χ and $?^\bullet T$ is the tail of χ, or Δ contains a lower link $(?^\bullet T, !^\bullet R)$ such that $?^\bullet T$ is the head of χ and $!^\bullet R$ its tail.

In other words, a circle can be seen as a chain without head or tail. Figure 8 shows an example for a circle. Observe that for every circle χ there is a number $n = n(\chi) \geqslant 1$ such that χ consists of n !-chains, n ?-chains, n upper links and n lower links. I will call this $n(\chi)$ the *characteristic number* of χ. For the example in Figure 8, we have $n = 2$.

6.9 Definition A circle χ is called a *promotion circle* if every upper link of χ is redex of a $\mathsf{p}\!\downarrow$-rule (called *link promotion*) and every lower link of χ is contractum of a $\mathsf{p}\!\uparrow$-rule (called *link copromotion*).

The example in Figure 8 is not a promotion circle because the upper link $[!^\bullet R_1, ?^\bullet T_1]$ is not redex of a $\mathsf{p}\!\downarrow$-rule and the lower link $(!^\bullet R_1, ?^\bullet T_2)$ is not contractum of a $\mathsf{p}\!\uparrow$-rule. Figure 9 shows an example for a promotion circle. Observe that it is not necessarily the case that all upper links are above all lower links in the derivation.

$$
\begin{array}{rc}
 & (![R_2,T_2],!R_1,?T_1) \\
\mathsf{p}\!\downarrow & \overline{([!^\bullet R_2,?^\bullet T_2],!R_1,?T_1)} \\
\mathsf{s} & \overline{([!^\bullet R_2,(!R_1,?^\bullet T_2)],?T_1)} \\
\mathsf{s} & \overline{[(!^\bullet R_2,?T_1),(!R_1,?^\bullet T_2)]} \\
\mathsf{w}\!\downarrow & \overline{[?(!R_2,[!^\bullet R_1,?^\bullet T_1],?T_2),(!^\bullet R_2,?T_1),(!R_1,?^\bullet T_2)]} \\
\mathsf{s} & \overline{[?(!R_2,[(!^\bullet R_1,?T_2),?^\bullet T_1]),(!^\bullet R_2,?T_1),(!R_1,?^\bullet T_2)]} \\
\mathsf{s} & \overline{[?[(!R_2,?^\bullet T_1),(!^\bullet R_1,?T_2)],(!^\bullet R_2,?T_1),(!R_1,?^\bullet T_2)]} \\
\mathsf{b}\!\downarrow & \overline{?[(!^\bullet R_2,?^\bullet T_1),(!^\bullet R_1,?^\bullet T_2)]} \\
\mathsf{p}\!\uparrow & \overline{?[?(R_2,T_1),(!^\bullet R_1,?^\bullet T_2)]} \\
\mathsf{w}\!\uparrow & \overline{?[?(R_2,T_1),?T_2]}
\end{array}
$$

Figure 8: Example for a circle χ with $n(\chi) = 2$

$$
\begin{array}{rc}
 & !(![R_1,T_1],![R_2,T_2],![R_3,T_3]) \\
\mathsf{p}\!\downarrow & \overline{!([!^\bullet R_1,?^\bullet T_1],![R_2,T_2],![R_3,T_3])} \\
\mathsf{p}\!\downarrow & \overline{!([!^\bullet R_1,?^\bullet T_1],![R_2,T_2],[!^\bullet R_3,?^\bullet T_3])} \\
\mathsf{b}\!\uparrow & \overline{(!([!^\bullet R_1,?T_1],![R_2,T_2],[!R_3,?^\bullet T_3]),[!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{w}\!\uparrow & \overline{(!([!^\bullet R_1,?T_1],[!R_3,?^\bullet T_3]),[!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{s} & \overline{(![(!^\bullet R_1,[!R_3,?^\bullet T_3]),?T_1],[!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{s} & \overline{(![(!^\bullet R_1,?^\bullet T_3),!R_3,?T_1],[!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{p}\!\uparrow & \overline{(![?(R_1,T_3),!R_3,?T_1],[!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{w}\!\uparrow & \overline{([!R_1,?^\bullet T_1]![R_2,T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{p}\!\downarrow & \overline{([!R_1,?^\bullet T_1][!^\bullet R_2,?^\bullet T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{s} & \overline{([(!^\bullet R_2,[!R_1,?^\bullet T_1]),?^\bullet T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{s} & \overline{([(!^\bullet R_2,?^\bullet T_1),!R_1,?^\bullet T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{p}\!\uparrow & \overline{([?(R_2,T_1),!R_1,?^\bullet T_2],[!^\bullet R_3,?T_3])} \\
\mathsf{s} & \overline{[?(R_2,T_1),!R_1,(?^\bullet T_2,[!^\bullet R_3,?T_3])]} \\
\mathsf{s} & \overline{[?(R_2,T_1),!R_1,(!^\bullet R_3,?^\bullet T_2),?T_3]} \\
\mathsf{p}\!\uparrow & \overline{[?(R_2,T_1),!R_1,?(R_3,T_2),?T_3]}
\end{array}
$$

Figure 9: Example for a promotion circle χ with $n(\chi) = 3$

6.10 Definition Let χ be a circle inside a derivation Δ, and let all !-links and ?-links of χ be marked with $!^\bullet$ or $?^\bullet$, respectively. Then, χ is called *forked* if one of the following holds:

(i) There is an instance $\mathsf{b}\!\downarrow \dfrac{S[?T,T]}{S\{?T\}}$ inside Δ, such that both substructures $?T$ and T of the contractum contain at least one marked $!^\bullet$- or $?^\bullet$-substructure.

(ii) There is an instance $\mathsf{b}\!\uparrow \dfrac{S!R}{S\{(!R,R)\}}$ inside Δ, such that both substructures $?R$ and R of the redex contain at least one marked $!^\bullet$- or $?^\bullet$-substructure.

A circle is called *nonforked* if it is not forked.

Both examples for circles, that I have shown, are forked circles. I am not able to give an example for a nonforked circle for the simple reason that there is none. The proof of this statement is not trivial and will be the main purpose if this section.

6.11 Definition If a context can be generated by the syntax

$$S ::= \{\ \} \mid [\underbrace{R,\dots,R}_{\geqslant 0},S,\underbrace{R,\dots,R}_{\geqslant 0}] \mid (\underbrace{R,\dots,R}_{\geqslant 0},S,\underbrace{R,\dots,R}_{\geqslant 0}) \quad ,$$

i.e. the hole does not occur inside an !- or ?-structure, it is called a *flat context*.

For example the contexts $[a,b,(\bar{a},[c,d,\bar{b},\{\ \},a],?c)]$ and $([!(b,?a),\{\ \}],b)$ are flat, whereas $([!(\{\ \},?a),?(\bar{a},c)],b)$ is not flat.

6.12 Lemma *Let $S\{\ \}$ be a flat context and R and T be any structures. Then there is a derivation*

$$\begin{array}{c} S[R,T] \\ \Delta \Big\| \{\mathsf{s}\} \\ [S\{R\},T] \end{array} \quad .$$

Proof: By structural induction on $S\{\ \}$.

- $S = \{\ \}$. Trivial because $S[R,T] = [R,T] = [S\{R\},T]$.
- $S = [S', S''\{\ \}]$. Then by induction hypothesis we have $\begin{array}{c} [S', S''[R,T]] \\ \Delta \Big\| \{\mathsf{s}\} \\ [S', S''\{R\}, T] \end{array}$.
- $S = (S', S''\{\ \})$. Then let Δ be $\begin{array}{c} (S', S''[R,T]) \\ \Delta' \Big\| \{\mathsf{s}\} \\ \mathsf{s}\,\dfrac{(S',[S''\{R\},T])}{[(S',S''\{R\}),T]} \end{array}$, where Δ' exists by induction hypothesis.

□

6.13 Definition A circle χ is called *pure* if (i) for each !-chain and each ?-chain contained in χ, head and tail are equal, and (ii) all upper links occur in the same structure and all lower links occur in the same structure.

In other words, if a derivation $\begin{array}{c} P \\ \Delta \Big\| \mathsf{SELS} \\ Q \end{array}$ contains a pure circle then there are structures $R_1,\dots,R_n,T_1,\dots,T_n$ (for some $n \geqslant 1$) and two n-ary contexts $S\{\ \}\dots\{\ \}$ and $S'\{\ \}\dots\{\ \}$, such that Δ is of the shape

$$
\begin{array}{c}
P \\
\Delta_1 \Vert \mathsf{SELS} \\
S[!^\bullet R_1, ?^\bullet T_1]\,[!^\bullet R_2, ?^\bullet T_2] \dots [!^\bullet R_n, ?^\bullet T_n] \\
\Delta_2 \Vert \mathsf{SELS} \\
S'(!^\bullet R_2, ?^\bullet T_1)(!^\bullet R_3, ?^\bullet T_2) \dots (!^\bullet R_1, ?^\bullet T_n) \\
\Delta_3 \Vert \mathsf{SELS} \\
Q
\end{array}
\quad ,
$$

where no rule inside Δ_2 has redex or contractum inside a $!^\bullet$- or $?^\bullet$-structure. The two circles in Figures 8 and 9 are not pure. Although in both cases condition (i) is fulfilled, condition (ii) is not. Figure 10 shows an example for a pure circle.

$$
\begin{array}{ll}
 & [?(!R_2, [!R_1, ?T_1], ?T_2), (![R_2, T_2], !R_1, ?T_1)] \\
\mathsf{p}{\downarrow} & \overline{[?(!R_2, [!^\bullet R_1, ?^\bullet T_1], ?T_2), ([!^\bullet R_2, ?^\bullet T_2], !R_1, ?T_1)]} \\
\mathsf{s} & \overline{[?(!R_2, [!^\bullet R_1, ?^\bullet T_1], ?T_2), ([!^\bullet R_2, (!R_1, ?^\bullet T_2)], ?T_1)]} \\
\mathsf{s} & \overline{[?(!R_2, [!^\bullet R_1, ?^\bullet T_1], ?T_2), (!^\bullet R_2, ?T_1), (!R_1, ?^\bullet T_2)]} \\
\mathsf{s} & \overline{[?(!R_2, [(!^\bullet R_1, ?T_2), ?^\bullet T_1]), (!^\bullet R_2, ?T_1), (!R_1, ?^\bullet T_2)]} \\
\mathsf{s} & \overline{[?[(!R_2, ?^\bullet T_1), (!^\bullet R_1, ?T_2)], (!^\bullet R_2, ?T_1), (!R_1, ?^\bullet T_2)]} \\
\mathsf{b}{\downarrow} & \overline{\mathsf{p}{\uparrow}\dfrac{?[(!^\bullet R_2, ?^\bullet T_1), (!^\bullet R_1, ?^\bullet T_2)]}{?[?(R_2, T_1), (!R_1, ?T_2)]}}
\end{array}
$$

Figure 10: Example for a pure circle χ with $n(\chi) = 2$

6.14 Proposition *If there is a derivation* $\begin{array}{c} P \\ \Delta \Vert \mathsf{SELS} \\ Q \end{array}$ *that contains a nonforked promotion circle, then there is a derivation* $\begin{array}{c} \tilde{P} \\ \tilde{\Delta} \Vert \{\mathsf{a}{\downarrow}, \mathsf{a}{\uparrow}, \mathsf{s}\} \\ \tilde{Q} \end{array}$ *that contains a pure circle.*

Proof: Let χ be the nonforked promotion circle inside Δ and let all !-links and ?-links of χ be marked with $!^\bullet$ and $?^\bullet$, respectively (see Figure 11, first derivation). Further, let all link promotion rules and all link copromotion rules be marked as $\mathsf{p}{\downarrow}^\bullet$ and $\mathsf{p}{\uparrow}^\bullet$ (see Figure 11, second derivation). Now I will stepwise construct $\tilde{\Delta}$ from Δ by adding some further markings and by permuting, adding and removing rules until the circle is pure. Observe that the transformations will not destroy the circle but might change premise and conclusion of the derivation.

(1) Let n be the characteristic number of χ. For each of the n marked instances of $\mathsf{p}{\downarrow}^\bullet \dfrac{S\{![R_i, T_i]\}}{S[!^\bullet R_i, ?^\bullet T_i]}$ proceed as follows: Mark the contractum $![R_i, T_i]$ as $!^\bullet[R_i, T_i]$ and continue the marking for all !-links of the (maximal) !-chain that has $!^\bullet[R_i, T_i]$ as tail. There is always a unique choice how to continue the marking (see Definition 6.3), except for one case: If the marking reaches a $\mathsf{b}{\downarrow} \dfrac{S[?U, U]}{S\{?U\}}$ and the last marked $!^\bullet$-structure is inside the redex $?U$. Then there are two possibilities: either continue inside $?U$ (case (4.ii) of Definition 6.3) or continue inside U (case (4.iii) of Definition 6.3).

$$
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_1\{![R_1,T_1]\}}{S_1[!^\bullet R_1,?^\bullet T_1]}\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_2\{![R_2,T_2]\}}{S_2[!^\bullet R_2,?^\bullet T_2]}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_1(!^\bullet R'_2,?^\bullet T'_1)}{S'_1\{?(R'_2,T'_1)\}}\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_3\{![R_3,T_3]\}}{S_3[!^\bullet R_3,?^\bullet T_3]}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_3(!^\bullet R'_1,?^\bullet T'_3)}{S'_3\{?(R'_1,T'_3)\}}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_2(!^\bullet R'_3,?^\bullet T'_2)}{S'_2\{?(R'_3,T'_2)\}}\\
\|\\
Q
\end{array}
\quad\leadsto\quad
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_1\{![R_1,T_1]\}}{S_1[!^\bullet R_1,?^\bullet T_1]}\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_2\{![R_2,T_2]\}}{S_2[!^\bullet R_2,?^\bullet T_2]}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_1(!^\bullet R'_2,?^\bullet T'_1)}{S'_1\{?(R'_2,T'_1)\}}\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_3\{![R_3,T_3]\}}{S_3[!^\bullet R_3,?^\bullet T_3]}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_3(!^\bullet R'_1,?^\bullet T'_3)}{S'_3\{?(R'_1,T'_3)\}}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_2(!^\bullet R'_3,?^\bullet T'_2)}{S'_2\{?(R'_3,T'_2)\}}\\
\|\\
Q
\end{array}
\quad\leadsto\quad
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_1\{!^\bullet[R_1,T_1]\}}{S_1[!^\bullet R_1,?^\bullet T_1]}\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_2\{!^\bullet[R_2,T_2]\}}{S_2[!^\bullet R_2,?^\bullet T_2]}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_1(!^\bullet R'_2,?^\bullet T'_1)}{S'_1\{?^\bullet(R'_2,T'_1)\}}\\
\|\\
\mathsf{p}{\downarrow}^\bullet\,\dfrac{S_3\{!^\bullet[R_3,T_3]\}}{S_3[!^\bullet R_3,?^\bullet T_3]}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_3(!^\bullet R'_1,?^\bullet T'_3)}{S'_3\{?^\bullet(R'_1,T'_3)\}}\\
\|\\
\mathsf{p}{\uparrow}^\bullet\,\dfrac{S'_2(!^\bullet R'_3,?^\bullet T'_2)}{S'_2\{?^\bullet(R'_3,T'_2)\}}\\
\|\\
Q
\end{array}
$$

Figure 11: Example (with $n(\chi)=3$) for the marking inside Δ

Choose that side that already contains a marked $!^\bullet$- or $?^\bullet$-structure. Since the circle χ is nonforked, it cannot happen that both sides already contain a marked $!^\bullet$- or $?^\bullet$-structure. If there is no marked $!^\bullet$- or $?^\bullet$-structure inside the contractum $[?T,T]$ of the $\mathsf{b}{\downarrow}$, then choose either one.

Proceed dually for all marked $\mathsf{p}{\uparrow}^\bullet\,\dfrac{S(!^\bullet R'_i,?^\bullet T'_i)}{S\{?(R'_i,T'_i)\}}$, i.e. mark the redex $?(R'_i,T'_i)$ as $?^\bullet(R'_i,T'_i)$ and mark also all links of the ?-chain that has $?^\bullet(R'_i,T'_i)$ as tail (see Figure 11, third derivation).

(2) Now consider all !-substructures and all ?-substructures that occur somewhere in the derivation Δ. They can be divided into three groups:

(a) those which are marked with $!^\bullet$ or $?^\bullet$,

(b) those which are substructure of a marked $!^\bullet$- or $?^\bullet$-structure, and

(c) all others.

In this step replace all substructures $!R$ and $?T$ that fall in group (c) by R and T respectively, i.e. remove the exponential. This rather drastic step will, of course, yield a nonvalid derivation because correct rule applications might become incorrect. Observe that all instance of $\mathsf{a}{\downarrow}$, $\mathsf{a}{\uparrow}$ and s inside Δ do not suffer from this step, i.e. they remain valid. Let us now inspect more closely what could happen to the instances of $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$ and $\mathsf{b}{\uparrow}$.

- Consider any instance of $\mathsf{p}\!\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}$ in Δ. Then the following cases exhaust all possibilities.

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\bullet}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\bullet}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{p}\!\downarrow$ remain unchanged and the rule remains valid.

  (ii) The $\mathsf{p}\!\downarrow$ is marked as $\mathsf{p}\!\downarrow^{\bullet} \dfrac{S\{!^{\bullet}[R,T]\}}{S[!^{\bullet}R,?^{\bullet}T]}$. Then it also remains unchanged.

  (iii) The $\mathsf{p}\!\downarrow$ is marked as $\mathsf{p}\!\downarrow \dfrac{S\{!^{\bullet}[R,T]\}}{S[!^{\bullet}R,?T]}$. Then all exponentials inside $?T$ are removed, and we obtain an instance of $\hat{\mathsf{p}}\!\downarrow \dfrac{S\{!^{\bullet}[R,T]\}}{S[!^{\bullet}R,T']}$.

  (iv) The $\mathsf{p}\!\downarrow$ is not marked and does not occur inside a marked structure. Then it becomes $\mathsf{p}\!\downarrow' \dfrac{S[R',T']}{S[R',T']}$.

  There are no other cases because there are no other markings possible. Observe that the rule $\mathsf{p}\!\downarrow'$ in case (iv) is vacuous and can therefore be removed in the whole derivation. Hence, it only remains to remove all instances of the rule $\hat{\mathsf{p}}\!\downarrow$ (case (iii)). This will be done in step (5).

- The rule $\mathsf{p}\!\uparrow \dfrac{S(!R,?T)}{S\{?(R,T)\}}$ is dual to the rule $\mathsf{p}\!\downarrow$. Hence the only problem lies in the new rule $\hat{\mathsf{p}}\!\uparrow \dfrac{S(R',?^{\bullet}T)}{S\{?^{\bullet}(R,T)\}}$, which will be removed in step (5).

- For the rule $\mathsf{w}\!\downarrow \dfrac{S\bot}{S\{?T\}}$ only two cases are possible.

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\bullet}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\bullet}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{w}\!\downarrow$ remain unchanged and the rule remains valid.

  (ii) The rule becomes $\hat{\mathsf{w}}\!\downarrow \dfrac{S\bot}{S\{T'\}}$, where T' is obtained from T by removing the exponentials.

  Observe that the marking $\mathsf{w}\!\downarrow \dfrac{S\bot}{S\{?^{\bullet}T\}}$ is not possible.

- For the rule $\mathsf{w}\!\uparrow \dfrac{S\{!R\}}{S1}$ the situation is dual and we obtain $\hat{\mathsf{w}}\!\uparrow \dfrac{S\{R'\}}{S1}$. The two rules $\hat{\mathsf{w}}\!\downarrow$ and $\hat{\mathsf{w}}\!\uparrow$ will be removed in step (4).

- For $\mathsf{b}\!\downarrow \dfrac{S[?T,T]}{S\{?T\}}$ the situation is more complicated.

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\bullet}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\bullet}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{b}\!\downarrow$ remain unchanged and the rule remains valid.

  (ii) The rule is marked as $\mathsf{b}\!\downarrow \dfrac{S[?^{\bullet}T,T]}{S\{?^{\bullet}T\}}$. Then it becomes $\mathsf{b}\!\downarrow' \dfrac{S[?^{\bullet}T,T']}{S\{?^{\bullet}T\}}$, where T' is obtained from T by removing the exponentials.

  (iii) Neither redex nor contractum of the rule contain any marked $!^{\bullet}$- or $?^{\bullet}$-structure, nor are they contained in a marked structure. Then the rule becomes $\mathsf{b}\!\downarrow'' \dfrac{S[T',T']}{S\{T'\}}$, where T' is obtained form T by removing the exponentials.

(iv) There are marked $!^\bullet$- or $?^\bullet$-structures inside the structure T in the redex. Then all those markings reoccur in one of the two substructures T in the contractum whereas the other T does not contain any marking (because the circle χ is nonforked). Hence the rule becomes $\mathsf{b}\!\downarrow''' \dfrac{S[T'',T']}{S\{T''\}}$, where in T' all exponentials are removed and in T'' some exponentials are removed and some remain.

Observe that all instances of $\mathsf{b}\!\downarrow'$, $\mathsf{b}\!\downarrow''$ and $\mathsf{b}\!\downarrow'''$ are instances of $\hat{\mathsf{b}}\!\uparrow \dfrac{S[T,T']}{S\{T\}}$, where $S\{\ \}$ is a flat context.

- Dually, for $\mathsf{b}\!\uparrow \dfrac{S\{!R\}}{S(!R,R)}$, we obtain $\hat{\mathsf{b}}\!\downarrow \dfrac{S\{R\}}{S(R,R')}$, where $S\{\ \}$ is a flat context. The new instances of $\hat{\mathsf{b}}\!\uparrow$ and $\hat{\mathsf{b}}\!\downarrow$ will be removed in the next step.

Let me summarize what is achieved after this step: The original derivation $\begin{array}{c}P\\ \Delta\big\Vert\mathsf{SELS}\\ Q\end{array}$ has been transformed into $\begin{array}{c}P\\ \hat{\Delta}\big\Vert\mathsf{SELS}\cup\{\mathsf{p}\!\downarrow^\bullet,\mathsf{p}\!\uparrow^\bullet,\hat{\mathsf{p}}\!\downarrow,\hat{\mathsf{p}}\!\uparrow,\hat{\mathsf{w}}\!\downarrow,\hat{\mathsf{w}}\!\uparrow,\hat{\mathsf{b}}\!\uparrow,\hat{\mathsf{b}}\!\downarrow\}\\ Q\end{array}$, where the circle together with the extentions of its chains is marked. In the following steps I will remove all rules (including all $\hat{\rho}$) that prevent the circle from being pure.

(3) First I will remove all instances of $\hat{\mathsf{b}}\!\uparrow$ and $\hat{\mathsf{b}}\!\downarrow$. Consider the bottommost occurance of $\hat{\mathsf{b}}\!\uparrow \dfrac{S[T,T']}{S\{T\}}$ inside $\hat{\Delta}$. Replace

$$\hat{\Delta} \quad = \quad \begin{array}{c}P\\ \Delta_1\big\Vert\\ \hat{\mathsf{b}}\!\uparrow \dfrac{S[T,T']}{S\{T\}}\\ \Delta_2\big\Vert\\ Q\end{array} \qquad \text{by} \qquad \begin{array}{c}P\\ \Delta_1\big\Vert\\ S[T,T']\\ \Delta_3\big\Vert\{\mathsf{s}\}\\ [S\{T\},T']\\ \Delta_2\big\Vert\\ [Q,T']\end{array} \quad ,$$

where Δ_2 does not contain any $\hat{\mathsf{b}}\!\uparrow$ and Δ_3 exists by Lemma 6.12. Repeat this until there are no more $\hat{\mathsf{b}}\!\uparrow$ in the derivation. Then proceed dually to remove all $\hat{\mathsf{b}}\!\downarrow$, i.e. start with the topmost $\hat{\mathsf{b}}\!\uparrow$. This gives us a derivation $\begin{array}{c}P'\\ \hat{\Delta}'\big\Vert\mathsf{SELS}\cup\{\mathsf{p}\!\downarrow^\bullet,\mathsf{p}\!\uparrow^\bullet,\hat{\mathsf{p}}\!\downarrow,\hat{\mathsf{p}}\!\uparrow,\hat{\mathsf{w}}\!\downarrow,\hat{\mathsf{w}}\!\uparrow\}\\ Q'\end{array}$. Observe that premise and conclusion have changed now.

(4) In this step I will remove all instances of $\hat{\mathsf{w}}\!\downarrow$ and $\hat{\mathsf{w}}\!\uparrow$. For this observe that the proof of Lemma 5.4 would also work for $\hat{\mathsf{w}}\!\downarrow$. Further observe that it can never happen that the contractum $\bot$ of $\hat{\mathsf{w}}\!\downarrow \dfrac{S\bot}{S\{T\}}$ is inside an active structure of the redex of $\mathsf{p}\!\uparrow$, $\hat{\mathsf{p}}\!\uparrow$, $\mathsf{b}\!\downarrow$, $\mathsf{b}\!\uparrow$ or $\mathsf{w}\!\downarrow$ because then the contractum T would be inside a marked $!^\bullet$- or $?^\bullet$-structure, which is not possible by the construction of $\hat{\mathsf{w}}\!\downarrow$ in (2). Hence, the rule $\hat{\mathsf{w}}\!\downarrow$ permutes up over all other rules in the derivation $\hat{\Delta}'$. Dually, $\hat{\mathsf{w}}\!\uparrow$

permutes down under all other rules in $\hat{\Delta}'$. This means that $\hat{\Delta}'$ can easily be decomposed into

$$\begin{array}{c} P' \\ \Delta'_1 \Vert \{\hat{\mathsf{w}}\downarrow\} \\ P'' \\ \hat{\Delta}'' \Vert \mathsf{SELS} \cup \{\mathsf{p}\downarrow^\bullet, \mathsf{p}\uparrow^\bullet, \hat{\mathsf{p}}\downarrow, \hat{\mathsf{p}}\uparrow\} \\ Q'' \\ \Delta'_2 \Vert \{\hat{\mathsf{w}}\uparrow\} \\ Q' \end{array}$$

by permuting stepwise all $\hat{\mathsf{w}}\downarrow$ up and all $\hat{\mathsf{w}}\uparrow$ down. Let us now consider only $\begin{array}{c} P'' \\ \hat{\Delta}'' \Vert \\ Q'' \end{array}$ in which the circle χ is still untouched.

(5) Inside $\hat{\Delta}''$ mark all rules ρ whose redex is inside a marked $!^\bullet$-structure as $\rho^\vartriangle$. Additionally, mark all instances of $\hat{\mathsf{p}}\downarrow$ as $\hat{\mathsf{p}}\downarrow^\vartriangle$. Dually, mark all rules $\hat{\mathsf{p}}\uparrow$ as well as all rules ρ whose contractum is inside a marked $?^\bullet$-structure as $\rho^\triangledown$. Now mark all remaining, i.e. not yet marked, rules ρ as ρ°. This means, we now have a derivation $\begin{array}{c} P'' \\ \hat{\Delta}'' \Vert \{\mathsf{p}\downarrow^\bullet, \mathsf{p}\uparrow^\bullet, \rho^\vartriangle, \rho^\triangledown, \rho^\circ\} \\ Q'' \end{array}$, which will in this step be decomposed into

$$\begin{array}{c} P'' \\ \hat{\Delta}''_1 \Vert \{\rho^\vartriangle\} \\ P''' \\ \hat{\Delta}''_2 \Vert \{\mathsf{p}\downarrow^\bullet\} \\ \tilde{P} \\ \tilde{\Delta} \Vert \{\rho^\circ\} \\ \tilde{Q} \\ \hat{\Delta}''_3 \Vert \{\mathsf{p}\uparrow^\bullet\} \\ Q''' \\ \hat{\Delta}''_4 \Vert \{\rho^\triangledown\} \\ Q'' \end{array}$$

only by permutation of rules. In order to obtain this decomposition, I only need to show that

(a) all rules marked as $\rho^\vartriangle$ permute up over all other rules,

(b) all rules marked as $\rho^\triangledown$ permute down under all other rules,

(c) all rules $\mathsf{p}\downarrow^\bullet$ permute up over all rules marked as ρ° or $\mathsf{p}\uparrow^\bullet$, and

(d) all rules $\mathsf{p}\uparrow^\bullet$ permute down under all rules marked as ρ° or $\mathsf{p}\downarrow^\bullet$.

I will apply the scheme of 5.3 to show the four statements.

(a) Consider $\pi \dfrac{Q}{\rho^\vartriangle \dfrac{S\{W\}}{S\{Z\}}}$, where π is not marked $\pi^\vartriangle$ and not trivial. Then the cases are:

(1) The redex of π is inside the context $S\{\ \}$ of $\rho^\vartriangle$. Trivial.

(2) The contractum W of $\rho^\vartriangle$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum W of $\rho^\vartriangle$. Trivial.

(4) The redex of π is inside an active structure of the contractum W of $\rho^\vartriangle$. Not possible because

(i) if the redex of $\rho^\triangle$ is inside a $!^\bullet$-structure, then the contractum of $\rho^\triangle$ is also inside a $!^\bullet$-structure, and hence, the redex of π is inside a $!^\bullet$-structure, and therefore π is $\pi^\triangle$;

(ii) if $\rho^\triangle = \hat{\mathsf{p}}\downarrow^\triangle$, then the redex of π is inside a $!^\bullet$-structure, and therefore π is $\pi^\triangle$.

(5) The contractum W of $\rho^\triangle$ is inside an active structure of the redex of π but not inside a passive one. There are the following subcases:

(i) The redex of $\rho^\triangle$ is inside a $!^\bullet$-structure. Not possible because then the contractum of $\rho^\triangle$ is also inside a $!^\bullet$-structure. Since it is also inside an active structure of the redex of π, we have that either this active structure is a $!^\bullet$-structure and therefore $\pi = \hat{\mathsf{p}}\downarrow^\triangle$, or the whole redex of π is inside a $!^\bullet$-structure and therefore π must be marked as $\pi^\triangle$.

(ii) $\rho^\triangle = \hat{\mathsf{p}}\downarrow^\triangle$ and $\pi = \hat{\mathsf{p}}\downarrow$. Not possible because then π is marked as $\pi^\triangle$.

(iii) $\rho^\triangle = \hat{\mathsf{p}}\downarrow^\triangle$ and $\pi = \mathsf{p}\downarrow$. Then $\pi = \mathsf{p}\downarrow^\bullet$ because there are no other $\mathsf{p}\downarrow$ that have a marked $!^\bullet$-structure in the redex. Hence, the situation

$$\hat{\mathsf{p}}\downarrow^\triangle \frac{\mathsf{p}\downarrow^\bullet \dfrac{S'\{!^\bullet[R,T_1,T_2]\}}{S'[!^\bullet[R,T_1],?^\bullet T_2]}}{S'[!^\bullet R, T_1', ?^\bullet T_2]} \quad \text{can be replaced by} \quad \mathsf{p}\downarrow^\bullet \frac{\hat{\mathsf{p}}\downarrow^\triangle \dfrac{S'\{!^\bullet[R,T_1,T_2]\}}{S'[!^\bullet[R,T_2],T_1']}}{S'[!^\bullet R, T_1', ?^\bullet T_2]} \quad .$$

(6) The contractum W of ρ° and the redex of π overlap. Not possible.

(b) Dual to (a).

(c) Consider $\mathsf{p}\downarrow^\bullet \dfrac{\pi \dfrac{Q}{S\{!^\bullet[R,T]\}}}{S[!^\bullet R, ?^\bullet T]}$, where $\pi \in \{\rho^\circ, \mathsf{p}\uparrow^\bullet\}$ is not trivial.

(1) The redex of π is inside the context $S\{\ \}$ of $\mathsf{p}\downarrow^\bullet$. Trivial.

(2) The contractum of $\mathsf{p}\downarrow^\bullet$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum of $\mathsf{p}\downarrow^\bullet$. Trivial.

(4) The redex of π is inside an active structure of the contractum of $\mathsf{p}\downarrow^\bullet$. Not possible because then the redex of π is inside a $!^\bullet$-structure, and therefore π is $\pi^\triangle$.

(5) The contractum $!^\bullet[R,T]$ of $\mathsf{p}\downarrow^\bullet$ is inside an active structure of the redex of π but not inside a passive one. Not possible because then π were a $\pi^\triangle$-rule.

(6) The contractum of $\mathsf{p}\downarrow^\bullet$ and the redex of π overlap. Not possible.

(d) Dual to (c).

Now it only remains to show that the subderivation $\begin{array}{c}\tilde{P}\\ \tilde{\Delta}\| \{\rho^\circ\}\\ \tilde{Q}\end{array}$ obtained in the last step has indeed the desired properties (i.e. contains a pure circle and consists only of the rules $\mathsf{a}\downarrow, \mathsf{a}\uparrow$ and s). Observe that all rules $\rho \in \{\mathsf{p}\downarrow, \mathsf{p}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow, \mathsf{b}\downarrow, \mathsf{b}\uparrow\}$ in Δ have either

- been transformed into $\hat{\rho}$ in Step (2) and then been removed in the Steps (3), (4) and (5),
- or they remained unchanged in step (2) (because they occured inside a marked $!^\bullet$- or $?^\bullet$-structure) and have then been marked as $\rho^\triangle$ or $\rho^\triangledown$ and removed in step (5).

This means that only the rules $\mathsf{a}\downarrow, \mathsf{a}\uparrow$ and s are left inside $\tilde{\Delta}$. Now consider the premise $\tilde{P}$ of $\tilde{\Delta}$. Since it is also the conclusion of $\hat{\Delta}_2''$ which consists only of $\mathsf{p}\downarrow^\bullet$, it is of the shape

$$S[!^\bullet R_1, ?^\bullet T_1][!^\bullet R_2, ?^\bullet T_2] \dots [!^\bullet R_n, ?^\bullet T_n]$$

for some structures $R_1,\dots,R_n,T_1,\dots,T_n$ and some n-ary context $S\{\ \}\{\ \}\dots\{\ \}$. Similarly, we have that

$$\tilde{Q} = S'(!^\bullet R'_1, ?^\bullet T'_1)(!^\bullet R'_2, ?^\bullet T'_2)\dots(!^\bullet R'_n, ?^\bullet T'_n)$$

for some structures $R'_1,\dots,R'_n,T'_1,\dots,T'_n$ and some n-ary context $S'\{\ \}\{\ \}\dots\{\ \}$. Since no transformation in the steps (2)-(5) destroyed the circle, it must still be present in $\tilde{\Delta}$. Since $\tilde{\Delta}$ contains no rule that operates inside a $!^\bullet$- or $?^\bullet$-structure, we have that $R'_1 = R_2$, $R'_2 = R_3$, ..., $R'_n = R_1$ and $T'_1 = T_1$, $T'_2 = T_2$, ..., $T'_n = T_n$. This means that $\tilde{\Delta}$ does indeed contain a pure circle. □

6.15 Definition Let S be a structure and R and T be substructures of S. Then the structures R and T are in *par-relation* in S if there are contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$ such that $S = S'[S''\{R\}, S'''\{T\}]$. Similarly, R and T are in *times-relation* in S if $S = S'(S''\{R\}, S'''\{T\})$ for some contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$.

6.16 Lemma *If there is a derivation* $\begin{array}{c}P\\ \Delta \Vert \{\mathsf{a}\downarrow,\mathsf{a}\uparrow,\mathsf{s}\}\\ Q\end{array}$ *that contains a pure circle* χ*, then there is a derivation* $\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \dots [!R_n, ?T_n])\\ \tilde{\Delta} \Vert \{\mathsf{s}\}\\ [(!R_2, ?T_1), [!R_3, ?T_2], \dots [!R_1, ?T_n]]\end{array}$ *for any structures* $R_1,\dots,R_n,T_1,\dots,T_n$*, where* n *is the characteristic number of* χ*.*

Proof: By Lemma 5.6 and Lemma 5.7, the derivation Δ can be decomposed into $\begin{array}{c}P\\ \Delta_1 \Vert \{\mathsf{a}\downarrow\}\\ P'\\ \Delta_2 \Vert \{\mathsf{s}\}\\ Q'\\ \Delta_3 \Vert \{\mathsf{a}\uparrow\}\\ Q\end{array}$. This transformation does not destroy the circle. Hence, the pure circle is contained in Δ_2. In other words, Δ_2 has a subderivation $\begin{array}{c}S[!^\bullet R_1, ?^\bullet T_1][!^\bullet R_2, ?^\bullet T_2]\dots[!^\bullet R_n, ?^\bullet T_n]\\ \Delta' \Vert \{\mathsf{s}\}\\ S'(!^\bullet R_2, ?^\bullet T_1)(!^\bullet R_3, ?^\bullet T_2)\dots(!^\bullet R_1, ?^\bullet T_n)\end{array}$ for some structures $R_1,\dots,R_n,T_1,\dots,T_n$ and two n-ary contexts $S\{\ \}\dots\{\ \}$ and $S'\{\ \}\dots\{\ \}$. In the premise of Δ', for every $i = 1,\dots,n$, the substructures $!^\bullet R_i$ and $?^\bullet T_i$ are in par-relation. No rule in systen SELS is able to transform a par-relation into a times relation while going down in a derivation. Hence, for every $i = 1,\dots,n$, the substructures $!^\bullet R_i$ and $?^\bullet T_i$ are also in par-relation in the conclusion of Δ'. This means that the context $S'\{\ \}\dots\{\ \} = S'_0[S'_1\{\ \},\dots,S'_n\{\ \}]$ for some contexts $S'_0\{\ \}, S'_1\{\ \},\dots,S'_n\{\ \}$. Dually, we have that $S\{\ \}\dots\{\ \} = S_0(S_1\{\ \},\dots,S_n\{\ \})$ for some contexts $S_0\{\ \}, S_1\{\ \},\dots,S_n\{\ \}$. Hence, the derivation Δ' has the shape

$$\begin{array}{c}S_0(S_1[!^\bullet R_1, ?^\bullet T_1], S_2[!^\bullet R_2, ?^\bullet T_2],\dots,S_n[!^\bullet R_n, ?^\bullet T_n])\\ \Delta' \Vert \{\mathsf{s}\}\\ S'_0[S'_1(!^\bullet R_2, ?^\bullet T_1), S'_2(!^\bullet R_3, ?^\bullet T_2),\dots,S'_n(!^\bullet R_1, ?^\bullet T_n)]\end{array}.$$

Observe that the two contexts $S_0(S_1\{\ \},\dots,S_n\{\ \})$ and $S'_0[S'_1\{\ \},\dots,S'_n\{\ \}]$ must contain the same atoms because Δ' contains no rules that could create or destroy any atoms. Hence, the derivation Δ' remains valid if those atoms are removed from the derivation, which gives us the derivation

$$\begin{array}{c}([!^\bullet R_1, ?^\bullet T_1], [!^\bullet R_2, ?^\bullet T_2], \dots [!^\bullet R_n, ?^\bullet T_n])\\ \tilde{\Delta} \Vert \{\mathsf{s}\}\\ [(!^\bullet R_2, ?^\bullet T_1), [!^\bullet R_3, ?^\bullet T_2], \dots [!^\bullet R_1, ?^\bullet T_n]]\end{array}.$$

The derivation also remains valid if the structures $!^{\bullet}R_1, \ldots, !^{\bullet}R_n, ?^{\bullet}T_1, \ldots, ?^{\bullet}T_n$ are replaced by any other strurctures $!R'_1, \ldots, !R'_n, ?T'_1, \ldots, ?T'_n$ inside the whole derivation because they can never be touched by an s-rule, i.e. they occur only inside passive structures. $\square$

6.17 Lemma *Let $n \geqslant 1$ and $R_1, \ldots, R_n, T_1, \ldots, T_n$ be any structures. Then there is no derivation* $\begin{array}{c}([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n])\\ \Delta\|\{\mathsf{s}\}\\ [(!R_2,?T_1),(!R_3,?T_2),\ldots(!R_1,?T_n)]\end{array}$.

Proof: By induction on n.

- Base case: Let $n = 1$. Then it is easy to see that there is no derivation $\begin{array}{c}[!R_1,?T_1]\\ \Delta\|\{\mathsf{s}\}\\ (!R_1,?T_1)\end{array}$ because a times-relation can never become a par-relation while going up in a derivation.
- Inductive case: Suppose there is a derivation $\begin{array}{c}([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n])\\ \Delta\|\{\mathsf{s}\}\\ [(!R_2,?T_1),(!R_3,?T_2),\ldots(!R_1,?T_n)]\end{array}$. Now consider the bottommost instance of $\mathsf{s}\,\dfrac{S([\tilde{R},\tilde{T}],\tilde{U})}{S[(\tilde{R},\tilde{U}),\tilde{T}]}$ in Δ. Without loss of generality we can assume that $\tilde{R} = !R_2$ and $\tilde{U} = ?T_1$. There are $n-1$ possibilities how $\tilde{T}$ can be matched:

$$\begin{array}{rcl}\tilde{T} &=& (!R_3,?T_2) \quad ,\\ \tilde{T} &=& [(!R_3,?T_2),(!R_4,?T_3)] \quad ,\\ &\vdots& \\ \tilde{T} &=& [(!R_3,?T_2),(!R_4,?T_3),\ldots,(!R_1,?T_n)] \quad .\end{array}$$

Hence, we get one of the following cases:

$$\begin{array}{c}([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n])\\ \Delta_2\|\{\mathsf{s}\}\\ \mathsf{s}\,\dfrac{[([!R_2,(!R_3,?T_2)],?T_1),(!R_4,?T_3),\ldots,(!R_1,?T_n)]}{[(!R_2,?T_1),(!R_3,?T_2),(!R_4,?T_3),\ldots(!R_1,?T_n)]}\end{array} \quad ,$$

$$\begin{array}{c}([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n])\\ \Delta_3\|\{\mathsf{s}\}\\ \mathsf{s}\,\dfrac{[([!R_2,(!R_3,?T_2),(!R_4,?T_3)],?T_1),\ldots,(!R_1,?T_n)]}{[(!R_2,?T_1),(!R_3,?T_2),(!R_4,?T_3),\ldots(!R_1,?T_n)]}\end{array} \quad ,$$

$$\vdots$$

$$\begin{array}{c}([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n])\\ \Delta_n\|\{\mathsf{s}\}\\ \mathsf{s}\,\dfrac{([!R_2,(!R_3,?T_2),(!R_4,?T_3),\ldots,(!R_1,?T_n)],?T_1),}{[(!R_2,?T_1),(!R_3,?T_2),(!R_4,?T_3),\ldots(!R_1,?T_n)]}\end{array} \quad .$$

This means that at least one of $\Delta_2, \Delta_3, \ldots, \Delta_n$ must exist. In all derivations $\Delta_2, \Delta_3, \ldots, \Delta_n$ the structures $!R_1, \ldots, !R_n, ?T_1, \ldots, ?T_n$ occur only inside passive structures of instances of s. Therefore, if we replace inside one derivation Δ_i any structure $!R_j$ or $?T_j$ by some other structure V, the derivation Δ_i must remain valid.

Now replace in Δ_2 the structure $!R_2$ everywhere by $\bot$ and the structure $?T_2$ by 1. Then we obtain

$$\begin{array}{c}([!R_1,?T_1],[\bot,1],[!R_3,?T_3],\dots[!R_n,?T_n])\\ \Delta'_1\Big\| \{s\}\\ [([\bot,(!R_3,1)],?T_1),(!R_4,?T_3),\dots,(!R_1,?T_n)]\end{array} = \begin{array}{c}([!R_1,?T_1],[!R_3,?T_3],\dots[!R_n,?T_n])\\ \Delta'_1\Big\| \{s\}\\ [(!R_3,?T_1),(!R_4,?T_3),\dots,(!R_1,?T_n)]\end{array},$$

which is impossible by induction hypothesis. Similarly, replace in Δ_3 the structures $!R_2$ and $!R_3$ by $\bot$ and the structures $?T_2$ and $?T_3$ by 1. Then we obtain

$$\begin{array}{c}([!R_1,?T_1],[\bot,1],[\bot,1],[!R_4,?T_4],\dots[!R_n,?T_n])\\ \Delta'_2\Big\| \{s\}\\ [([\bot,(\bot,1),(!R_4,1)],?T_1),(!R_5,?T_4),\dots,(!R_1,?T_n)]\end{array} = \begin{array}{c}([!R_1,?T_1],[!R_4,?T_4],\dots[!R_n,?T_n])\\ \Delta'_2\Big\| \{s\}\\ [(!R_4,?T_1),(!R_5,?T_4),\dots,(!R_1,?T_n)]\end{array},$$

which is again impossible by induction hypothesis. Similarly, we can for every $i = 2,\dots,n$ replace in Δ_i the structures $!R_2,\dots,!R_i$ by $\bot$ and the structures $?T_2,\dots,?T_i$ by 1, which yields a derivation

$$\begin{array}{c}([!R_1,?T_1],[!R_{i+1},?T_{i+1}],\dots[!R_n,?T_n])\\ \Delta'_i\Big\| \{s\}\\ [(!R_{i+1},?T_1),(!R_{i+2},?T_{i+1}),\dots,(!R_1,?T_n)]\end{array},$$

which cannot exist by induction hypothesis.

□

6.18 Theorem *There exists no derivation containing an unforked promotion circle.*

Proof: Suppose there is a derivation Δ containing an unforked promotion circle χ. By Proposition 6.14, Lemma 6.16 and Lemma 6.17, this is impossible. □

6.19 Corollary *There exists no derivation containing an unforked circle.*

Proof: Any (unforked) circle can easily be transformed into an (unforked) promotion circle by adding instances of $p\downarrow$ and $p\uparrow$. □

7 Decomposition of Derivations

This section I will use the results of the previous two sections in order to prove the decomposition theorem, which is as follows:

7.1 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ R \end{array}$ *there is a derivation* $\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\downarrow\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{a}\downarrow\} \\ T_3 \\ \Delta_4 \Vert \{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{a}\uparrow\} \\ R_2 \\ \Delta_2 \Vert \{\mathsf{w}\uparrow\} \\ R_1 \\ \Delta_6 \Vert \{\mathsf{b}\downarrow\} \\ R \end{array}$ *for some structures* T_1, T_2, T_3, R_1, R_2 *and* R_3.

Proof: The decomposition is done in three steps:

$$\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ R \end{array} \overset{1}{\leadsto} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Delta' \Vert \{\mathsf{a}\downarrow,\mathsf{a}\uparrow,\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow,\mathsf{w}\downarrow,\mathsf{w}\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\downarrow\} \\ R \end{array} \overset{2}{\leadsto} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\downarrow\} \\ T_2 \\ \Delta'' \Vert \{\mathsf{a}\downarrow,\mathsf{a}\uparrow,\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\downarrow\} \\ R \end{array} \overset{3}{\leadsto} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\downarrow\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{a}\downarrow\} \\ T_3 \\ \Delta_4 \Vert \{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{a}\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\downarrow\} \\ R \end{array} .$$

The first step is done by Proposition 7.7. For the second step, we can repeatedly apply Lemma 5.4, Lemma 5.5 and Lemma 5.9 (a) and (b). For the last step use Lemma 5.6, Lemma 5.7 and Lemma 5.9 (c) and (d). □

Before I complete the proof (i.e. prove Proposition 7.7), let me make some informal remarks about the theorem. The statement of Theorem 7.1 can be read in two different ways. The first observation is the decomposition of any derivation into five parts:

$$\text{non-core} - \text{interaction} - \text{core} - \text{interaction} - \text{non-core} \quad .$$

The second reading gives us a decomposition of every derivation into three parts: a first part, in which matter is created (rules $\mathsf{b}\uparrow$, $\mathsf{w}\downarrow$ and $\mathsf{a}\downarrow$), i.e. each rule increases the size of the structure, a second part (rules s, $\mathsf{p}\downarrow$, $\mathsf{p}\uparrow$), in which the size of the formula does not chance, i.e. the number of atoms in the structure is constant, and a third part (rules $\mathsf{a}\uparrow$, $\mathsf{w}\uparrow$ and $\mathsf{b}\downarrow$), in which matter is destroyed, i.e. each rule decreases the size of the structure. Theorem 7.1 endorses the fact that in the calculus of structures derivations are symmetric objects in the vertical perspective.

The remainder of this section is devoted to the proof of Proposition 7.7, which states that

$$
\begin{array}{c} T \\ \Big\| \mathsf{SELS} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}}
\begin{array}{c} T \\ \| \{\mathsf{b}\uparrow\} \\ T' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\uparrow\} \\ R \end{array}
\overset{\mathsf{b}\downarrow}{\underset{\mathsf{down}}{\leadsto}}
\begin{array}{c} T \\ \| \{\mathsf{b}\uparrow\} \\ T' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\downarrow\} \\ R' \\ \| \{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}}
\begin{array}{c} T \\ \| \{\mathsf{b}\uparrow\} \\ T'' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\uparrow\} \\ R' \\ \| \{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\downarrow}{\underset{\mathsf{down}}{\leadsto}}
\begin{array}{c} T \\ \| \{\mathsf{b}\uparrow\} \\ T'' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\downarrow\} \\ R'' \\ \| \{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}} \cdots \leadsto
\begin{array}{c} T \\ \| \{\mathsf{b}\uparrow\} \\ T''' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R''' \\ \| \{\mathsf{b}\downarrow\} \\ R \end{array}
$$

Figure 12: Permuting $\mathsf{b}\uparrow$ up and $\mathsf{b}\downarrow$ down

I. If there are no subderivations of the shape $\pi \dfrac{Q}{\mathsf{b}\uparrow \dfrac{U}{P}}$, where $\pi \neq \mathsf{b}\uparrow$, or of the shape $\mathsf{b}\downarrow \dfrac{Q}{\rho \dfrac{V}{P}}$, where $\rho \neq \mathsf{b}\downarrow$, then terminate.

II. Permute all instances of $\mathsf{b}\uparrow$ up by applying $\mathsf{b}\uparrow\mathsf{up}$.

III. Permute all instances of $\mathsf{b}\downarrow$ down by applying $\mathsf{b}\downarrow\mathsf{down}$.

IV. Go to step I.

Figure 13: The algorithm $\mathsf{b}\uparrow\downarrow\mathsf{split}$ for separating $\mathsf{b}\uparrow$ and $\mathsf{b}\downarrow$

for every derivation $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ R \end{array}$ there is a derivation $\begin{array}{c} T \\ \Delta_1 \| \{\mathsf{b}\uparrow\} \\ T' \\ \Delta' \| \mathsf{SELS}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R' \\ \Delta_2 \| \{\mathsf{b}\downarrow\} \\ R \end{array}$. The idea of the proof is to permute all instances of $\mathsf{b}\uparrow$ up and all instances of $\mathsf{b}\downarrow$ down, according to the scheme of 5.3. The problem that occurs is that while permuting up the rule $\mathsf{b}\uparrow$ over the rule $\mathsf{p}\downarrow$, it can happen that new instances of $\mathsf{b}\downarrow$ are introduced. Dually, while permuting $\mathsf{b}\downarrow$ down under $\mathsf{p}\uparrow$ new instances of $\mathsf{b}\uparrow$ are introduced. Figure 12 shows the working principle of the algorithm $\mathsf{b}\uparrow\downarrow\mathsf{split}$ (Figure 13) that is used to obtain the desired decomposition.

The task of the proof is now to show that the process of permuting $\mathsf{b}\uparrow$ up and $\mathsf{b}\downarrow$ down does terminate. I will show this by way of contradiction. Since the derivation to be considered is of finite length and all structures occuring in this derivation are of finite size, the assumption of a non-termination means that there must be a loop. This loop turns out to be a nonforked promotion circle as defined in the previous section. Since a nonforked circle cannot exist, the algorithm in Figure 13 must terminate. This argument will now be made formal.

The processes of permuting up all instances of $\mathsf{b}\uparrow$ in a given derivation $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ S \end{array}$ is realized by the the procedure $\mathsf{b}\uparrow\mathsf{up}$ shown in Figure 14. It is easy to see that if this terminates, the resulting derivation has the shape $\begin{array}{c} T \\ \| \{\mathsf{b}\!\uparrow\} \\ T' \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\!\uparrow\} \\ S \end{array}$. However, it is not obvious that this procedure does indeed terminate, because while permuting the rule b↑ up, it might happen that new instances of b↑ as well as new instances of b↓ are introduced.

In order to show that b↑up terminates, we need the following definition.

7.2 Definition Let $S\{\ \}$ be a context in normal form, i.e. all occurances of ?? are replaced by ? and all occurances of !! are replaced by !. For $S\{\ \}$, define the *?-depth* $d_?(S\{\ \})$ inductively as follows:

$$\begin{array}{rcl} d_?(\ \{\ \}\ ) &=& 0 \\ d_?(\ [R_1,\dots,R_{i-1},S\{\ \},R_{i+1},\dots,R_h]\ ) &=& d_?(\ S\{\ \}\ ) \\ d_?(\ (R_1,\dots,R_{i-1},S\{\ \},R_{i+1},\dots,R_h)\ ) &=& d_?(\ S\{\ \}\ ) \\ d_?(\ !S\{\ \}\ ) &=& d_?(\ S\{\ \}\ ) \\ d_?(\ ?S\{\ \}\ ) &=& 1 + d_?(\ S\{\ \}\ ) \end{array}$$

For a given instance of a rule $\rho\,\dfrac{S\{W\}}{S\{V\}}$, the ?-depth $d_?(\rho)$ of ρ is the ?-depth $d_?(S\{\ \})$ of the context. The notion of !-depth is defined dually.

7.3 Lemma *For an input derivation* $\begin{array}{c} V \\ \| \mathsf{SELS}\setminus\{\mathsf{b}\!\uparrow\} \\ \mathsf{b}\!\uparrow\,\dfrac{S\{!R\}}{S(!R,R)} \end{array}$ *, the* b↑up *procedure does terminate.*

Proof: The following four observation suffice to show the termination.

- While the b↑ is permuted up, its ?-depth never increases and after each step it is closer to the top of the derivation. (Except for case (5.ii) where the b↑ disappears.)
- The only case where another b↑ is introduced is (5.iii). Then the ?-depth of the new b↑ is either smaller than or equal to the ?-depth of the first b↑.
- Case (5.iii) can only occur if the b↑ meets a b↓ with smaller ?-depth.
- The only cases where new instances of b↓ can be introduced are (4) and (5.i). Whenever in case (4) a new b↓ is introduced (i.e. $\pi = \mathsf{b}\!\downarrow$), it has a ?-depth that is bigger or equal to the ?-depth of the b↑ that is permuted up. Whenever in case (5.i) a new b↓ is introduced it has the same ?-depth as the b↑ that is permuted up.

This means that whenever a new b↓ is introduced, it has never smaller ?-depth than any of the b↑ that are below in the derivation and introduced by case (5.iii). Hence, all instances of b↓ that cause an increase of the number of b↑ in the derivation must already be present in Δ. Since Δ is of finite length, there are only finitely many b↓ in Δ, say n. Each of them could cause a duplication of the number of b↑ in the derivation. This means that there are at most 2^n many b↑, that all reach the top eventually. □

7.4 Lemma *The* b↑up *algorithm terminates for any input derivation* $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ S \end{array}$.

Proof: Apply Lemma 7.3 to every instance of $\mathsf{b}\uparrow$ in Δ. $\square$

The dual procedure to $\mathsf{b}\uparrow\mathsf{up}$ is $\mathsf{b}\downarrow\mathsf{down}$ (see Figure 15) in which all occurances of $\mathsf{b}\downarrow$ are moved down in the derivation.

7.5 Lemma *The* $\mathsf{b}\downarrow\mathsf{down}$ *procedure terminates for every input derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ S \end{array}$ *and yields a derivation* $\begin{array}{c} T \\ \Delta' \Vert \mathsf{SELS} \setminus \{\mathsf{b}\downarrow\} \\ S' \\ \Delta'' \Vert \{\mathsf{b}\downarrow\} \\ S \end{array}$.

Proof: Dual to Lemma 7.4. $\square$

Consider the topmost occurrence of a subderivation $\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{Q}{S\{!R\}}}{S(!R,R)}$, where $\pi \neq \mathsf{b}{\uparrow}$. According to 5.3 there are the following cases (cases (3) and (6) are not possible):

(1) If the redex of π is inside $S\{\ \}$, or

(2) if the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside a passive structure of the redex of π, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S'\{!R\}}{S\{!R\}}}{S(!R,R)} \qquad \text{by} \qquad \pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!R\}}{S'(!R,R)}}{S(!R,R)} \quad .$$

(4) If the redex of π is inside the contractum $!R$ of $\mathsf{b}{\uparrow}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S\{!R'\}}{S\{!R\}}}{S(!R,R)} \qquad \text{by} \qquad \pi\,\dfrac{\pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S\{!R'\}}{S(!R',R')}}{S(!R',R)}}{S(!R,R)} \quad .$$

(5) If the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside an active structure of the redex of π but not inside a passive one, then there are three subcases:

(i) If $\pi = \mathsf{p}{\downarrow}$ and $S\{!R\} = S'[!R,?T]$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,R),?T]} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{![R,T]\}}{S'(![R,T],[R,T])}}{S'([!R,?T],[R,T])}}{S'[([!R,?T],R),T]}}{S'[(!R,R),?T,T]}}{S'[(!R,R),?T]} \quad .$$

(ii) If $\pi = \mathsf{w}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''(!R,R)\}} \quad .$$

(iii) If $\pi = \mathsf{b}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'[?S''(!R,R),S''\{!R\}]}}{S'[?S''(!R,R),S''(!R,R)]}}{S'\{?S''(!R,R)\}} \quad .$$

Repeat until all instances of $\mathsf{b}{\uparrow}$ are at the top of the derivation.

Figure 14: The $\mathsf{b}{\uparrow}\mathsf{up}$ procedure

Consider the bottommost occurrence of a sub-derivation $\begin{array}{rc} & S[?T,T] \\ \mathsf{b}{\downarrow} & \hline S\{?T\} \\ \rho & \hline P \end{array}$, where $\rho \neq \mathsf{b}{\downarrow}$, until all instances of $\mathsf{b}{\downarrow}$ are at the bottom of the derivation. The possible cases are:

(1) The contractum of ρ is inside $S\{\ \}$, or

(2) the redex $?T$ of $\mathsf{b}{\downarrow}$ is inside a passive structure of the contractum of ρ. Then replace

$$\begin{array}{rc} & S[?T,T] \\ \mathsf{b}{\downarrow} & \hline S\{?T\} \\ \rho & \hline S'\{?T\} \end{array} \quad \text{by} \quad \begin{array}{rc} & S[?T,T] \\ \rho & \hline S'[?T,T] \\ \mathsf{b}{\downarrow} & \hline S'\{?T\} \end{array} \quad .$$

(4) The contractum of ρ is inside the redex $?T$ of $\mathsf{b}{\downarrow}$ Then replace

$$\begin{array}{rc} & S[?T,T] \\ \mathsf{b}{\downarrow} & \hline S\{?T\} \\ \rho & \hline S\{?T'\} \end{array} \quad \text{by} \quad \begin{array}{rc} & S[?T,T] \\ \rho & \hline S[?T,T'] \\ \rho & \hline S[?T',T'] \\ \mathsf{b}{\downarrow} & \hline S\{?T'\} \end{array} \quad .$$

(5) The redex $?T$ of $\mathsf{b}{\downarrow}$ is inside an active structure of the contractum of ρ but not inside a passive one. Then there are three cases:

(i) If $\rho = \mathsf{p}{\uparrow}$ and $S\{?T\} = S'(!R, ?T)$, then replace

$$\begin{array}{rc} & S'(!R,[?T,T]) \\ \mathsf{b}{\downarrow} & \hline S'(!R,?T) \\ \mathsf{p}{\uparrow} & \hline S'\{?(R,T)\} \end{array} \quad \text{by} \quad \begin{array}{rc} & S'(!R,[?T,T]) \\ \mathsf{b}{\uparrow} & \hline S'(!R,R,[?T,T]) \\ \mathsf{s} & \hline S'([(!R,R),?T],T) \\ \mathsf{s} & \hline S'[(!R,?T),(R,T)] \\ \mathsf{p}{\uparrow} & \hline S'[?(R,T),(R,T)] \\ \mathsf{b}{\downarrow} & \hline S'\{?(R,T)\} \end{array} \quad .$$

(ii) If $\rho = \mathsf{w}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\begin{array}{rc} & S'\{!S''[?T,T]\} \\ \mathsf{b}{\downarrow} & \hline S'\{!S''\{?T\}\} \\ \mathsf{w}{\uparrow} & \hline S'\{1\} \end{array} \quad \text{by} \quad \begin{array}{rc} & S'\{!S''[?T,T]\} \\ \mathsf{w}{\uparrow} & \hline S'\{1\} \end{array} \quad .$$

(iii) If $\rho = \mathsf{b}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\begin{array}{rc} & S'\{!S''[?T,T]\} \\ \mathsf{b}{\downarrow} & \hline S'\{!S''\{?T\}\} \\ \mathsf{b}{\uparrow} & \hline S'(!S''\{?T\},S''\{?T\}) \end{array} \quad \text{by} \quad \begin{array}{rc} & S'\{!S''[?T,T]\} \\ \mathsf{b}{\uparrow} & \hline S'(!S''[?T,T],S''[?T,T]) \\ \mathsf{b}{\downarrow} & \hline S'(!S''[?T,T],S''\{?T\}) \\ \mathsf{b}{\downarrow} & \hline S'(!S''\{?T\},S''\{?T\}) \end{array} \quad .$$

Remark: Cases (3) and (6) are not possible.

Figure 15: The $\mathsf{b}{\downarrow}\mathsf{down}$ procedure

$$
\mathsf{e}{\downarrow}\,\frac{}{1} \qquad \mathsf{a}{\downarrow}\,\frac{S1}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]}
$$

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{w}{\downarrow}\,\frac{S\bot}{S\{?R\}} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}}
$$

Figure 16: System ELS

Lemma 7.4 and Lemma 7.5 ensure that each step of the algorithm b↑↓split depicted in Figure 12 does terminate. It remains to show that the whole algorithm does terminate eventually.

7.6 Lemma *The algorithm* b↑↓split *does terminate.*

Proof: Suppose the algorithm does not terminate. This means that there is a derivation $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ R \end{array}$ such that in each run of b↑up at least one new instance of b↓ is introduced, and in each run of b↓down at least one new instance of b↑ is introduced. Now run b↑up and b↓down once, which transformes Δ into

$$
\begin{array}{c} T \\ \Delta_1 \| \{\mathsf{b}\uparrow\} \\ T' \\ \Delta' \| \mathsf{SELS} \setminus \{\mathsf{b}\downarrow\} \\ R' \\ \Delta_2 \| \{\mathsf{b}\downarrow\} \\ R \end{array} .
$$

□

7.7 Proposition *For every* $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ S \end{array}$ *there is a derivation* $\begin{array}{c} T \\ \Delta_1 \| \{\mathsf{b}\uparrow\} \\ T' \\ \Delta' \| \{\mathsf{a}\downarrow,\mathsf{a}\uparrow,\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow,\mathsf{w}\downarrow,\mathsf{w}\uparrow\} \\ S' \\ \Delta_2 \| \{\mathsf{b}\downarrow\} \\ S \end{array}$.

Proof: Apply the algorithm b↑↓split, which terminates by Lemma 7.6. □

8 Cut-Elimination in the Calculus of Structures

8.1 Definition The system {e↓, a↓, s, p↓, w↓, b↓}, shown in Figure 16, which is obtained from the "down-fragment" of system SELS together with the axiom, is called *multiplicative exponential linear logic in the calculus of structures*, or system ELS.

As an immediate consequence of Propositions 4.5 and 4.8 we get the following:

8.2 Theorem *The systems* ELS ∪ {i↑} *and* SELS ∪ {e↓} *are strongly equivalent.*

The proof of Theorem 4.13 shows that for every cut-free proof in MELL we can obtain a proof in system ELS. Therefore, it follows from the cut-elimination theorem for MELL [4], that the rule i↑ is admissible for system ELS, or equivalently, systems SELS ∪ {e↓} and ELS are equivalent, i.e.

the whole "up-fragment" of system SELS is admissible for system ELS.

We obtained this cut-elimination result for the calculus of structures by using the calculus of sequents. In this section I will show that it is possible to prove cut-elimination within the calculus of structures, without the detour of using the sequent calculus of MELL.

A good reason to study cut-elimination in the calculus of structures is the hope for modularity. As already pointed out by Girard in [5], p. 15, a central problem in structural proof theory is the non-modularity of the cut-elimination proofs. Whenever a new pair of rules is added to a system, the one has to redo the whole proof. The methodology used in this paper is very different from all known cut-elimination proofs for MELL. The proof will be more complicated for the simple reason that the calculus of structures is finer than the sequent calculus and has not the power of the main connective. However, since it consists of several lemmata which still hold if the system is extended, we have a methodology that could allow for a modular cut-elimination proof.

The full cut-elimination proof will consist of three main steps. First, the non-core part of the up-fragment (i.e. the rules $\mathsf{b}\uparrow$ and $\mathsf{w}\uparrow$) will be eliminated. In the second step, I will eliminate the core part of the up-fragment (i.e. the rule $\mathsf{p}\uparrow$). The last step is then the atomic cut-elimination (i.e. the rule $\mathsf{a}\uparrow$ is eliminated).

8.3 Theorem *For every proof* $\begin{array}{c} \Pi \| \mathsf{SELS}\cup\{\mathsf{e}\downarrow\} \\ R \end{array}$, *there is a proof* $\begin{array}{c} \mathsf{e}\downarrow \dfrac{}{1} \\ \| \{\mathsf{w}\downarrow\} \\ R_4 \\ \| \{\mathsf{a}\downarrow\} \\ R_3 \\ \| \{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\ R_2 \\ \| \{\mathsf{a}\uparrow\} \\ R_1 \\ \| \{\mathsf{b}\downarrow\} \\ R \end{array}$ *for some structures* R_1, R_2, R_3 *and* R_4.

Proof: By Propositions 4.5 and 4.8, every occurrence of the rule $\mathsf{w}\uparrow$ can be replaced by a derivation containing only the rules $\mathsf{w}\downarrow, \mathsf{a}\downarrow, \mathsf{a}\uparrow, \mathsf{s}, \mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$. Apply Theorem 7.1 to the derivation $\begin{array}{c} 1 \\ \| \mathsf{SELS}\setminus\{\mathsf{w}\uparrow\} \\ S \end{array}$ which is obtained from Π by removing the axiom. Now every application of $\mathsf{b}\uparrow$ in $\begin{array}{c} 1 \\ \Delta_1 \| \{\mathsf{b}\uparrow\} \\ T' \end{array}$ must be trivial, i.e. of the shape $\mathsf{b}\uparrow \dfrac{(!1,1)}{!1}$. Hence $T' = 1$. □

This theorem shows that the non-core rules $\mathsf{w}\uparrow$ and $\mathsf{b}\uparrow$ are admissible. In order to show that the rule $\mathsf{p}\uparrow$ is admissible, we need to add the following two rules to system SELS:

$$\mathsf{r}\downarrow \frac{S\{?[R,T]\}}{S[?R,?T]} \quad \text{and} \quad \mathsf{r}\uparrow \frac{S(!R,!T)}{S\{!(R,T)\}} \quad .$$

It is easy to see that both rules are sound.

For technical reasons, I was not able to simply eliminate the rule $\mathsf{p}\uparrow$. Instead I will eliminate the rules p↑ and r↑ simultaneously, with the result that instances of r↓ might be introduced. Those instances will be eliminated afterwards. Finally, the rule a↑ will also be eliminated.

All three rules p↑, r↑ and a↑ are removed by a technique that has already been employed in [6] for proving the cut-elimination for system BV. Namely, for all three rules p↑, r↑ and a↑, there are super-rules sp↑, sr↑ and sa↑, repectively:

$$\text{sa}\uparrow \frac{S([a,U],[\bar{a},V])}{S[U,V]} \quad , \quad \text{sp}\uparrow \frac{S([?R,U],[!T,V])}{S[?(R,T),U,V]} \quad \text{and} \quad \text{sr}\uparrow \frac{S([!R,U],[!T,V])}{S[!(R,T),U,V]} \quad .$$

Observe that each “up-rule” x↑ is an instance of its “super-up-rule” sx↑. I will now show that every sx↑-rule, can be permuted up in the proof until it disappears or its application becomes trivial.

Before we can start, a few more definitions are necessary.

8.4 Definition structure R is called a *proper par* if there are two structures R' and R'' with $R = [R', R'']$ and $R' \neq \bot \neq R''$. Similarly, a structure R is a *proper times*, if there are two structures R' and R'' with $R = (R', R'')$ and $R' \neq 1 \neq R''$.

Let *deep switch* be the rule $\text{ds}\,\dfrac{S([R,T],U)}{S[(R,U),T]}$, where the structure R is not a proper times. The rule $\text{ns}\,\dfrac{S([(R,R'),T],U)}{S[(R,R',U),T]}$, where $R \neq 1 \neq R'$, will be called *non-deep switch*. Both rules are instances of the switch-rule, and every instance of the switch-rule is either an instance of deep switch or an instance of non-deep switch.

8.5 This is sufficient to outline the scheme (shown in Figure 17) of the full cut-elimination proof. First, all instances of the rule s are replaced by ds or ns, and all instances of p↑ and a↑ are replaced by their super rules. While permuting up the rules ns and sp↑ over ds and p↓ in Step 2, the rules r↓ and sr↑ are introduced. In Step 3, the rules ns, sp↑ and sr↑ are eliminated. Then, the rule r↓ is eliminated in Step 4. Finally, the atomic cut is eliminated.

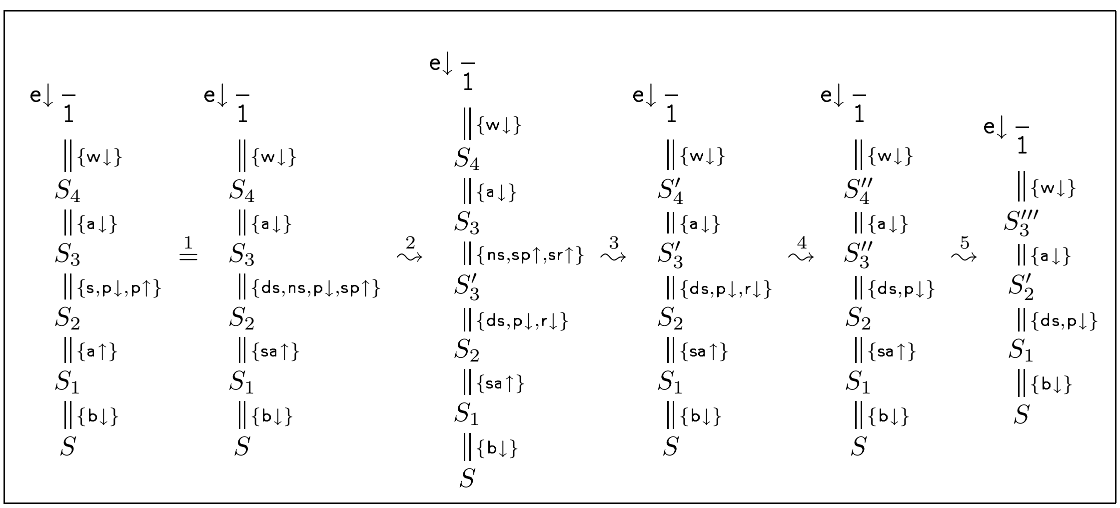

Figure 17: Cut-elimination for system SELS ∪ {e↓}

8.6 Lemma *The rule* ns *permutes up over the rules* ds, p↓ *and* r↓ *by the rule* ds.

Proof: Following the scheme of 5.3, let us consider a derivation $\pi \frac{\mathsf{ns}\frac{Q}{S([(R,R'),T],U)}}{S[(R,R',U),T]}$, where the application of $\pi \in \{\mathsf{ds},\mathsf{p}\!\downarrow,\mathsf{r}\!\downarrow\}$ is not trivial. Without loss of generality we can assume that R is not a proper times. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([(R,R'),T],U)$ of ns is inside a passive structure in the redex of π. Trivial.

(3) The redex of π is inside one of the passive structures R, R', T or U of the contractum of ns. Trivial.

(4) The redex of π is inside the contractum $([(R,R'),T],U)$ of ns, but not inside R, R', T or U. Only one case is possible ($\pi = \mathsf{ds}$):

$$\mathsf{ds}\frac{S([R,T],R',U)}{\mathsf{ns}\dfrac{S([(R,R'),T],U)}{S[(R,R',U),T]}} \quad \text{yields} \quad \mathsf{ds}\frac{S([R,T],R',U)}{S[(R,R',U),T]} \quad .$$

(5) The contractum $([(R,R'),T],U)$ of ns is inside an active structure of the redex of π but not inside a passive one. Then $\pi = \mathsf{ds}$ and $S([(R,R'),T],U) = S'[([(R,R'),T],U,V),W]$. There are two possibilities:

$$\mathsf{ds}\frac{S'([(R,R'),T,W],U,V)}{\mathsf{ns}\dfrac{S'[([(R,R'),T],U,V),W]}{S'[([(R,R',U),T],V),W]}} \quad \text{yields} \quad \mathsf{ns}\frac{S'([(R,R'),T,W],U,V)}{\mathsf{ds}\dfrac{S'([(R,R',U),T,W],V)}{S'[([(R,R',U),T],V),W]}} \quad \text{and}$$

$$\mathsf{ds}\frac{S'([(R,R'),T],[U,W],V)}{\mathsf{ns}\dfrac{S'[([(R,R'),T],U,V),W]}{S'[([(R,R',U),T],V),W]}} \quad \text{yields} \quad \mathsf{ns}\frac{S'([(R,R'),T],[U,W],V)}{\mathsf{ds}\dfrac{S'([(R,R',[U,W]),T],V)}{\mathsf{ds}\dfrac{S'([(R,R',U),T,W],V)}{S'[([(R,R',U),T],V),W]}}} \quad .$$

Note: The second case is only possible if U is not a proper times.

(6) The redex of π and the contractum $([(R,R'),T],U)$ of ns overlap. Not possible.

□

8.7 Lemma *The rules* sa↑, sp↑ *and* sr↑ *permute up over the rule* ds.

Proof: All three rules are of the shape $\mathsf{sx}\!\uparrow \frac{S([P,U],[P',V])}{S[P'',U,V]}$, where neither P nor P' is a proper par nor a proper times. Now consider the derivation $\mathsf{ds}\frac{Q}{\mathsf{sx}\!\uparrow\dfrac{S([P,U],[P',V])}{S[P'',U,V]}}$, where the application of ds is not trivial.

(1) The redex of ds is inside $S\{\ \}$. Trivial.

(2) The contractum $([P,U],[P',V])$ of sx↑ is inside a passive structure of the redex of ds. Trivial.

(3) The redex of ds is inside a passive structure of the contractum of sx↑. Trivial. (Remark: The passive structures are U and V, and, if sx↑ is sp↑ or sr↑, also R and T are passive structures.)

(4) The redex of ds is inside the contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$, but not inside a passive structure U, V, R or T. Observe that the redex of ds cannot be inside P or P' because they are neither a proper par nor a proper times. Therefore, there are only two remaining cases.

(i) $U=(U',U'')$. Without loss of generality assume that U' is not a proper times. Then

$$\mathsf{sx}\uparrow\frac{\mathsf{ds}\frac{S([P,U'],[P',V],U'')}{S([P,(U',U'')],[P',V])}}{S[P'',(U',U''),V]} \quad \text{yields} \quad \mathsf{ds}\frac{\mathsf{sx}\uparrow\frac{S([P,U'],[P',V],U'')}{S([P'',U',V],U'')}}{S[P'',(U',U''),V]} \quad .$$

(ii) $V=(V',V'')$. Similar.

(5) The contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$ is inside an active structure of the redex of ds but not inside a passive one. Let $S([P,U],[P',V]) = S'[([P,U],[P',V],W),Z]$. Then

$$\mathsf{sx}\uparrow\frac{\mathsf{ds}\frac{S([P,U,Z],[P',V],W)}{S'[([P,U],[P',V],W),Z]}}{S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \mathsf{ds}\frac{\mathsf{sx}\uparrow\frac{S([P,U,Z],[P',V],W)}{S'([P'',U,V,Z],W)}}{S'[([P'',U,V],W),Z]} \quad \text{and}$$

$$\mathsf{sx}\uparrow\frac{\mathsf{ds}\frac{S([P,U],[P',V,Z],W)}{S'[([P,U],[P',V],W),Z]}}{S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \mathsf{ds}\frac{\mathsf{sx}\uparrow\frac{S([P,U],[P',V,Z],W)}{S'([P'',U,V,Z],W)}}{S'[([P'',U,V],W),Z]} \quad .$$

(6) The redex of ds and the contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$ overlap. Not possible.

□

8.8 Observe that the rules $\mathsf{sa}\uparrow$, $\mathsf{sp}\uparrow$ and $\mathsf{sr}\uparrow$ do not permute over the rule s. For example in the derivation

$$\mathsf{sa}\uparrow\frac{\mathsf{s}\frac{S([a,U],[([\bar{a},V],W),Z])}{S[([a,U],[\bar{a},V],W),Z]}}{S[([U,V],W),Z]}$$

the rule $\mathsf{sa}\uparrow$ cannot be permuted up over the *switch*. This is the reason why the rule ds has been introduced in [6] in the first place.

8.9 Lemma *For every derivation* $\rho\frac{\pi\frac{Q}{Z}}{P}$ *with* $\rho\in\{\mathsf{sp}\uparrow,\mathsf{sr}\uparrow\}$ *and* $\pi\in\{\mathsf{p}\downarrow,\mathsf{r}\downarrow\}$*, there is either a derivation* $\pi\frac{\rho\frac{Q}{Z'}}{P}$ *for some structure* Z' *or a derivation* $\pi'\frac{\mathsf{s}\frac{\rho'\frac{Q}{Z'}}{Z''}}{P}$ *for some structures* Z' *and* Z'' *and rules* $\rho'\in\{\mathsf{sp}\uparrow,\mathsf{sr}\uparrow\}$ *and* $\pi'\in\{\mathsf{p}\downarrow,\mathsf{r}\downarrow\}$.

Proof: Consider the derivation $\rho\frac{\pi\frac{Q}{S([*R,U],[!T,V])}}{S[*(R,T),U,V]}$, where $\rho\in\{\mathsf{sp}\uparrow,\mathsf{sr}\uparrow\}$, $*\in\{?,!\}$ and the application of $\pi\in\{\mathsf{p}\downarrow,\mathsf{r}\downarrow\}$ is not trivial. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([*R,U],[!T,V])$ of ρ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure R,U,T or V of the contractum of ρ. Trivial.

(4) The redex of π is inside the contractum $([*R,U],[!T,V])$ of ρ but not inside R,U,T or V. There are the following five subcases:

(i) $\rho = \mathsf{sp}\!\uparrow, * = ?, \pi = \mathsf{p}\!\downarrow$ and $U = [!U',U'']$. Then

$$\begin{array}{rc} & S([![R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{p}\!\downarrow & S([?R,!U',U''],[!T,V]) \\ \cline{2-2} \mathsf{sp}\!\uparrow & S[?(R,T),!U',U'',V] \end{array} \qquad \text{yields} \qquad \begin{array}{rc} & S([![R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{sr}\!\uparrow & S[!([R,U'],T),U'',V] \\ \cline{2-2} \mathsf{s} & S[![(R,T),U'],U'',V] \\ \cline{2-2} \mathsf{p}\!\downarrow & S[?(R,T),!U',U'',V] \end{array} \quad .$$

(ii) $\rho = \mathsf{sp}\!\uparrow, * = ?, \pi = \mathsf{p}\!\downarrow$ and $V = [?V',V'']$. Then

$$\begin{array}{rc} & S([?R,U],[![T,V'],V'']) \\ \cline{2-2} \mathsf{p}\!\downarrow & S([?R,U],[!T,?V',V'']) \\ \cline{2-2} \mathsf{sp}\!\uparrow & S[?(R,T),U,?V',V''] \end{array} \qquad \text{yields} \qquad \begin{array}{rc} & S([?R,U],[![T,V'],V'']) \\ \cline{2-2} \mathsf{sp}\!\uparrow & S[?(R,[T,V']),U,V''] \\ \cline{2-2} \mathsf{s} & S[?[(R,T),V'],U,V''] \\ \cline{2-2} \mathsf{r}\!\downarrow & S[?(R,T),U,?V',V''] \end{array} \quad .$$

(iii) $\rho = \mathsf{sp}\!\uparrow, * = ?, \pi = \mathsf{r}\!\downarrow$ and $U = [?U',U'']$. Then

$$\begin{array}{rc} & S([?[R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{r}\!\downarrow & S([?R,?U',U''],[!T,V]) \\ \cline{2-2} \mathsf{sp}\!\uparrow & S[?(R,T),?U',U'',V] \end{array} \qquad \text{yields} \qquad \begin{array}{rc} & S([?[R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{sp}\!\uparrow & S[?([R,U'],T),U'',V] \\ \cline{2-2} \mathsf{s} & S[?[(R,T),U'],U'',V] \\ \cline{2-2} \mathsf{r}\!\downarrow & S[?(R,T),?U',U'',V] \end{array} \quad .$$

(iv) $\rho = \mathsf{sr}\!\uparrow, * = !, \pi = \mathsf{p}\!\downarrow$ and $U = [?U',U'']$. Then

$$\begin{array}{rc} & S([![R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{p}\!\downarrow & S([!R,?U',U''],[!T,V]) \\ \cline{2-2} \mathsf{sr}\!\uparrow & S[!(R,T),?U',U'',V] \end{array} \qquad \text{yields} \qquad \begin{array}{rc} & S([![R,U'],U''],[!T,V]) \\ \cline{2-2} \mathsf{sr}\!\uparrow & S[!([R,U'],T),U'',V] \\ \cline{2-2} \mathsf{s} & S[![(R,T),U'],U'',V] \\ \cline{2-2} \mathsf{p}\!\downarrow & S[!(R,T),?U',U'',V] \end{array} \quad .$$

(v) $\rho = \mathsf{sr}\!\uparrow, * = !, \pi = \mathsf{p}\!\downarrow$ and $V = [?V',V'']$. Then

$$\begin{array}{rc} & S([!R,U],[![T,V'],V'']) \\ \cline{2-2} \mathsf{p}\!\downarrow & S([!R,U],[!T,?V',V'']) \\ \cline{2-2} \mathsf{sr}\!\uparrow & S[!(R,T),U,?V',V''] \end{array} \qquad \text{yields} \qquad \begin{array}{rc} & S([!R,U],[![T,V'],V'']) \\ \cline{2-2} \mathsf{sr}\!\uparrow & S[!(R,[T,V']),U,V''] \\ \cline{2-2} \mathsf{s} & S[![(R,T),V'],U,V''] \\ \cline{2-2} \mathsf{p}\!\downarrow & S[!(R,T),U,?V',V''] \end{array} \quad .$$

(5) The contractum $([*R,U],[!T,V])$ of ρ is inside an active structure of the redex of π, but not inside a passive one. Not possible.

(6) The redex of π and the contractum $([*R,U],[!T,V])$ of ρ overlap. Not possible.

$\square$

8.10 Lemma *For every derivation* $\begin{array}{c} S_3 \\ \Vert\{\mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\} \\ S_2 \end{array}$ *there is a derivation* $\begin{array}{c} S_3 \\ \Vert\{\mathsf{ns},\mathsf{sp}{\uparrow},\mathsf{sr}{\uparrow}\} \\ S'_3 \\ \Vert\{\mathsf{ds},\mathsf{p}{\downarrow},\mathsf{r}{\downarrow}\} \\ S_2 \end{array}$.

Proof: All occurrences of the rules p↑ and r↑ are instances of the rules sp↑ and sr↑, respectively; and all occurrences of the rule s are either instances of ds or of ns. Now apply the following algorithm:

I. If there is no occurrence of a rule ns, sp↑ or sr↑ below a rule ds, p↓ or r↓ in the derivation, then terminate.

II. Otherwise, let ρ be the topmost occurrence of a rule ns, sp↑ or sr↑ that is below a ds, p↓ or r↓.

(1) If ρ is ns, then (by Lemma 8.6) this occurrence can be permuted up (by possibly introducing new instances of ds).

(2) If ρ is sp↑ or sr↑, then (by Lemmata 8.7 and 8.9) it can be permuted up over all occurrences of the rules ds, p↓ and r↓ (by possibly introducing new instances of ds and ns).

Go to step I.

It is easy to see that this algorithm does indeed terminate (since the derivation is finite). □

8.11 Lemma *The rules* ns*,* sp↑ *and* sr↑ *permute up over the rule* a↓*.*

Proof: Consider the derivation $\rho\,\dfrac{\mathsf{a}{\downarrow}\,\dfrac{Q}{S\{W\}}}{S\{Z\}}$, where the application of $\rho \in \{\mathsf{ns}, \mathsf{sr}{\uparrow}, \mathsf{sp}{\uparrow}\}$ is not trivial. The cases are:

(1) The redex of a↓ is inside the context $S\{\ \}$ of ρ. Trivial.

(2) The contractum of ρ is inside a passive structure of the redex of a↓. Trivial.

(3) The redex of a↓ is inside a passive structure of the contractum W of ρ. Trivial.

(4) The redex of a↓ is inside an active structure of the contractum W of ρ but not inside a passive one. Not possible.

(5) The contractum W of ρ is inside an active structure of the redex of a↓. Not possible because the application of ρ is not trivial.

(6) The contractum W of ρ and the redex of π overlap. Not possible.

□

8.12 Lemma *For every* $\rho\,\dfrac{\begin{array}{c} 1 \\ \Vert\{\mathsf{w}{\downarrow}\} \\ P' \end{array}}{P}$ *, where* $\rho \in \{\mathsf{ns}, \mathsf{sp}{\uparrow}, \mathsf{sr}{\uparrow}\}$*, there is a derivation* $\begin{array}{c} 1 \\ \Vert\{\mathsf{w}{\downarrow}\} \\ P \end{array}$.

Proof: Let me introduce the rule $\mathsf{spw}{\uparrow}\,\dfrac{S(U, [!T, V])}{S[?(R, T), U, V]}$, which is a combination of sp↑ and w↓. Now consider a derivation $\rho\,\dfrac{\mathsf{a}{\downarrow}\,\dfrac{Q}{S\{W\}}}{S\{Z\}}$, where $\rho \in \{\mathsf{ns}, \mathsf{sp}{\uparrow}, \mathsf{sr}{\uparrow}, \mathsf{spw}{\uparrow}\}$ and permute ρ up over w↓ by applying the scheme of 5.3:

(1) The redex of $\mathsf{w}\!\downarrow$ is inside the context $S\{\ \}$ of ρ. Trivial.

(2) The contractum of ρ is inside a passive structure of the redex of $\mathsf{w}\!\downarrow$. Trivial.

(3) The redex of $\mathsf{w}\!\downarrow$ is inside a passive structure of the contractum W of ρ. Trivial.

(4) The redex of $\mathsf{w}\!\downarrow$ is inside an active structure of the contractum W of ρ but not inside a passive one. Then $W = ([?R, U], [!T, V])$ and

$$\mathsf{sp}\!\uparrow \frac{\mathsf{w}\!\downarrow \dfrac{S([\bot, U], [!T, V])}{S([?R, U], [!T, V])}}{S[?(R,T), U, V]} \qquad \text{yields} \qquad \mathsf{spw}\!\uparrow \frac{S(U, [!T, V])}{S[?(R,T), U, V]} \quad .$$

(5) The contractum W of ρ is inside an active structure of the redex of $\mathsf{w}\!\downarrow$. Then $S\{\ \} = S'\{?S''\{\ \}\}$ and

$$\rho \frac{\mathsf{w}\!\downarrow \dfrac{S'\bot}{S'\{?S''\{W\}\}}}{S'\{?S''\{Z\}\}} \qquad \text{yields} \qquad \mathsf{w}\!\downarrow \frac{S'\bot}{S'\{?S''\{Z\}\}} \quad .$$

(6) The contractum W of ρ and the redex of π overlap. Not possible.

So, we either obtain a derivation $\begin{array}{c} 1 \\ \| \{\mathsf{w}\!\downarrow\} \\ P \end{array}$ or a derivation $\begin{array}{c} \rho' \dfrac{1}{P''} \\ \| \{\mathsf{w}\!\downarrow\} \\ P \end{array}$ with $\rho' \in \{\mathsf{ns}, \mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow, \mathsf{spw}\!\uparrow\}$. In the former case the proof is finished. In the latter, there are two possibilities.

(1) $\rho' \in \{\mathsf{ns}, \mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$. Then the application of ρ' must be trivial, because its premise is 1. Hence its conclusion $P'' = 1$ and we have the desired derivation by leaving out ρ'.

(2) $\rho' = \mathsf{spw}\!\uparrow$. Then the application of ρ' must be an instance of $\mathsf{w}\!\downarrow$ because its premise is 1. Hence, it can be replaced by an application of $\mathsf{w}\!\downarrow$.

□

8.13 Lemma *For every derivation* $\begin{array}{c} 1 \\ \| \{\mathsf{w}\!\downarrow\} \\ U \\ \| \{\mathsf{a}\!\downarrow\} \\ V \\ \| \{\mathsf{ds}, \mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\} \\ W \end{array}$ *there is a derivation* $\begin{array}{c} 1 \\ \| \{\mathsf{w}\!\downarrow\} \\ U' \\ \| \{\mathsf{a}\!\downarrow\} \\ V' \\ \| \{\mathsf{ds}, \mathsf{p}\!\downarrow\} \\ W \end{array}$.

Proof: Instead of eliminating the rule $\mathsf{r}\!\downarrow$, I will eliminate the rule $\mathsf{sr}\!\downarrow \dfrac{S\{?U\}}{S[?R, ?T]}$, where U is any structure such that there is a derivation $\begin{array}{c} U \\ \Delta \| \{\mathsf{w}\!\downarrow, \mathsf{a}\!\downarrow, \mathsf{ds}, \mathsf{p}\!\downarrow\} \\ [R, T] \end{array}$. Note that $\mathsf{r}\!\downarrow$ is an instance of $\mathsf{sr}\!\downarrow$. Now consider a derivation $\mathsf{sr}\!\downarrow \dfrac{\pi \dfrac{Q}{S\{?U\}}}{S[?R, ?T]}$, where $\pi \in \{\mathsf{w}\!\downarrow, \mathsf{a}\!\downarrow, \mathsf{ds}, \mathsf{p}\!\downarrow\}$ is not trivial, and permute $\mathsf{sr}\!\downarrow$ up by applying the scheme in 5.3:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $?U$ of $\mathsf{sr}\!\downarrow$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum of $\mathsf{sr}\!\downarrow$. Not possible because there is no passive structure.

(4) The redex of π is inside the contractum $?U$ of $\mathsf{sr}\!\downarrow$. Then replace

$$\mathsf{sr}\!\downarrow \frac{\pi \dfrac{S\{?U'\}}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{sr}\!\downarrow \frac{S\{?U'\}}{S[?R,?T]} \quad .$$

(5) The contractum $?U$ of $\mathsf{sr}\!\downarrow$ is inside an active structure of the redex of π but not inside a passive one. Then the following subcases are possible:

(i) $\pi = \mathsf{p}\!\downarrow$ and $S\{?U\} = S'[!V,?U]$. Then replace

$$\mathsf{sr}\!\downarrow \frac{\mathsf{p}\!\downarrow \dfrac{S\{![V,U]\}}{S[!V,?U]}}{S[!V,?R,?T]} \qquad \text{by} \qquad \mathsf{p}\!\downarrow \frac{\mathsf{p}\!\downarrow \dfrac{\begin{array}{c} S\{![V,U]\} \\ \Delta \| \\ S\{![V,R,T]\} \end{array}}{S[![V,R],?T]}}{S[!V,?R,?T]} \quad ,$$

where Δ is the derivation that exists by definition of $\mathsf{sr}\!\downarrow$.

(ii) $\pi = \mathsf{w}\!\downarrow$ and $?U$ is the redex. Then replace

$$\mathsf{sr}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S\bot}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S\bot}{S\{?R\}}}{S[?R,?T]} \quad .$$

(iii) $\pi = \mathsf{w}\!\downarrow$ and $S\{?U\} = S'\{?S''\{?U\}\}$. Then replace

$$\mathsf{sr}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S'\bot}{S'\{?S''\{?U\}\}}}{S'\{?S''[?R,?T]\}} \qquad \text{by} \qquad \mathsf{w}\!\downarrow \frac{S'\bot}{S'\{?S''[?R,?T]\}} \quad .$$

(6) The redex of π and the contractum $?U$ of $\mathsf{sr}\!\downarrow$ overlap. Not possible.

□

8.14 Lemma (Atomic cut-elimination) *The rule* $\mathsf{sa}\!\uparrow$ *permutes up over the rules* $\mathsf{w}\!\downarrow, \mathsf{a}\!\downarrow, \mathsf{ds}$ *and* $\mathsf{p}\!\downarrow$ *by the rule* ds*.*

Proof: Consider the derivation $\mathsf{sa}\!\uparrow \dfrac{\pi \dfrac{Q}{S([a,U],[\bar{a},V])}}{S[U,V]}$. If $\pi = \mathsf{ds}$, then Lemma 8.7 applies. Now let $\pi \in \{\mathsf{w}\!\downarrow, \mathsf{a}\!\downarrow, \mathsf{p}\!\downarrow\}$ be not trivial. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([a,U],[\bar{a},V])$ of $\mathsf{sa}\!\uparrow$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure U or V of the contractum of $\mathsf{sa}\!\uparrow$. Trivial.

(4) The redex of π is inside the contractum $([a,U],[\bar{a},V])$ of $\mathsf{sa}\!\uparrow$ but not inside U or V.

(i) $\pi = \mathsf{a}\!\downarrow$ and $U = [\bar{a}, U']$. Then

$$\mathsf{sa}\!\uparrow \dfrac{\mathsf{a}\!\downarrow \dfrac{S([1,U'],[\bar{a},V])}{S([a,\bar{a},U'],[\bar{a},V])}}{S[\bar{a},U',V]} \qquad \text{yields} \qquad \mathsf{ds}\,\dfrac{S([1,U'],[\bar{a},V])}{S[(1,[\bar{a},V]),U']} \quad .$$

(ii) $\pi = \mathsf{a}\!\downarrow$ and $V = [a, V']$. Similar.

(5) The contractum $([a,U],[\bar{a},V])$ of $\mathsf{sa}\!\uparrow$ is inside an active structure of the redex of π but not inside a passive one. The only possible case is $\pi = \mathsf{w}\!\downarrow$ and $S([a,U],[\bar{a},V]) = S'\{?S''([a,U],[\bar{a},V])\}$. Then

$$\mathsf{sa}\!\uparrow \dfrac{\mathsf{w}\!\downarrow \dfrac{S'\bot}{S'\{?S''([a,U],[\bar{a},V])\}}}{S'\{?S''[U,V]\}} \qquad \text{yields} \qquad \mathsf{w}\!\downarrow \dfrac{S'\bot}{S'\{?S''[U,V]\}} \quad .$$

(6) The redex of π and the contractum of $\mathsf{sa}\!\uparrow$ overlap. Not possible.

□

8.15 Theorem (Cut-elimination) *The systems* $\mathsf{SELS} \cup \{\mathsf{e}\!\downarrow\}$ *and* ELS *are equivalent.*

Proof: The proof follows the scheme of 8.5 and Figure 17, where Step 2 is realized by Lemma 8.10, Step 3 by Lemmata 8.11 and 8.12, and Steps 4 and 5 by Lemma 8.13 and Lemma 8.14, respectively. □

8.16 Remark Observe that although a proof $\begin{array}{c}\Pi \Vert \mathsf{SELS}\cup\{\mathsf{e}\downarrow\} \\ S\end{array}$, obtained by this cut-elimination process has not the shape

$$\begin{array}{c} \mathsf{e}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ V \\ \Vert\{\mathsf{a}\!\downarrow\} \\ W \\ \Vert\{\mathsf{ds},\mathsf{p}\!\downarrow\} \\ W' \\ \Vert\{\mathsf{b}\!\downarrow\} \\ S \end{array}$$

, it can easly transformed into such a one by Lemma 5.6.

8.17 Theorem 7.1 is of great value for the proof of cut-elimination. First, it shows that the non-core part of the "up-fragment" is admissible. And second, the rule $\mathsf{b}\!\downarrow$ is moved below the core rules of the "up-fragment" (namely, the rules $\mathsf{p}\!\uparrow$ and $\mathsf{a}\!\uparrow$). This means that in the cut-elimination process we do not have to deal with contraction, which is known to be most problematic in cut-elimination proofs.

9 Conclusion

The calculus of structures has originally been developed to describe a system that could not be observed in the sequent calculus [6]. In this paper I showed that it is also suitable for formal systems that have a well-known sequent calculus representation. Although the calculus of structures is more general than the sequent calculus, it is not more complicated. I even dare to say that in the case of the multiplicative exponential fragment of linear logic (MELL), the calculus of structures yields a *better* system than the sequent calculus, especially because of the *local* promotion rule.

The price we have to pay is a more complicated proof of the cut-elimination theorem because the calculus of structures is much finer than the sequent calculus. But this fineness of the calculus of structures enables it to unveil properties of formal systems that cannot be observed in the sequent calculus. One example is the decomposition theorem, whose relationship to cut-elimination is still a mystery.

Furthermore, the methodology for proving cut-elimination that is presented in this paper raises hope for modularity. This is of great importance for the field of structural proof theory for which the non-modularity of cut-elimination proofs is a central problem [5], p.15.

It is already under investigation, how the calculus of structures performs on classical logic [1].

The next step is to give a conservative extension of BV and the system presented in this paper. It will be essentially MELL together with a self-dual non-commutative connective. The proof of the cut-elimination theorem will only be slightly more complicated than proof I presented here. We conjecture that this system is undecidable.

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