TECHNISCHE UNIVERSITÄT DRESDEN

# COMPLEXITY THEORY

**Lecture 10: Polynomial Space**

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## The Class PSpace

We defined PSpace as:

$$\text{PSpace} = \bigcup_{d \geq 1} \text{DSpace}(n^d)$$

and we observed that

$$\text{P} \subseteq \text{NP} \subseteq \text{PSpace} = \text{NPSpace} \subseteq \text{ExpTime}.$$

We can also define a corresponding notion of PSpace-hardness:

**Definition 10.1:**

- A language **H** is PSpace-hard, if $\mathbf{L} \leq_p \mathbf{H}$ for every language $\mathbf{L} \in \text{PSpace}$.
- A language **C** is PSpace-complete, if **C** is PSpace-hard and $\mathbf{C} \in \text{PSpace}$.



## Quantified Boolean Formulae (QBF)

A QBF is a formula of the following form:

$$Q_1 X_1 . Q_2 X_2 . \cdots Q_\ell X_\ell . \varphi[X_1, \ldots, X_\ell]$$

where $Q_i \in \{\exists, \forall\}$ are quantifiers, $X_i$ are propositional logic variables, and $\varphi$ is a propositional logic formula with variables $X_1, \ldots, X_\ell$ and constants $\top$ (true) and $\bot$ (false)

**Semantics:**

- Propositional formulae without variables (only constants $\top$ and $\bot$) are evaluated as usual
- $\exists X . \varphi[X]$ is true if either $\varphi[X/\top]$ or $\varphi[X/\bot]$ are true
- $\forall X . \varphi[X]$ is true if both $\varphi[X/\top]$ and $\varphi[X/\bot]$ are true

  (where $\varphi[X/\top]$ is "$\varphi$ with $X$ replaced by $\top$, and similar for $\bot$)



## Deciding QBF Validity

**TRUE QBF**

| | |
|---|---|
| Input: | A quantified Boolean formula $\varphi$. |
| Problem: | Is $\varphi$ true (valid)? |

**Observation:** We can assume that the quantified formula is in CNF or 3-CNF (same transformations possible as for propositional logic formulae)

Consider a propositional logic formula $\varphi$ with variables $X_1, \ldots, X_\ell$:

**Example 10.2:** The QBF $\exists X_1 . \cdots \exists X_\ell . \varphi$ is true if and only if $\varphi$ is satisfiable.

**Example 10.3:** The QBF $\forall X_1 . \cdots \forall X_\ell . \varphi$ is true if and only if $\varphi$ is a tautology.

## The Power of QBF

**Theorem 10.4: TRUE QBF** is PSpace-complete.

**Proof:**

(1) **TRUE QBF** $\in$ PSpace:
Give an algorithm that runs in polynomial space.

(2) **TRUE QBF** is PSpace-hard:
Proof by reduction from the word problem for polynomially space-bounded TMs.

□



## Solving TRUE QBF in PSpace

```
01 TrueQBF(φ) {
02   if φ has no quantifiers :
03     return "evaluation of φ"
04   else if φ = ∃X.ψ :
05     return (TrueQBF(ψ[X/⊤]) OR TrueQBF(ψ[X/⊥]))
06   else if φ = ∀X.ψ :
07     return (TrueQBF(ψ[X/⊤]) AND TrueQBF(ψ[X/⊥]))
08 }
```

- Evaluation in line 03 can be done in polynomial space
- Recursions in lines 05 and 07 can be executed one after the other, reusing space
- Maximum depth of recursion = number of variables (linear)
- Store one variable assignment per recursive call

$\rightsquigarrow$ polynomial space algorithm



## PSpace-Hardness of TRUE QBF

Express TM computation in logic, similar to Cook-Levin

### Given:

- a polynomial $p$
- a $p$-space bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$
- a word $w$

### Intended reduction

Define a QBF $\varphi_{p,\mathcal{M},w}$ such that
$\varphi_{p,\mathcal{M},w}$ is true if and only if $\mathcal{M}$ accepts $w$ in space $p(|w|)$.

### Note

We show the reduction for NTMs, which is more than needed, but makes little difference in logic and allows us to reuse our previous formulae from Cook-Levin



## Review: Encoding Configurations

### Use propositional variables for describing configurations:

$Q_q$ for each $q \in Q$ means "$\mathcal{M}$ is in state $q \in Q$"

$P_i$ for each $0 \leq i < p(n)$ means "the head is at Position $i$"

$S_{a,i}$ for each $a \in \Gamma$ and $0 \leq i < p(n)$ means "tape cell $i$ contains Symbol $a$"

### Represent configuration $(q, p, a_0 \dots a_{p(n)})$

by assigning truth values to variables from the set

$$\overline{C} := \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

using the truth assignment $\beta$ defined as

$$\beta(Q_s) := \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases} \qquad \beta(P_i) := \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \qquad \beta(S_{a,i}) := \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

## Review: Validating Configurations

We define a formula $\mathrm{Conf}(\overline{C})$ for a set of configuration variables

$$\overline{C} = \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

as follows:

$\mathrm{Conf}(\overline{C})$ := "the assignment is a valid configuration":

$$\bigvee_{q \in Q} \Big( Q_q \wedge \bigwedge_{q' \neq q} \neg Q_{q'} \Big)$$ "TM in exactly one state $q \in Q$"

$$\wedge \bigvee_{p < p(n)} \Big( P_p \wedge \bigwedge_{p' \neq p} \neg P_{p'} \Big)$$ "head in exactly one position $p < p(n)$"

$$\wedge \bigwedge_{0 \leq i < p(n)} \bigvee_{a \in \Gamma} \Big( S_{a,i} \wedge \bigwedge_{b \neq a \in \Gamma} \neg S_{b,i} \Big)$$ "exactly one $a \in \Gamma$ in each cell"



## Review: Validating Configurations

For an assignment $\beta$ defined on variables in $\overline{C}$ define

$$\mathrm{conf}(\overline{C}, \beta) := \left\{ (q, p, w_0 \dots w_{p(n)}) \;\middle|\; \begin{array}{l} \beta(Q_q) = 1, \\ \beta(P_p) = 1, \\ \beta(S_{w_i,i}) = 1 \text{ for all } 0 \leq i < p(n) \end{array} \right\}$$

Note: $\beta$ may be defined on other variables besides those in $\overline{C}$.

**Lemma 10.5:** If $\beta$ satisfies $\mathrm{Conf}(\overline{C})$ then $|\mathrm{conf}(\overline{C}, \beta)| = 1$.
We can therefore write $\mathrm{conf}(\overline{C}, \beta) = (q, p, w)$ to simplify notation.

Observations:

- $\mathrm{conf}(\overline{C}, \beta)$ is a potential configuration of $\mathcal{M}$, but it may not be reachable from the start configuration of $\mathcal{M}$ on input $w$.
- Conversely, every configuration $(q, p, w_1 \dots w_{p(n)})$ induces a satisfying assignment $\beta$ or which $\mathrm{conf}(\overline{C}, \beta) = (q, p, w_1 \dots w_{p(n)})$.



## Review: Transitions Between Configurations

Consider the following formula $\mathrm{Next}(\overline{C}, \overline{C}')$ defined as

$$\mathrm{Conf}(\overline{C}) \wedge \mathrm{Conf}(\overline{C}') \wedge \mathrm{NoChange}(\overline{C}, \overline{C}') \wedge \mathrm{Change}(\overline{C}, \overline{C}').$$

$$\mathrm{NoChange} := \bigvee_{0 \leq p < p(n)} \Big( P_p \wedge \bigwedge_{i \neq p, a \in \Gamma} (S_{a,i} \to S'_{a,i}) \Big)$$

$$\mathrm{Change} := \bigvee_{0 \leq p < p(n)} \Big( P_p \wedge \bigvee_{\substack{q \in Q \\ a \in \Gamma}} (Q_q \wedge S_{a,p} \wedge \bigvee_{(q', b, D) \in \delta(q,a)} (Q'_{q'} \wedge S'_{b,p} \wedge P'_{D(p)})) \Big)$$

where $D(p)$ is the position reached by moving in direction $D$ from $p$.

**Lemma 10.6:** For any assignment $\beta$ defined on $\overline{C} \cup \overline{C}'$:
$\beta$ satisfies $\mathrm{Next}(\overline{C}, \overline{C}')$ if and only if $\mathrm{conf}(\overline{C}, \beta) \vdash_{\mathcal{M}} \mathrm{conf}(\overline{C}', \beta)$



## Review: Start and End

### Defined so far:

- $\mathrm{Conf}(\overline{C})$: $\overline{C}$ describes a potential configuration
- $\mathrm{Next}(\overline{C}, \overline{C}')$: $\mathrm{conf}(\overline{C}, \beta) \vdash_{\mathcal{M}} \mathrm{conf}(\overline{C}', \beta)$

### Start configuration: Let $w = w_0 \cdots w_{n-1} \in \Sigma^*$ be the input word

$$\mathrm{Start}_{\mathcal{M},w}(\overline{C}) := \mathrm{Conf}(\overline{C}) \wedge Q_{q_0} \wedge P_0 \wedge \textstyle\bigwedge_{i=0}^{n-1} S_{w_i,i} \wedge \bigwedge_{i=n}^{p(n)-1} S_{\sqcup,i}$$

Then an assignment $\beta$ satisfies $\mathrm{Start}_{\mathcal{M},w}(\overline{C})$ if and only if $\overline{C}$ represents the start configuration of $\mathcal{M}$ on input $w$.

### Accepting stop configuration:

$$\text{Acc-Conf}(\overline{C}) := \mathrm{Conf}(\overline{C}) \wedge Q_{q_{\text{accept}}}$$

Then an assignment $\beta$ satisfies $\text{Acc-Conf}(\overline{C})$ if and only if $\overline{C}$ represents an accepting configuration of $\mathcal{M}$.

## Simulating Polynomial Space Computations

For Cook-Levin, we used one set of configuration variables for every computating step:
polynomially time $\rightsquigarrow$ polynomially many variables

Problem: For polynomial space, we have $2^{O(p(n))}$ possible steps ...

**What would Savitch do?**

Define a formula $\text{CanYield}_i(\overline{C}_1, \overline{C}_2)$ to state that $\overline{C}_2$ is reachable from $\overline{C}_1$ in at most $2^i$ steps:

$$\text{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \text{Next}(\overline{C}_1, \overline{C}_2)$$
$$\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\text{Conf}(\overline{C}) \wedge \text{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \text{CanYield}_i(\overline{C}, \overline{C}_2)$$

But what is $\overline{C}_1 = \overline{C}_2$ supposed to mean here? It is short for:

$$\bigwedge_{q \in Q} Q^1_q \leftrightarrow Q^2_q \wedge \bigwedge_{0 \le i < p(n)} P^1_i \leftrightarrow P^2_i \wedge \bigwedge_{a \in \Gamma, 0 \le i < p(n)} S^1_{a,i} \leftrightarrow S^2_{a,i}$$



## Putting Everything Together

We define the formula $\varphi_{p,\mathcal{M},w}$ as follows:

$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\text{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \text{Acc-Conf}(\overline{C}_2) \wedge \text{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

where we select $d$ to be the least number such that $\mathcal{M}$ has less than $2^{dp(n)}$ configurations in space $p(n)$.

**Lemma 10.7:** $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if $\mathcal{M}$ accepts $w$ in space $p(|w|)$.



## Did we do it?

Note: we used only existential quantifiers when defining $\varphi_{p,\mathcal{M},w}$:

$$\text{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \text{Next}(\overline{C}_1, \overline{C}_2)$$
$$\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\text{Conf}(\overline{C}) \wedge \text{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \text{CanYield}_i(\overline{C}, \overline{C}_2)$$
$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\text{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \text{Acc-Conf}(\overline{C}_2) \wedge \text{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

Now that's quite interesting ...

- With only (non-negated) $\exists$ quantifiers, **True QBF** coincides with **Sat**
- **Sat** is in NP
- So we showed that the word problem for PSpace NTMs to be in NP

So we found that NP = PSpace!

Strangely, most textbooks claim that this is not known to be true ...
Are we up for the next Turing Award, or did we make a mistake?



## Size

How big is $\varphi_{p,\mathcal{M},w}$?

$$\text{CanYield}_0(\overline{C}_1, \overline{C}_2) := (\overline{C}_1 = \overline{C}_2) \vee \text{Next}(\overline{C}_1, \overline{C}_2)$$
$$\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) := \exists \overline{C}.\text{Conf}(\overline{C}) \wedge \text{CanYield}_i(\overline{C}_1, \overline{C}) \wedge \text{CanYield}_i(\overline{C}, \overline{C}_2)$$
$$\varphi_{p,\mathcal{M},w} := \exists \overline{C}_1.\exists \overline{C}_2.\text{Start}_{\mathcal{M},w}(\overline{C}_1) \wedge \text{Acc-Conf}(\overline{C}_2) \wedge \text{CanYield}_{dp(n)}(\overline{C}_1, \overline{C}_2)$$

Size of $\text{CanYield}_{i+1}$ is more than twice the size of $\text{CanYield}_i$
$\rightsquigarrow$ Size of $\varphi_{p,\mathcal{M},w}$ is in $2^{O(p(n))}$. Oops.

A correct reduction: We redefine CanYield by setting

$$\text{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) :=$$
$$\exists \overline{C}.\text{Conf}(\overline{C}) \wedge$$
$$\forall \overline{Z}_1.\forall \overline{Z}_2.(((\overline{Z}_1 = \overline{C}_1 \wedge \overline{Z}_2 = \overline{C}) \vee (\overline{Z}_1 = \overline{C} \wedge \overline{Z}_2 = \overline{C}_2)) \rightarrow \text{CanYield}_i(\overline{Z}_1, \overline{Z}_2))$$

## Size

Let's analyse the size more carefully this time:

$$\begin{aligned}&\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2) :=\\ &\exists \overline{C}.\mathsf{Conf}(\overline{C}) \wedge\\ &\forall \overline{Z}_1.\forall \overline{Z}_2.(((\overline{Z}_1 = \overline{C}_1 \wedge \overline{Z}_2 = \overline{C}) \vee (\overline{Z}_1 = \overline{C} \wedge \overline{Z}_2 = \overline{C}_2)) \rightarrow \mathsf{CanYield}_i(\overline{Z}_1, \overline{Z}_2))\end{aligned}$$

- $\mathsf{CanYield}_{i+1}(\overline{C}_1, \overline{C}_2)$ extends $\mathsf{CanYield}_i(\overline{C}_1, \overline{C}_2)$ by parts that are linear in the size of configurations $\rightsquigarrow$ growth in $O(p(n))$
- Maximum index $i$ used in $\varphi_{p,\mathcal{M},w}$ is $dp(n)$, that is in $O(p(n))$
- Therefore: $\varphi_{p,\mathcal{M},w}$ has size $O(p^2(n))$ – and thus can be computed in polynomial time

### Exercise:

Why can we just use $dp(n)$ in the reduction? Don't we have to compute it somehow? Maybe even in polynomial time?



## The Power of QBF

**Theorem 10.4: True QBF** is PSpace-complete.

**Proof:**

(1) **True QBF** $\in$ PSpace:
Give an algorithm that runs in polynomial space.

(2) **True QBF** is PSpace-hard:
Proof by reduction from the word problem for polynomially space-bounded TMs.

□



## A More Common Logical Problem in PSpace

Recall standard first-order logic:

- Instead of propositional variables, we have atoms (predicates with constants and variables)
- Instead of propositional evaluations we have first-order structures (or interpretations)
- First-order quantifiers can be used on variables
- Sentences are formulae where all variables are quantified
- A sentence can be satisfied or not by a given first-order structure

**FOL Model Checking**

Input: A first-order sentence $\varphi$ and a finite first-order structure $\mathcal{I}$.

Problem: Is $\varphi$ satisfied by $\mathcal{I}$?



## First-Order Logic is PSpace-complete

**Theorem 10.8: FOL Model Checking** is PSpace-complete.

**Proof:**

(1) **FOL Model Checking** $\in$ PSpace:
Give algorithm that runs in polynomial space.

(2) **FOL Model Checking** is PSpace-hard:
Proof by reduction **True QBF** $\leq_p$ **FOL Model Checking**.

□

## Checking FOL Models in Polynomial Space (Sketch)

```
01 Eval(φ, I) {
02   switch (φ) :
03     case p(c_1, ..., c_n) :  return ⟨c_1, ..., c_n⟩ ∈ p^I
04     case ¬ψ :  return NOT Eval(ψ, I)
05     case ψ_1 ∧ ψ_2 :  return Eval(ψ_1, I) AND Eval(ψ_2, I)
06     case ∃x.ψ :
07       for c ∈ Δ^I :
08         if Eval(ψ[x ↦ c], I) :  return TRUE
09       // eventually, if no success:
10       return FALSE
11 }
```

- We can assume $\varphi$ only uses $\neg$, $\wedge$ and $\exists$ (easy to get)
- We use $\Delta^{\mathcal{I}}$ to denote the (finite!) domain of $\mathcal{I}$
- We allow domain elements to be used like constants in the formula



## Hardness of **FOL Model Checking**

Given: a QBF $\varphi = Q_1 X_1. \cdots Q_\ell X_\ell. \psi$

FOL Model Checking Problem:

- Interpretation domain $\Delta^{\mathcal{I}} := \{0, 1\}$
- Single predicate symbol $\mathrm{true}$ with interpretation $\mathrm{true}^{\mathcal{I}} = \{\langle 1 \rangle\}$
- FOL formula $\varphi'$ is obtained by replacing variables in input QBF with corresponding first-order expressions:

$$Q_1 x_1. \cdots Q_\ell x_\ell. \psi[X_1 \mapsto \mathrm{true}(x_1), \ldots, X_\ell \mapsto \mathrm{true}(x_\ell)]$$

**Lemma 10.9:** $\langle \mathcal{I}, \varphi' \rangle \in$ **FOL Model Checking** if and only if $\varphi \in$ **True QBF**.



## First-Order Logic is PSpace-complete

**Theorem 10.8:** **FOL Model Checking** is PSpace-complete.

**Proof:**

(1) **FOL Model Checking** $\in$ PSpace:
Give algorithm that runs in polynomial space.

(2) **FOL Model Checking** is PSpace-hard:
Proof by reduction **True QBF** $\leq_p$ **FOL Model Checking**.

□



## **FOL Model Checking**: Practical Significance

Why is **FOL Model Checking** a relevant problem?

Correspondence with database query answering:

- Finite first-order interpretation = database
- First-order logic formula = database query
- Satisfying assignments (for non-sentences) = query results

Known correspondence:
As a query language, FOL has the same expressive power as (basic) SQL (relational algebra).

**Corollary 10.10:** Answering SQL queries over a given database is PSpace-complete.

# Games



## Games as Computational Problems

Many single-player games relate to NP-complete problems:

- Sudoku
- Minesweeper
- Tetris
- ...

Decision problem: Is there a solution?
(For Tetris: is it possible to clear all blocks?)

What about two-player games?

- Two players take moves in turns
- The players have different goals
- The game ends if a player wins

Decision problem: Does Player 1 have a winning strategy?
In other words: can Player 1 enforce winning, whatever Player 2 does?



## Example: The Formula Game

A contrived game, to illustrate the idea:

- Given: a propositional logic formula $\varphi$ with consecutively numbered variables $X_1, \dots X_\ell$.
- Two players take turns in selecting values for the next variable:
  - Player 1 sets $X_1$ to true or false
  - Player 2 sets $X_2$ to true or false
  - Player 1 sets $X_3$ to true or false
  - ...

  until all variables are set.
- Player 1 wins if the assignment makes $\varphi$ true.
  Otherwise, Player 2 wins.



## Deciding the Formula Game

**Formula Game**

Input: A formula $\varphi$.

Problem: Does Player 1 have a winning strategy on $\varphi$?

**Theorem 10.11:** **Formula Game** is PSpace-complete.

**Proof sketch:** **Formula Game** is essentially the same as **True QBF**.

Having a winning strategy means: there is a truth value for $X_1$, such that, for all truth values of $X_2$, there is a truth value of $X_3$, ... such that $\varphi$ becomes true.

If we have a QBF where quantifiers do not alternate, we can add dummy quantifiers and variables that do not change the semantics to get the same alternating form as for the Formula Game. □

## Example: The Geography Game

A children's game:

- Two players are taking turns naming cities.
- Each city must start with the last letter of the previous.
- Repetitions are not allowed.
- The first player who cannot name a new city looses.

A mathematicians' game:

- Two players are marking nodes on a directed graph.
- Each node must be a successor of the previous one.
- Repetitions are not allowed.
- The first player who cannot mark a new node looses.

Decision problem **(Generalised) Geography**:
given a graph and start node, does Player 1 have a winning strategy?



## Geography is PSpace-hard

Let $\varphi$ with variables $X_1, \ldots, X_\ell$ be an instance of **Formula Game**.
Without loss of generality, we assume:

- $\ell$ is odd (Player 1 gets the first and last turn)
- $\varphi$ is in CNF

We now build a graph that encodes **Formula Game** in terms of **Geography**

- The left-hand side of the graph is a chain of diamond structures that represent the choices that players have when assigning truth values
- The right-hand side of the graph encodes the structure of $\varphi$: Player 2 may choose a clause (trying to find one that is not true under the assignment); Player 1 may choose a literal (trying to find one that is true under the assignment).

(see board or [Sipser, Theorem 8.14]) □



## Geography is PSpace-complete

**Theorem 10.12:** **Generalised Geography** is PSpace-complete.

**Proof:**

(1) **Geography** $\in$ PSpace:
Give algorithm that runs in polynomial space.
It is not difficult to provide a recursive algorithm similar to the one for **True QBF** or **FOL Model Checking**.

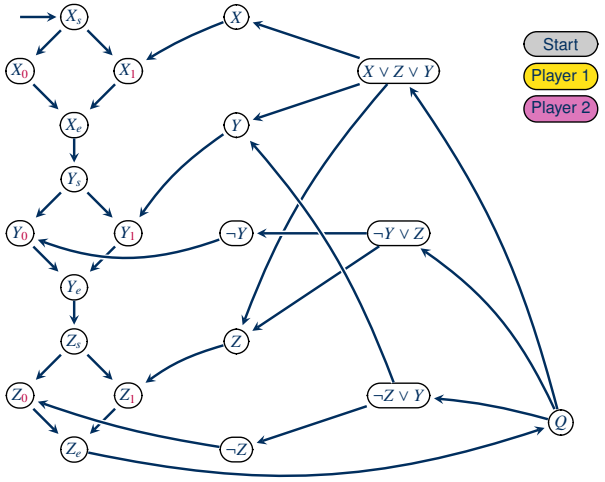
(2) **Geography** is PSpace-hard:
Proof by reduction **Formula Game** $\leq_p$ **Geography**.

□



## Geography is PSpace-hard: Example

We consider the formula $\exists X.\forall Y.\exists Z.(X \vee Z \vee Y) \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y)$



# Summary and Outlook

**True QBF** is PSpace-complete

**FOL Model Checking** and the related problem of SQL query answering are PSpace-complete

Some games are PSpace-complete

**What's next?**

- Some more remarks on games
- Logarithmic space
- Complements of space classes