Prefix-vocabulary classes

$$[\Pi, (p_1, p_2, \ldots), (f_1, f_2, \ldots)]_{(=)}$$

- Π is a word over $\{\exists, \forall, \exists^*, \forall^*\}$, describing set of quantifier prefixes
- $p_m, f_m \leq \omega$ indicate how many relation and function symbols of arity m may occur
- presence or absence of $=$ indicates whether the formulae may contain equality

Example: $[\exists^*\forall\exists^*, (\omega, 1), all]_=$

sentences $\exists x_1 \ldots \exists x_m \forall y \exists z_1 \ldots \exists z_n \varphi$ where φ is quantifier-free and

- contains at most one binary predicate, and no predicates of arity ≥ 3,
- may contain any number of monadic predicates,
- may contain any number of function symbols of any arity,
- may contain equality.

The complete classification: undecidable cases

A: Pure predicate logic (without functions, without =)

(1)	$[\forall\exists\forall, (\omega, 1), (0)]$	(Kahr 1962)
(2)	$[\forall^3\exists, (\omega, 1), (0)]$	(Surányi 1959)
(3)	$[\forall^*\exists, (0, 1), (0)]$	(Kalmár-Surányi 1950)
(4)	$[\forall\exists\forall^*, (0, 1), (0)]$	(Denton 1963)
(5)	$[\forall\exists\forall\exists^*, (0, 1), (0)]$	(Gurevich 1966)
(6)	$[\forall^3\exists^*, (0, 1), (0)]$	(Kalmár-Surányi 1947)
(7)	$[\forall\exists^*\forall, (0, 1), (0)]$	(Kostyrko-Genenz 1964)
(8)	$[\exists^*\forall\exists\forall, (0, 1), (0)]$	(Surányi 1959)
(9)	$[\exists^*\forall^3\exists, (0, 1), (0)]$	(Surányi 1959)

The complete classification: undecidable cases

B: Classes with functions or equality

(10)	$[\forall, (0), (2)]_=$	(Gurevich 1976)
(11)	$[\forall, (0), (0, 1)]_=$	(Gurevich 1976)
(12)	$[\forall^2, (0, 1), (1)]$	(Gurevich 1969)
(13)	$[\forall^2, (1), (0, 1)]$	(Gurevich 1969)
(14)	$[\forall^2\exists, (\omega, 1), (0)]_=$	(Goldfarb 1984)
(15)	$[\exists^*\forall^2\exists, (0, 1), (0)]_=$	(Goldfarb 1984)
(16)	$[\forall^2\exists^*, (0, 1), (0)]_=$	(Goldfarb 1984)

The complete classification: decidable cases

(Exclude the trivial classes: finite prefix and finite relational vocabulary)

A: Classes with the finite model property

(1)	$[\exists^*\forall^*, all, (0)]_=$	(Bernays, Schönfinkel 1928)
(2)	$[\exists^*\forall^2\exists^*, all, (0)]$	(Gödel 1932, Kalmár 1933, Schütte 1934)
(3)	$[all, (\omega), (\omega)]$	(Löb 1967, Gurevich 1969)
(4)	$[\exists^*\forall\exists^*, all, all]$	(Gurevich 1973)
(5)	$[\exists^*, all, all]_=$	(Gurevich 1976)

Monadic Fragment of FO

B: Classes with infinity axioms

(6)	$[all, (\omega), (1)]_=$	(Rabin 1969)
(7)	$[\exists^*\forall\exists^*, all, (1)]_=$	(Shelah 1977)

$\mathcal{FO}^1$ WITH COUNTING, $\mathcal{C}^1$

$\mathcal{C}^1$: extension of $\mathcal{FO}^1$ with *counting quantifiers*: $\exists^{\leq m}$, $\exists^{\geq m}$, $\exists^{=m}$ meaning that there exists *at most*, *at least*, *exactly m* elements satisfying some property.

$$\exists^{\geq 125} x \top \wedge \exists^{=50} x \,\text{French}(x) \wedge \exists^{=36} x \,\text{German}(x) \wedge \exists^{=36} x \,\text{Spanish}(x)$$

SATISFIABLE

$$\begin{aligned}\exists^{=122} x \top \wedge \exists^{=50} x \,\text{French}(x) \wedge \exists^{=36} x \,\text{German}(x) \wedge \exists^{=36} x \,\text{Spanish}(x) \wedge \\ \forall x\, (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \\ \exists^{=38} x\, (\text{French}(x) \wedge \neg\text{German}(x)) \wedge \\ \exists^{=18} x\, (\text{French}(x) \wedge \text{Spanish}(x)) \wedge \\ \exists^{=21} x\, (\text{German}(x) \wedge \text{Spanish}(x)) \wedge \\ \exists^{=10} x\, (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))\end{aligned}$$

SATISFIABLE !

$\mathcal{FO}^1$ WITH COUNTING, $\mathcal{C}^1$

$\mathcal{C}^1$: extension of $\mathcal{FO}^1$ with *counting quantifiers*: $\exists^{\leq m}$, $\exists^{\geq m}$, $\exists^{=m}$ meaning that there exists *at most*, *at least*, *exactly* m elements satisfying some property.

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NOT SATISFIABLE

$$\exists^{=122} x \top \wedge \exists^{=50} x\, \text{French}(x) \wedge \exists^{=36} x\, \text{German}(x) \wedge \exists^{=36} x\, \text{Spanish}(x) \wedge \forall x\, (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \exists^{=38} x\, (\text{French}(x) \wedge \neg \text{German}(x)) \wedge \exists^{=18} x\, (\text{French}(x) \wedge \text{Spanish}(x)) \wedge \exists^{=21} x\, (\text{German}(x) \wedge \text{Spanish}(x)) \wedge \exists^{=10} x\, (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))$$

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NOT SATISFIABLE

$$\exists^{=122} x \top \wedge \exists^{=50} x\, \text{French}(x) \wedge \exists^{=36} x\, \text{German}(x) \wedge \exists^{=36} x\, \text{Spanish}(x) \wedge \forall x\, (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge$$
$$\exists^{=38} x\, (\text{French}(x) \wedge \neg\text{German}(x)) \wedge$$
$$\exists^{=18} x\, (\text{French}(x) \wedge \text{Spanish}(x)) \wedge$$
$$\exists^{=21} x\, (\text{German}(x) \wedge \text{Spanish}(x)) \wedge$$
$$\exists^{=10} x\, (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))$$

SATISFIABLE !

$\mathcal{FO}^1$ WITH COUNTING, $\mathcal{C}^1$

$\mathcal{C}^1$: extension of $\mathcal{FO}^1$ with *counting quantifiers*: $\exists^{\leq m}$, $\exists^{\geq m}$, $\exists^{=m}$ meaning that there exists *at most*, *at least*, *exactly m* elements satisfying some property.

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NOT SATISFIABLE

$$\begin{aligned}\exists^{=122} x \top \wedge \exists^{=50} x \,\text{French}(x) \wedge \exists^{=36} x \,\text{German}(x) \wedge \exists^{=36} x \,\text{Spanish}(x) \wedge \\ \forall x \,(\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \\ \exists^{=38} x \,(\text{French}(x) \wedge \neg\text{German}(x)) \wedge \\ \exists^{=18} x \,(\text{French}(x) \wedge \text{Spanish}(x)) \wedge \\ \exists^{=21} x \,(\text{German}(x) \wedge \text{Spanish}(x)) \wedge \\ \exists^{=10} x \,(\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))\end{aligned}$$

STILL SATISFIABLE ???

Lemma (Normal form for $\mathcal{C}^1$)

For every $\mathcal{C}^1$ formula φ we can compute in polynomial time a formula φ' of the form

$$\varphi' := \bigwedge_{i=1}^{m} \exists^{\bowtie_i C_i} x \varphi_i(x),$$

satisfiable over the same domains as φ, where:

- $1 \leq m \leq |\varphi|$,
- *each φ_i is quantifier free,*
- *each $\bowtie_i$ is any of the symbols $\leq$, $\geq$ or $=$, and*
- *the C_i are either one or occur as a quantifier subscript in φ.*

Proof: similarly to $\mathcal{FO}^1$ we replace subformulas of the form $\exists^{\bowtie C} x \chi(x)$ with $\chi(x)$-quantifier-free, by new predicate symbols and add appropriate definitions.

Theorem (FMP for $\mathcal{C}^1$)

Let φ be a formula in $\mathcal{C}^1$. If φ is satisfiable, then it is satisfiable over a domain of size at most $2^{|\varphi|}$.

WARNING! $\underline{\varphi = \exists^{\geq 2^n}_x \top}$

Proof.

By the normal form Lemma we may assume that φ has the form

$$\varphi := \bigwedge_{i=1}^{m} \exists^{\geq C_i} x\ \theta_i \quad \wedge \quad \bigwedge_{j=1}^{m'} \exists^{\leq D_j} x\ \chi_j.$$

Let $\mathfrak{A} \models \varphi$. For all i ($1 \leq i \leq m$) select distinct elements $a_{i,1}, \ldots, a_{i,C_i} \in A$ satisfying θ_i in $\mathfrak{A}$.
Let $B = \{a_{i,k} \mid 1 \leq i \leq m, 1 \leq k \leq C_i\}$, and let $\mathfrak{B}$ be the restriction of $\mathfrak{A}$ to B. Then $\mathfrak{B} \models \varphi$. $\square$

Corollary

SAT($\mathcal{C}^1$) is in NEXPTIME.

Goal: NP

Complexity of $\mathcal{C}^1$

Our aim is to prove

Theorem
SAT($\mathcal{C}^1$) is NP-complete.

$tp_1^{\mathfrak{A}}(a) = \{B, R\}$

We cannot improve the bound on the size of minimal models: the formula $\exists^{\geq n} x Px$ has only models of exponential size with respect to $|\varphi|$.

$tp_1^{\mathfrak{A}}(a) = B(x) \wedge R(x) \wedge \neg G(x)$

Definition
A 1-type of an element a in a model $\mathfrak{A}$ is the conjunction of all literals satisfied by a.

$\Sigma = \{B, R, G\}$ $\quad a^{B, R, \neg G}$

Idea: with each normal form φ we associate a system of linear inequalities $\mathcal{E}_\varphi$ describing constraints on the number of distinct 1-types realized in some model of φ.

Systems of inequalities - Example

$$
\begin{aligned}
\varphi := \exists^{=122} x \top \wedge \exists^{=50} x\ \mathrm{French}(x) \wedge \exists^{=36} x\ \mathrm{German}(x) \wedge \exists^{=36} x\ \mathrm{Spanish}(x) \wedge \\
\forall x\ (\mathrm{French}(x) \vee \mathrm{German}(x) \vee \mathrm{Spanish}(x)) \wedge \\
\exists^{=38} x\ (\mathrm{French}(x) \wedge \neg \mathrm{German}(x)) \wedge \\
\exists^{=18} x\ (\mathrm{French}(x) \wedge \mathrm{Spanish}(x)) \wedge \\
\exists^{=21} x\ (\mathrm{German}(x) \wedge \mathrm{Spanish}(x)) \wedge \\
\exists^{=10} x\ (\mathrm{French}(x) \wedge \mathrm{German}(x) \wedge \mathrm{Spanish}(x))
\end{aligned}
$$

Denote the 1-types over the signature French, German, Spanish by $t_\emptyset, t_F, t_G, t_S, t_{FG}, t_{FS}, t_{GS}, t_{FGS}$ (the letters in the subscript indicate the positive subformulas of the type). $\mathcal{E}_\varphi$ contains:

$$x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} = 122$$

$$x_F + x_{FG} + x_{FS} + x_{FGS} = 50$$

$$x_\emptyset = 0$$

$$x_F + x_{FS} = 38$$

Systems of inequalities - Example

$$\begin{aligned}\varphi := \exists^{=122}x\top \wedge\ & \exists^{=50}x\ \text{French}(x) \wedge \exists^{=36}x\ \text{German}(x) \wedge \exists^{=36}x\ \text{Spanish}(x) \wedge \\ & \forall x\ (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \\ & \exists^{=38}x\ (\text{French}(x) \wedge \neg\text{German}(x)) \wedge \\ & \exists^{=18}x\ (\text{French}(x) \wedge \text{Spanish}(x)) \wedge \\ & \exists^{=21}x\ (\text{German}(x) \wedge \text{Spanish}(x)) \wedge \\ & \exists^{=10}x\ (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))\end{aligned}$$

Denote the 1-types over the signature French, German, Spanish by $t_\emptyset, t_F, t_G, t_S, t_{FG}, t_{FS}, t_{GS}, t_{FGS}$ (the letters in the subscript indicate the positive subformulas of the type). $\mathcal{E}_\varphi$ contains:

$$x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} = 122$$

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$$x_\emptyset = 0$$

$$x_F + x_{FS} = 38$$

Systems of inequalities - Example

$$\varphi := \exists^{=122} x \top \wedge \exists^{=50} x \text{ French}(x) \wedge \exists^{=36} x \text{ German}(x) \wedge \exists^{=36} x \text{ Spanish}(x) \wedge$$
$$\forall x \, (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge$$
$$\exists^{=38} x \, (\text{French}(x) \wedge \neg \text{German}(x)) \wedge$$
$$\exists^{=18} x \, (\text{French}(x) \wedge \text{Spanish}(x)) \wedge$$
$$\exists^{=21} x \, (\text{German}(x) \wedge \text{Spanish}(x)) \wedge$$
$$\exists^{=10} x \, (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))$$

Denote the 1-types over the signature French, German, Spanish by $t_\emptyset, t_F, t_G, t_S, t_{FG}, t_{FS}, t_{GS}, t_{FGS}$ (the letters in the subscript indicate the positive subformulas of the type). $\mathcal{E}_\varphi$ contains:

$$x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} = 122$$
$$x_F + x_{FG} + x_{FS} + x_{FGS} = 50$$
$$x_\emptyset = 0$$
$$x_F + x_{FS} = 38$$

Systems of inequalities - Example

$$
\begin{aligned}
\varphi := \exists^{=122} x \top \wedge \exists^{=50} x \text{ French}(x) \wedge \exists^{=36} x \text{ German}(x) \wedge \exists^{=36} x \text{ Spanish}(x) \wedge \\
\forall x \, (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \\
\exists^{=38} x \, (\text{French}(x) \wedge \neg \text{German}(x)) \wedge \\
\exists^{=18} x \, (\text{French}(x) \wedge \text{Spanish}(x)) \wedge \\
\exists^{=21} x \, (\text{German}(x) \wedge \text{Spanish}(x)) \wedge \\
\exists^{=10} x \, (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))
\end{aligned}
$$

Denote the 1-types over the signature French, German, Spanish by $t_\emptyset, t_F, t_G, t_S, t_{FG}, t_{FS}, t_{GS}, t_{FGS}$ (the letters in the subscript indicate the positive subformulas of the type). $\mathcal{E}_\varphi$ contains:

$$
\begin{aligned}
x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} &= 122 \\
x_F + x_{FG} + x_{FS} + x_{FGS} &= 50 \\
x_\emptyset &= 0 \\
x_F + x_{FS} &= 38
\end{aligned}
$$

Systems of inequalities - Example

$$
\begin{aligned}
\varphi := {} & \exists^{=122}x\top \wedge \exists^{=50}x\ \text{French}(x) \wedge \exists^{=36}x\ \text{German}(x) \wedge \exists^{=36}x\ \text{Spanish}(x) \wedge \\
& \forall x\ (\text{French}(x) \vee \text{German}(x) \vee \text{Spanish}(x)) \wedge \\
& \exists^{=38}x\ (\text{French}(x) \wedge \neg\text{German}(x)) \wedge \\
& \exists^{=18}x\ (\text{French}(x) \wedge \text{Spanish}(x)) \wedge \\
& \exists^{=21}x\ (\text{German}(x) \wedge \text{Spanish}(x)) \wedge \\
& \exists^{=10}x\ (\text{French}(x) \wedge \text{German}(x) \wedge \text{Spanish}(x))
\end{aligned}
$$

$2^{|\varphi|}$

$|\varphi|$

Denote the 1-types over the signature French, German, Spanish by $t_\emptyset, t_F, t_G, t_S, t_{FG}, t_{FS}, t_{GS}, t_{FGS}$ (the letters in the subscript indicate the positive subformulas of the type). $\mathcal{E}_\varphi$ contains:

$$
\begin{cases}
x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} = 122 \\
x_F + x_{FG} + x_{FS} + x_{FGS} = 50 \\
x_\emptyset = 0 \\
x_F + x_{FS} = 38
\end{cases}
$$

$\mathcal{E}_\varphi$ FOR OUR EXAMPLE

$$\begin{aligned}
x_\emptyset + x_F + x_G + x_S + x_{FG} + x_{FS} + x_{GS} + x_{FGS} &= 122 \\
x_F + x_{FG} + x_{FS} + x_{FGS} &= 50 \\
x_\emptyset &= 0 \\
x_F + x_{FS} &= 38 \\
x_G + x_{FG} + x_{GS} + x_{FGS} &= 36 \\
x_S + x_{FS} + x_{GS} + x_{FGS} &= 36 \\
x_{FS} + x_{FGS} &= 18 \\
x_{GS} + x_{FGS} &= 21 \\
x_{FGS} &= 10
\end{aligned}$$

Lemma: $\mathcal{E}_\varphi$ has a non-negative integer solution iff φ has a model.

Systems of inequalities - formalized

$$\varphi := \bigwedge_{i=1}^{m} \exists^{\bowtie C_i} x\ \theta_i$$

Let $\sigma = \{P_1, \ldots, P_l\}$. A 1-type (over σ) is any of the formulas:

$$\pm P_1 x \wedge \ldots \wedge \pm P_l x$$

Let $\mathfrak{A}$ be a finite σ-structure and $t_1, \ldots, t_L$ be an enumeration of all 1-types, $L = 2^l$. We characterize $\mathfrak{A}$ by the sequence of natural numbers $(\alpha_1, \ldots, \alpha_L)$ where $a_j = \mid \{a \in A : \mathfrak{A} \models t_j(a)\} \mid$.
The system $\mathcal{E}_\varphi$ contains for each conjunct $\exists^{\bowtie_i C_i} x\ \theta_i$ the inequality:

$$c_{i,1} x_1 + \ldots + c_{i,L} x_L \bowtie_i C_i,$$

where $c_{i,j} = 1$ if the 1-type t_j entails θ_i and $c_{i,j} = 0$, otherwise.

COMPLEXITY OF $\mathcal{C}^1$

Lemma (Reduction property)

$\mathcal{E}_\varphi$ has a non-negative integer solution iff φ has a model. Moreover, every solution of $\mathcal{E}_\varphi$ characterizes some model of φ.

The problem *integer programming* is as follows:

- given: a system $\mathcal{E}$ of linear equations and inequalities
  check whether $\mathcal{E}$ has a solution over $\mathbb{N}$.

Theorem (Borosh and Treybig 1976)

Integer programming is in NPTIME.

$\mathcal{E}_\varphi$ has m inequalities and $L = 2^l$ variables. Recall $m, l \leq |\varphi|$.

$$|\varphi| \left\{ \underbrace{\boxed{}}_{2^{|\varphi|}} \right. \qquad \varphi := \bigwedge_{i=1}^{m} \exists^{\bowtie C_i} x\ \theta_i$$

Optimal Complexity for $\mathcal{C}^1$

$\varphi := \bigwedge_{i=1}^{m} \exists^{\bowtie C_i} x\ \theta_i$; $\mathcal{E}_\varphi$: m inequalities, $L = 2^l$ variables.

Lemma (linear algebra)

If $\mathcal{E}_\varphi$ has a solution over $\mathbb{N}$, then $\mathcal{E}_\varphi$ has a solution over $\mathbb{N}$ with at most $m \log(L+1)$ non-zero entries.

polynomial in $|\varphi|$

Corollary

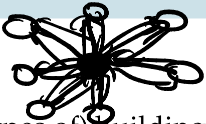
SAT($\mathcal{C}^1$)$\in$ NP.

Proof.

Let $C = \max\{C_i : 1 \leq i \leq m\}$. If $(\alpha_1, \ldots, \alpha_L)$ is a solution of $\mathcal{E}_\varphi$, then so is $(\beta_1, \ldots, \beta_L)$, where $\beta_j = \min(\alpha_j, C)$.
The linear algebra Lemma allows one to first guess a polynomial number of non-zero variables and write down the system $\mathcal{E}_\varphi$ only over these variables; since Integer Programming is in NPTIME, solutions of such systems can be guessed and verified in time bounded by a polynomial function of $|\varphi|$. □

Reduction to Integer Programming

$C^2 \quad \exists^{=1}_y R(x,y)$



Ian Pratt Hartmann

Idea: Depending on the logic:
identify (finitely many types of) building blocks of a potential model and connecting conditions for them, describe them in a succinct way by a set of (in)equalities.

Advantages:

- Useful for solving *simultaneously* SAT and FINSAT.
  We look for solutions over $\mathbb{N}$ (FINSAT) or over $\mathbb{N} \cup \{\infty\}$ (SAT), e.g.

$$x + 1 = x$$

  has a solution $x = \infty$.
- Gives better (optimal) complexity bounds.

We will see more about this approach later in the course.

FO^2 : Two-variable fragment of FO

x, y
relational symbols of arity ≤ 2
no constants

NExp-complete
FMP

Exponential model property

Def. Normal forms for FO^2
(Scott)

$$\varphi = \forall x \forall y\, \psi \wedge \bigwedge_{i=1}^{n} \forall x \exists y\, \underbrace{\psi_i(x,y)}_{\text{quantifier-free}}$$

quantifier free

computable in PTime

$$\forall y \left(S(y) \Longleftrightarrow \exists x \left(A(x) \wedge B(y)\right)\right) \equiv$$

$$\forall y \left[\left(S(y) \longrightarrow \exists x \ A(x) \wedge B(y)\right)\right] \wedge \left[\left(\exists x \ A(x) \wedge B(y)\right) \rightarrow S(y)\right] \equiv$$

$$\equiv \left(\forall x \ \exists y \ S(\cancel{x}) \rightarrow A(y) \wedge B(\cancel{x})\right) \wedge \left(\forall y \left(\exists x \ A(x) \wedge B(y)\right) \rightarrow S(y)\right)$$

$$\equiv$$

$$\forall y \ \neg(\text{ —"— }) \vee S(y)$$

$$\equiv$$

$$\forall y \left(\forall x \ \neg A(x) \vee \neg B(y) \right) \vee S(y)$$

$$\equiv$$

$$\forall x \ \forall y \ \left(\neg A(x) \vee \neg B(y) \vee S(y)\right)$$