Fuzzy Ontologies over Lattices with T-norms

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1 Introduction

In some knowledge domains, a correct handling of vagueness and imprecision is fundamental for adequate knowledge representation and reasoning. For example, when trying to diagnose a disease, medical experts need to confront symptoms described by the patient, which are by definition subjective, and hence vague. Moreover, a single malady may present a diversity of clinical manifestations in different patients, which leads to imprecise (partial) diagnoses.

Fuzzy logic [15] is a prominent approach for dealing with imprecise knowledge. It is based on the notion of fuzzy sets [25], where elements are assigned a membership degree from the real interval $[0, 1]$. So-called t-norms are used to define the interpretation of the logical connectives. The notion of membership degrees and the operators used can be generalized to lattices, giving rise to L-fuzzy sets [13] and lattice-based t-norms [26, 12].

During the last two decades, several fuzzy DLs have been defined by enriching classical DLs first with fuzzy set semantics [24, 20, 19] and then t-norms [16, 7, 11]. Attempts have also been made at using L-fuzzy set semantics [21, 17]. However, all these approaches either disregard the terminological knowledge, or allow only for a limited class of TBoxes. In fact, it is still unknown whether standard reasoning in fuzzy DLs with general TBoxes is decidable [5, 3]. To the best of our knowledge, the only approaches capable of dealing with full fuzzy TBoxes are based on a finite total order with the Łukasiewicz t-norm [6, 8] or finite De Morgan lattices with the minimum t-norm [9].

In this paper we introduce the lattice-based fuzzy DL $\mathcal{ALC}_L$, where L is a complete De Morgan lattice equipped with a t-norm operator. We show that satisfiability in this logic is undecidable if L is infinite. Undecidability holds even if L is a countable, residuated total order. On the other hand, if L is finite, then satisfiability becomes decidable and, under some conditions on the lattice and the t-norm, ExpTime-complete, i.e. not harder than satisfiability in crisp $\mathcal{ALC}$.

Our reasoning procedure is in fact general enough to handle any kind of truth-functional semantics, as long as the functions defining the connectives are computable.

2 Lattices

We now give a brief introduction to lattices and t-norms. For a more comprehensive description of these notions, see e.g. [14, 12].

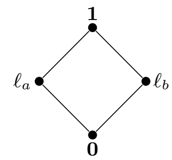

Fig. 1. The De Morgan lattice L_2 with $\sim\ell_a = \ell_a$ and $\sim\ell_b = \ell_b$. This lattice was first considered by Belnap [4] for reasoning with incomplete and inconsistent knowledge.

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ over a *carrier set* L with two binary operations supremum $\vee$ and infimum $\wedge$ that are idempotent, associative, and commutative and satisfy the absorption laws $\ell \vee (\ell \wedge m) = \ell = \ell \wedge (\ell \vee m)$ for all $\ell, m \in L$. The order $\leq$ on L is defined by $\ell \leq m$ iff $\ell \wedge m = \ell$ for all $\ell, m \in L$. A lattice is *distributive* if $\vee$ and $\wedge$ distribute over each other, *finite* if L is finite, and *bounded* if it has a *minimum* and a *maximum* element, denoted as $\mathbf{0}$ and $\mathbf{1}$, respectively. It is *complete* if suprema and infima of arbitrary subsets $T \subseteq L$ exist; these are denoted by $\bigvee_{t \in T} t$ and $\bigwedge_{t \in T} t$, respectively. Notice that every finite lattice is also bounded and complete. Whenever it is clear from the context, we will simply use the carrier set L to represent the lattice $(L, \vee, \wedge)$.

A *De Morgan lattice* is a bounded distributive lattice extended with an involutive and anti-monotonic unary operation $\sim$, called *(De Morgan) negation*, satisfying the De Morgan laws $\sim(\ell \vee m) = \sim\ell \wedge \sim m$ and $\sim(\ell \wedge m) = \sim\ell \vee \sim m$ for all $\ell, m \in L$. Figure 1 shows a simple De Morgan lattice.

In fuzzy logics, conjunctions and disjunctions are interpreted with the help of t-norms and t-conorms. Given a De Morgan lattice L, a *t-norm on* L is an associative and commutative binary operator $\otimes : L \times L \to L$ which has the unit $\mathbf{1}$, and is monotonic in both arguments. Given a t-norm $\otimes$, its associated *t-conorm* $\oplus$ is constructed using the negation as follows: $\ell \oplus m := \sim(\sim\ell \otimes \sim m)$. For example, the infimum operator $\ell \otimes m := \ell \wedge m$ defines a t-norm; its associated t-conorm is then given by $\ell \oplus m := \ell \vee m$.

Another important operator is the residuum, which is used for interpreting implications in the logic. The *residuum* of a t-norm $\otimes$ on a complete lattice L is the binary operator $\Rightarrow$ defined by $\ell \Rightarrow m := \bigvee\{x \mid \ell \otimes x \leq m\}$. If $\ell \otimes (\ell \Rightarrow m) \leq m$ for all $\ell, m \in L$ (that is, if the supremum in the definition of residuum is always a maximum), then $\otimes$ is called *residuated* and L a *residuated lattice*.[1]

In the following we will use two important properties of the residuum: for every $\ell, m \in L$, (i) $\mathbf{1} \Rightarrow \ell = \ell$, and (ii) if $\ell \leq m$, then $\ell \Rightarrow m = \mathbf{1}$. Additionally, if $\otimes$ is residuated, then $\ell \Rightarrow m = \mathbf{1}$ implies that $\ell \leq m$.

In the next section, we describe the multi-valued description logic $\mathcal{ALC}_L$, whose semantics uses the residuum $\Rightarrow$ and the De Morgan negation $\sim$. We emphasize, however, that the reasoning algorithm presented in Section 5 can be used with any choice of operators, as long as these are computable. In particular this means that our algorithm could also deal with other variants of multi-valued semantics, e.g. [9, 21].

[1] Residua are usually only defined for residuated lattices. However, as $\ell \Rightarrow m$ is well-defined for t-norms on complete De Morgan lattices, we remove this restriction.

3 The Fuzzy Logic $\mathcal{ALC}_L$

In the following, we will assume that L is a complete De Morgan lattice and $\otimes$ is a t-norm on L. The multi-valued description logic $\mathcal{ALC}_L$ is a generalization of the crisp DL $\mathcal{ALC}$ that allows the use of the elements of a complete De Morgan lattice as truth values, instead of just the Boolean values *true* and *false*. The syntax of concept descriptions in $\mathcal{ALC}_L$ is the same as in $\mathcal{ALC}$; that is, $\mathcal{ALC}_L$ concept descriptions are built from a set of concept names and role names through the constructors $\sqcap, \sqcup, \neg, \top, \bot, \exists$ and $\forall$.

The semantics of this logic is based on interpretation functions that not simply describe whether an element of the domain belongs to a concept or not, but give a lattice value describing the membership degree of the element to this concept; more formally, the semantics is based on L-fuzzy sets.

Definition 1 (semantics of $\mathcal{ALC}_L$). *An* interpretation *is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty (crisp)* domain *and $\cdot^\mathcal{I}$ is a function that assigns to every concept name A and every role name r functions $A^\mathcal{I} : \Delta^\mathcal{I} \to L$ and $r^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to L$, respectively. The function $\cdot^\mathcal{I}$ is extended to $\mathcal{ALC}_L$ concept descriptions as follows for every $x \in \Delta^\mathcal{I}$:*

- $\top^\mathcal{I}(x) = \mathbf{1}$, $\bot^\mathcal{I}(x) = \mathbf{0}$,
- $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$, $(C \sqcup D)^\mathcal{I}(x) = C^\mathcal{I}(x) \oplus D^\mathcal{I}(x)$,
- $(\neg C)^\mathcal{I}(x) = \sim C^\mathcal{I}(x)$,
- $(\exists r.C)^\mathcal{I}(x) = \bigvee_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x,y) \otimes C^\mathcal{I}(y)$,
- $(\forall r.C)^\mathcal{I}(x) = \bigwedge_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x,y) \Rightarrow C^\mathcal{I}(y)$.

Notice that, unlike crisp $\mathcal{ALC}$, the existential and universal quantifiers are not dual to each other, i.e. in general $\neg\exists r.C$ and $\forall r.\neg C$ have different semantics.

The axioms in a TBox are also associated to a lattice value, allowing for a general notion of subsumption between concepts that is based on the residuum.

Definition 2 (TBox). *A* TBox *is a finite set of* (labeled) general concept inclusions *(GCIs) of the form $\langle C \sqsubseteq D, \ell \rangle$, where C, D are $\mathcal{ALC}_L$ concept descriptions and $\ell \in L$.*

An interpretation $\mathcal{I}$ satisfies *a GCI $\langle C \sqsubseteq D, \ell \rangle$ if $\bigwedge_{x \in \Delta^\mathcal{I}} C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq \ell$. $\mathcal{I}$ is called a* model *of the TBox $\mathcal{T}$ if it satisfies all axioms in $\mathcal{T}$.*

We emphasize here that $\mathcal{ALC}$ is a special case of $\mathcal{ALC}_L$, where the underlying lattice contains only the elements $\mathbf{0}$ and $\mathbf{1}$, which may be interpreted as *false* and *true*, respectively, and the t-norm and t-conorm are just conjunction and disjunction, respectively. Accordingly, one can think of generalizing the reasoning problems for $\mathcal{ALC}$ to the use of other lattices. We will focus on the problem of deciding satisfiability of a concept. We are further interested in computing the highest degree with which an individual may belong to a concept.

Definition 3 (satisfiability). *Let C, D be $\mathcal{ALC}_L$ concept descriptions, $\mathcal{T}$ a TBox and $\ell \in L$. C is ℓ-*satisfiable *w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $\bigvee_{x \in \Delta^\mathcal{I}} C^\mathcal{I}(x) \geq \ell$. The* best satisfiability degree *for C w.r.t. $\mathcal{T}$ is the largest ℓ such that C is ℓ-satisfiable w.r.t. $\mathcal{T}$.*

Notice that if C is ℓ-satisfiable and ℓ'-satisfiable w.r.t. $\mathcal{T}$, then C is also $\ell \vee \ell'$-satisfiable. Hence, the notion of the best satisfiability degree is well defined.

In some cases, however, this definition of satisfiability turns out to be too weak, since a concept C may be ℓ-satisfiable even if no element of the domain may ever belong to C with a value $\geq \ell$. Consider the following example.

Example 4. We use the lattice L_2 from Figure 1 with t-norm $\ell \otimes \ell' := \ell \wedge \ell'$ and the TBox $\mathcal{T} = \{\langle \top \sqsubseteq (A \sqcap \neg A) \sqcup (B \sqcap \neg B), \mathbf{1} \rangle\}$. The concept A is $\mathbf{1}$-satisfiable w.r.t. $\mathcal{T}$ since the interpretation $\mathcal{I}_0 = (\{x_1, x_2\}, \cdot^{\mathcal{I}_0})$ with

$$A^{\mathcal{I}_0}(x_1) = B^{\mathcal{I}_0}(x_2) = \ell_a \text{ and } B^{\mathcal{I}_0}(x_1) = A^{\mathcal{I}_0}(x_2) = \ell_b$$

is a model of $\mathcal{T}$ and $\ell_a \vee \ell_b = \mathbf{1}$. However, since $\ell \wedge {\sim}\ell \neq \mathbf{1}$ for every $\ell \in L_2$, the axiom can only be satisfied for any $y \in \Delta^{\mathcal{I}}$ if $\{A^{\mathcal{I}}(y), B^{\mathcal{I}}(y)\} = \{\ell_a, \ell_b\}$. Thus, we always have $A^{\mathcal{I}}(y) < \mathbf{1}$.

For this reason, we consider a stronger notion of satisfiability that requires at least one element of the domain to satisfy the concept with the given value. A concept C is *strongly ℓ-satisfiable* w.r.t. a TBox $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ and an $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) \geq \ell$. Obviously, strong ℓ-satisfiability implies ℓ-satisfiability. As shown in Example 4, the converse does not hold.

Recall that the semantics of the quantifiers require the computation of a supremum or infimum of the membership degrees of a possibly infinite set of elements of the domain. If the lattice is finite, then this is in fact a computation over a finite set of values, but it may be a costly one. If the lattice is infinite, then the problem is more pronounced. For that reason, it is customary in fuzzy description logics to restrict reasoning to witnessed models [16].

Definition 5 (witnessed model). *Let $\eta \in \mathbb{N}$. A model $\mathcal{I}$ of a TBox $\mathcal{T}$ is called* η-witnessed *if for every $x \in \Delta^{\mathcal{I}}$ and every concept description of the form $\exists r.C$ there are η elements $x_1, \ldots, x_\eta \in \Delta^{\mathcal{I}}$ such that*

$$(\exists r.C)^{\mathcal{I}}(x) = \bigvee_{i=1}^{\eta} r^{\mathcal{I}}(x, x_i) \otimes C^{\mathcal{I}}(x_i),$$

and analogously for the universal restrictions $\forall r.C$. In particular, if $\eta = 1$, then the suprema and infima from the semantics of $\exists r.C$ and $\forall r.C$ become maxima and minima, respectively. In this case, we simply say that $\mathcal{I}$ is witnessed.

As we will show, ℓ-satisfiability, even w.r.t. η-witnessed models, is undecidable in general. For finite De Morgan lattices, however, this problem is decidable and belongs to the same complexity class as deciding satisfiability of crisp $\mathcal{ALC}$ concepts, if the lattice operations are easily computable.

4 Undecidability

Consider the lattice L_∞ over the domain $([0,1] \cap \mathbb{Q}) \cup \{-\infty, \infty\}$ with the usual total order, the De Morgan negation ${\sim}\ell = 1 - \ell$ if $\ell \in [0,1]$, ${\sim}\infty = -\infty$, and

$\sim(-\infty) = \infty$, and the t-norm $\otimes$ defined by

$$\ell \otimes m := \begin{cases} \max\{\ell + m - 1, 0\} & \text{if } \ell, m \in [0,1] \text{ and } \ell + m \neq 0, \\ -\infty & \text{if } \ell = m = 0, \text{ and} \\ \min\{\ell, m\} & \text{otherwise.} \end{cases}$$

That is, $\otimes$ is the Łukasiewicz t-norm on the rationals in $(0, 1]$ extended with two extreme elements $-\infty$ and ∞. One can easily confirm that this is in fact a residuated lattice and its t-conorm $\oplus$ is given by

$$\ell \oplus m := \begin{cases} \min\{l + m, 1\} & \text{if } \ell, m \in [0,1] \text{ and } \ell + m \neq 2, \\ \infty & \text{if } \ell = m = 1, \text{ and} \\ \max\{\ell, m\} & \text{otherwise.} \end{cases}$$

We will reduce the well-known undecidable Post Correspondence Problem [18] to decidability of ∞-satisfiability. Notice that for every $T \subseteq L_\infty$, $\bigvee_{t \in T} t = \infty$ iff $\infty \in T$. Thus, a concept is ∞-satisfiable iff it is strongly ∞-satisfiable and it suffices to prove that strong ∞-satisfiability is undecidable.

Definition 6 (PCP). *Let $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ be two finite lists of words over an alphabet $\Sigma = \{1, \ldots, s\}$. The* Post Correspondence Problem *(PCP) asks whether there is a non-empty sequence $i_1, i_2, \ldots, i_k$, $1 \leq i_j \leq p$ such that $v_{i_1} v_{i_2} \cdots v_{i_k} = w_{i_1} w_{i_2} \cdots w_{i_k}$. Such a sequence, if it exists, is called a* solution *of the problem instance.*

For a word $\nu = i_1 i_2 \cdots i_k \in \{1, \ldots, p\}^*$ we will use v_ν, w_ν to denote the words $v_{i_1} v_{i_2} \cdots v_{i_k}$ and $w_{i_1} w_{i_2} \cdots w_{i_k}$, respectively. Given an instance $\mathcal{P}$ of PCP, we will construct a TBox $\mathcal{T}_\mathcal{P}$ and a concept name S such that S is strongly ∞-satisfiable iff $\mathcal{P}$ has no solution. For doing this, we will encode words w from the alphabet Σ as rational numbers $0.w$ in $[0, 1]$ in base $s + 1$; exceptionally, the empty word will be encoded by the number 0. The two concept names V and W will store the encoding of the concatenated words v_ν and w_ν, respectively.

Given two $\mathcal{ALC}_L$ concept descriptions C, D and a role name r, the expression $\langle C \equiv D \rangle$ abbreviates the two axioms $\langle C \sqsubseteq D, \infty \rangle$,$\langle D \sqsubseteq C, \infty \rangle$ and the expression $\langle C \overset{r}{\rightsquigarrow} D \rangle$ abbreviates the two axioms $\langle C \sqsubseteq \forall r.D, \infty \rangle$, $\langle \neg C \sqsubseteq \forall r.\neg D, \infty \rangle$. For an interpretation $\mathcal{I}$, $\langle C \equiv D \rangle$ expresses that $C^\mathcal{I}(x) = D^\mathcal{I}(x)$ for every $x \in \Delta^\mathcal{I}$, while $\langle C \overset{r}{\rightsquigarrow} D \rangle$ expresses that, for every $x, y \in \Delta^\mathcal{I}$ such that $r^\mathcal{I}(x, y) = \infty$, it holds that $C^\mathcal{I}(x) = D^\mathcal{I}(y)$. We will also use $n \cdot C$ as abbrevation for the n-ary disjunction $C \sqcup \cdots \sqcup C$, which is interpreted at $x \in \Delta^\mathcal{I}$ as the value $\min\{C^\mathcal{I}(x) + \cdots + C^\mathcal{I}(x), 1\} = \min\{n \cdot C^\mathcal{I}(x), 1\}$ whenever $C^\mathcal{I}(x) \in [0, 1]$.

We now define the TBoxes $\mathcal{T}_\mathcal{P}^i$ for $0 \leq i \leq p$ as follows:

$$\begin{aligned} \mathcal{T}_\mathcal{P}^0 := \{&\langle S \sqsubseteq V, 0 \rangle, \langle S \sqsubseteq \neg V, 1 \rangle, \langle S \sqsubseteq W, 0 \rangle, \langle S \sqsubseteq \neg W, 1 \rangle\} \cup \\ \{&\langle S \sqsubseteq V_i, 0.v_i \rangle, \langle S \sqsubseteq \neg V_i, 1 - 0.v_i \rangle, \\ &\langle S \sqsubseteq W_i, 0.w_i \rangle, \langle S \sqsubseteq \neg W_i, 1 - 0.w_i \rangle \mid 1 \leq i \leq p\}, \end{aligned}$$

$$\mathcal{T}^i_{\mathcal{P}} := \{\langle \top \sqsubseteq \exists r_i.\top, \infty\rangle, \langle V \sqcup V_i \overset{r_i}{\rightsquigarrow} V\rangle, \langle W \sqcup W_i \overset{r_i}{\rightsquigarrow} W\rangle\} \cup$$
$$\{\langle V_j \equiv (s+1)^{|v_i|} \cdot F_{ij}\rangle, \langle W_j \equiv (s+1)^{|w_i|} \cdot G_{ij}\rangle,$$
$$\langle F_{ij} \overset{r_i}{\rightsquigarrow} V_j\rangle, \langle G_{ij} \overset{r_i}{\rightsquigarrow} W_j\rangle \mid 1 \leq j \leq p\}.$$

Intuitively, $\mathcal{T}^0_{\mathcal{P}}$ initializes a search tree for a solution of $\mathcal{P}$, by setting both V and W to the empty word, and describing each pair (v_i, w_i) by the concepts V_i and W_i. Each TBox $\mathcal{T}^i_{\mathcal{P}}$ then extends the search tree by concatenating each pair of words v, w produced so far with v_i and w_i, respectively. More formally, consider the interpretation $\mathcal{I}_{\mathcal{P}} = (\Delta^{\mathcal{I}_{\mathcal{P}}}, \cdot^{\mathcal{I}_{\mathcal{P}}})$ where

- $\Delta^{\mathcal{I}_{\mathcal{P}}} = \{1, \ldots, p\}^*$,
- $V^{\mathcal{I}_{\mathcal{P}}}(\nu) = 0.v_\nu, W^{\mathcal{I}_{\mathcal{P}}}(\nu) = 0.w_\nu, V_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = \frac{0.v_i}{(s+1)^{|v_\nu|}}, W_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = \frac{0.w_i}{(s+1)^{|w_\nu|}}$
- $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu i) = \infty$ and $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu') = -\infty$ if $\nu' \neq \nu i$, and
- $S^{\mathcal{I}_{\mathcal{P}}}(\varepsilon) = \infty$.

It is easy to see that $\mathcal{I}_{\mathcal{P}}$ is in fact a model of the TBox $\mathcal{T}_0 := \bigcup_{i=0}^{p} \mathcal{T}^i_{\mathcal{P}}$. More interesting, however, is that *every* model of this TBox where S is ∞-satisfiable must include $\mathcal{I}_{\mathcal{P}}$, as stated in the following lemma.

Lemma 7. *Let $\mathcal{I}$ be a model of $\mathcal{T}_0$ such that $S^{\mathcal{I}}(x) = \infty$ for some $x \in \Delta^{\mathcal{I}}$. There exists a function $f : \Delta^{\mathcal{I}_{\mathcal{P}}} \to \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}_{\mathcal{P}}}(\nu) = C^{\mathcal{I}}(f(\nu))$ holds for every concept name C occurring in $\mathcal{T}_0$ and $\nu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$.*

Proof (Sketch). The function f is constructed by induction on the length of ν. We can define $f(\varepsilon) := x$ since $S^{\mathcal{I}}(x) = \infty$ and $\mathcal{I}$ is a model of $\mathcal{T}^0_{\mathcal{P}}$. Let now ν be such that $f(\nu)$ is already defined. The axioms $\langle \top \sqsubseteq \exists r_i.\top, \infty\rangle$ ensure that, for every $i, 1 \leq i \leq p$ there is a $\gamma \in \Delta^{\mathcal{I}}$ such that $r_i^{\mathcal{I}}(f(\nu), \gamma) = \infty$. The definition $f(\nu i) := \gamma$ satisfies the required property. □

This lemma shows that every model of $\mathcal{T}_0$ must include a search tree for a solution of $\mathcal{P}$. Thus, in order to know whether a solution exists, we need to decide if there is a node of this tree where the concept names V and W are interpreted by the same value. Notice that, for any two values $\ell, m \in [0, 1]$, $\ell \neq m$ iff $(\sim \ell \oplus m) \otimes (\ell \oplus \sim m) < 1$. Moreover, $\ell < 1$ iff $\ell \oplus \ell \leq 1$ or, equivalently, $\sim \ell \otimes \sim \ell \geq 0$. Thus, as $\mathcal{I}_{\mathcal{P}}$ always interprets the concept names V and W in the interval $[0, 1]$, it is a model of the TBox

$$\mathcal{T}' := \{\langle E \equiv (\neg A \sqcup B) \sqcap (A \sqcup \neg B)\rangle\} \cup \{\langle \top \sqsubseteq \forall r_i.\neg(E \sqcup E), 0\rangle \mid 1 \leq i \leq p\}$$

iff $A^{\mathcal{I}_{\mathcal{P}}}(\nu) \neq B^{\mathcal{I}_{\mathcal{P}}}(\nu)$ holds for every $\nu \in \{1, \ldots, p\}^+$.

Theorem 8. *The instance $\mathcal{P}$ of the PCP has a solution iff S is not ∞-satisfiable w.r.t. $\mathcal{T}_{\mathcal{P}} := \mathcal{T}_0 \cup \mathcal{T}'$.*

Notice that the interpretation $\mathcal{I}_{\mathcal{P}}$ is witnessed, which means that undecidability holds even if we restrict reasoning to η-witnessed models, for any $\eta \in \mathbb{N}$.

Corollary 9. *(Strong) satisfiability is undecidable, even if the lattice is a countable, residuated total order and reasoning is restricted to η-witnessed models, with $\eta \in \mathbb{N}$.*

5 Deciding Strong Satisfiability

In the previous section, we have shown that satisfiability is undecidable in general. We now show that if we consider only *finite* De Morgan lattices L, then satisfiability in $\mathcal{ALC}_L$ can be effectively decided. As the following lemmata show, in this case we can restrict to strong ℓ-satisfiability w.r.t. η-witnessed models.

Lemma 10. *The best satisfiability degree for C w.r.t. $\mathcal{T}$ is the supremum of all ℓ such that C is strongly ℓ-satisfiable.*

Proof (Sketch). If C is strongly ℓ-satisfiable and strongly ℓ'-satisfiable, there are two models $\mathcal{I}, \mathcal{I}'$ of $\mathcal{T}$ and $x \in \Delta, x' \in \Delta'$ with $C^{\mathcal{I}}(x) \geq \ell$ and $C^{\mathcal{I}'}(x') \geq \ell'$. The disjoint union of $\mathcal{I}$ and $\mathcal{I}'$ gives a model $\mathcal{J}$ where $\bigvee_{y \in \Delta^{\mathcal{J}}} C^{\mathcal{J}}(y) \geq \ell \vee \ell'$. □

We can then find out whether C is ℓ-satisfiable by comparing ℓ to the best satisfiability degree of C. We will thus focus on finding all the lattice elements that witness the strong ℓ-satisfiability of a given concept.

Lemma 11. *If L has width $\eta \in \mathbb{N}$, i.e. the cardinality of the largest antichain of L is η, then $\mathcal{ALC}_L$ has the η-witnessed model property.*

To simplify the description, we consider $\eta = 1$ only. The algorithm and the proofs of correctness can be easily adapted for any other $\eta \in \mathbb{N}$.

Our approach reduces strong ℓ-satisfiability to the emptiness problem of an automaton on infinite trees. Before giving the details of this reduction, we present a brief introduction to these automata. The automata work over the infinite k-ary tree K^* for $K := \{1, \ldots, k\}$ with $k \in \mathbb{N}$. The positions of the *nodes* in this tree are represented through words in K^*: the empty word ε represents the root node, and ui represents the i-th successor of the node u.

Definition 12 (looping automaton). *A* looping automaton (LA) *is a tuple $\mathcal{A} = (Q, I, \Delta)$ consisting of a finite set Q of* states*, a set $I \subseteq Q$ of* initial states*, and a* transition relation $\Delta \subseteq Q \times Q^k$. *A* run *of $\mathcal{A}$ is a mapping $r : K^* \to Q$ assigning states to each node of K^* such that (i) $r(\varepsilon) \in I$ and (ii) for every $u \in K^*$ we have $(r(u), r(u1), \ldots, r(uk)) \in \Delta$. The* emptiness problem *for LA is to decide whether a given LA has a run.*

The emptiness problem for LA can be solved in polynomial time [23]. It is worth to point out that this procedure not only decides emptiness, but actually computes *all* the states that can be used as initial states to accept a non-empty language. We will later exploit this for computing the best satisfiability degree.

The following automata-based algorithm uses the fact that a concept is strongly ℓ-satisfiable iff it has a well-structured tree model, called a *Hintikka tree*. Intuitively, Hintikka trees are abstract representations of tree models that explicitly express the membership value of all "relevant" concept descriptions. The automaton we construct will have exactly these Hintikka trees as its runs. Strong ℓ-satisfiability is hence reduced to an emptiness test of this automaton.

We denote as $\mathsf{sub}(C, \mathcal{T})$ the set of all subconcepts of C and of the concept descriptions D and E for all $\langle D \sqsubseteq E, \ell\rangle \in \mathcal{T}$. The states of the automaton will be so-called Hintikka sets. These are L-fuzzy sets over the domain $\mathsf{sub}(C, \mathcal{T}) \cup \{\rho\}$, where ρ is an arbitrary new element.

Definition 13 (Hintikka set). *A function $H : \mathsf{sub}(C, \mathcal{T}) \cup \{\rho\} \to L$ is called a* (fuzzy) Hintikka set *for $C, \mathcal{T}$ if the following four conditions are satisfied:*

(i) $H(D \sqcap E) = H(D) \otimes H(E)$ *for every* $D \sqcap E \in \mathsf{sub}(C, \mathcal{T})$*,*
(ii) $H(D \sqcup E) = H(D) \oplus H(E)$ *for every* $D \sqcup E \in \mathsf{sub}(C, \mathcal{T})$*,*
(iii) $H(\neg D) = \sim H(D)$ *for every* $\neg D \in \mathsf{sub}(C, \mathcal{T})$*, and*
(iv) $H(D) \Rightarrow H(E) \geq \ell$ *for every GCI* $\langle D \sqsubseteq E, \ell\rangle$ *in* $\mathcal{T}$*.*

The arity k of our automaton is determined by the number of existential and universal restrictions, i.e. concept descriptions of the form $\exists r.D$ or $\forall r.D$, contained in $\mathsf{sub}(C, \mathcal{T})$. Intuitively, each successor will act as the witness for one of these restrictions. The additional domain element ρ will be used to express the degree with which the role relation to the parent node holds. Since we need to know which successor in the tree corresponds to which restriction, we fix an arbitrary bijection $\varphi : \{E \mid E \in \mathsf{sub}(C, \mathcal{T}) \text{ is of the form } \exists r.D \text{ or } \forall r.D\} \to K$. The following conditions define the transitions of our automaton.

Definition 14 (Hintikka condition). *The tuple $(H_0, H_1, \ldots, H_k)$ of Hintikka sets for $C, \mathcal{T}$ satisfies the* Hintikka condition *if:*

(i) $H_0(\exists r.D) = H_{\varphi(\exists r.D)}(\rho) \otimes H_{\varphi(\exists r.D)}(D)$ *for every existential restriction* $\exists r.D \in \mathsf{sub}(C, \mathcal{T})$*, and additionally* $H_0(\exists r.D) \geq H_{\varphi(F)}(\rho) \otimes H_{\varphi(F)}(D)$ *for every restriction* $F \in \mathsf{sub}(C, \mathcal{T})$ *of the form* $\exists r.E$ *or* $\forall r.E$*,*
(ii) $H_0(\forall r.D) = H_{\varphi(\forall r.D)}(\rho) \Rightarrow H_{\varphi(\forall r.D)}(D)$ *for every universal restriction* $\forall r.D \in \mathsf{sub}(C, \mathcal{T})$*, and additionally* $H_0(\forall r.D) \leq H_{\varphi(F)}(\rho) \Rightarrow H_{\varphi(F)}(D)$ *for every restriction* $F \in \mathsf{sub}(C, \mathcal{T})$ *of the form* $\exists r.E$ *or* $\forall r.E$*.*

A *Hintikka tree* for $C, \mathcal{T}$ is an infinite k-ary tree $\mathbf{T}$ labeled with Hintikka sets where, for every node $u \in K^*$, the tuple $(\mathbf{T}(u), \mathbf{T}(u1), \ldots, \mathbf{T}(uk))$ satisfies the Hintikka condition. The definition of Hintikka sets ensures that all axioms are satisfied at any node of the Hintikka tree, while the Hintikka condition makes sure that the tree is in fact a witnessed model.

The proof of the following theorem uses arguments similar to those in [2]. The main difference is that one also has to find witnesses for the universal restrictions.

Theorem 15. *Let C be a concept description and $\mathcal{T}$ a TBox. Then C is strongly ℓ-satisfiable w.r.t. $\mathcal{T}$ (in a witnessed model) iff there is a Hintikka tree $\mathbf{T}$ for $C, \mathcal{T}$ such that $\mathbf{T}(\varepsilon)(C) \geq \ell$.*

Proof (Sketch). A Hintikka tree can be seen as a witnessed model with domain K^* and interpretation function given by the Hintikka sets. The conditions satisfied by the Hintikka sets and the Hintikka condition ensure that this interpretation is well-defined. Thus, if there is a Hintikka tree $\mathbf{T}$ for $C, \mathcal{T}$ with $\mathbf{T}(\varepsilon)(C) \geq \ell$, then C is strongly ℓ-satisfiable w.r.t. $\mathcal{T}$.

On the other hand, every witnessed model $\mathcal{I}$ with a domain element $x \in \Delta^{\mathcal{I}}$ for which $C^{\mathcal{I}}(x) \geq \ell$ holds can be *unraveled* into a Hintikka tree $\mathbf{T}$ for $C, \mathcal{T}$ as follows. We start by labeling the root node by the Hintikka set that records the membership values of x for each concept from $\mathsf{sub}(C, \mathcal{T})$. We then create successors of the root by considering every element of $\mathsf{sub}(C, \mathcal{T})$ of the form $\exists r.D$ or $\forall r.D$ and finding the witness $y \in \Delta^{\mathcal{I}}$ for this restriction. We create a new node for y which is an r-successor of the root node with degree $r^{\mathcal{I}}(x, y)$. By continuing this process, we construct a Hintikka tree $\mathbf{T}$ for $C, \mathcal{T}$ for which $\mathbf{T}(\varepsilon)(C) \geq \ell$ holds. □

Thus, strong ℓ-satisfiability w.r.t. witnessed models is equivalent to the non-emptiness of the following automaton.

Definition 16 (Hintikka automaton). *Let C be an $\mathcal{ALC}_L$ concept description, $\mathcal{T}$ a TBox, and $\ell \in L$. The* Hintikka automaton *for $C, \mathcal{T}, \ell$ is the LA $\mathcal{A}_{C,\mathcal{T},\ell} = (Q, I, \Delta)$ where Q is the set of all Hintikka sets for $C, \mathcal{T}$, I contains all Hintikka sets H with $H(C) \geq \ell$, and Δ is the set of all $(k+1)$-tuples of Hintikka sets that satisfy the Hintikka condition.*

The runs of $\mathcal{A}_{C,\mathcal{T},\ell}$ are exactly the Hintikka trees $\mathbf{T}$ having $\mathbf{T}(\varepsilon)(C) \geq \ell$. Thus, C is strongly ℓ-satisfiable w.r.t. $\mathcal{T}$ iff $\mathcal{A}_{C,\mathcal{T},\ell}$ is not empty.

The size of the automaton $\mathcal{A}_{C,\mathcal{T},\ell}$ is exponential in $C, \mathcal{T}$ and polynomial in L. Hence, the emptiness test for this automaton uses time exponential in $C, \mathcal{T}$ and polynomial in the complexity of the lattice operations on L. Notice however that in general the encoding $\mathrm{enc}(L)$ of a lattice L may be much smaller than the whole lattice L. For this reason we need to consider the complexity of the lattice operations w.r.t. this encoding.

Theorem 17. *If $|L|$ is at most exponential in $|\mathrm{enc}(L)|$ and the lattice operations are in a complexity class* C *w.r.t. the size of* $\mathrm{enc}(L)$,[2] *then strong ℓ-satisfiability (w.r.t. witnessed models) is in* $\textsc{ExpTime}^{\mathrm{C}}$.

Furthermore, the emptiness test of $\mathcal{A}_{C,\mathcal{T},\ell}$ can be used to compute the set of *all* Hintikka sets that may appear at the root of a Hintikka tree. From this set we can extract the set of all values ℓ such that $\mathbf{T}(\epsilon)(C) \geq \ell$ for some Hintikka tree $\mathbf{T}$. From the presented results it follows that the best satisfiability degree can also be computed in $\textsc{ExpTime}^{\mathrm{C}}$.

Corollary 18. *If L is fixed or of size polynomial in $|\mathrm{enc}(L)|$ and $\sim$, $\otimes$ can be computed in time polynomial in $|L|$, then (strong) ℓ-satisfiability (w.r.t. witnessed models) is* ExpTime*-complete.*

Proof. ExpTime-hardness follows from ExpTime-hardness of concept satisfiability in crisp $\mathcal{ALC}$ [1]. By assumption, all lattice operations can be computed in at most polynomial time by several nested iterations over L. Applying Theorem 17 yields inclusion in $\textsc{ExpTime}^{\textsc{PTime}} = \textsc{ExpTime}$. □

[2] More formally, deciding $\ell \leq m$, $\ell \otimes m = n$, etc. for given $\ell, m, n \in L$ is in C.

Notice that the definitions of Hintikka sets and Hintikka trees are independent of the operators used. One could have chosen the residual negation $\ominus\ell := \ell \Rightarrow \mathbf{0}$ to interpret the constructor $\neg$, or the Kleene-Dienes implication $\ell \Rightarrow m := \sim\ell \vee m$ instead of the residuum. The only restrictions are that the semantics must be truth functional, i.e. the value of a formula must depend only on the values of its direct subformulas, and the underlying operators must be computable.

As a last remark, we want to point out that the algorithm can be modified for reasoning w.r.t. η-witnessed models with $\eta > 1$. One needs only extend the arity of the Hintikka trees to account for η witnesses for each quantified formula in $\mathsf{sub}(C, \mathcal{T})$. The emptiness test of the automaton, and hence also satisfiability w.r.t. η-witnessed models, is exponential in η.

6 Conclusions

We have introduced the fuzzy DL $\mathcal{ALC}_L$ whose semantics is based on arbitrary complete De Morgan lattices and t-norms. To the best of our knowledge, all previously existing approaches for fuzzy $\mathcal{ALC}$, either based on total orders or on lattices, are special cases of $\mathcal{ALC}_L$.

We showed that reasoning in this logic is undecidable, even if restricted to a very simple infinite lattice and t-norm. This result suggests, but does not prove, that reasoning with the Łukasiewicz t-norm over the interval $[0, 1]$ may, contrary to previous claims [22], be undecidable.

For the special case of finite lattices, we showed decidability by presenting an automata-based decision procedure that runs in exponential time, assuming a polynomial-time oracle for computing the lattice and t-norm operations. An advantage of our decision procedure is that it can easily be adapted to deal with different kinds of truth-functional semantics, and hence is useful for different applications. Given the promising first steps towards an automata-based implementation of $\mathcal{ALC}$ reasoning shown in [10], we believe that our algorithm not only yields an interesting theoretical result, but may be useful for a future implementation. We intend to further study this possibility by developing adequate optimizations and analyzing low-complexity instances of lattice operators.

There are three issues that we will pursue in future work. The first is to explore the limits of undecidability: are there classes of infinite lattices and t-norms in which reasoning is decidable? As said before, it is still unknown whether reasoning in fuzzy $\mathcal{ALC}$ with continuous t-norms over $[0, 1]$ is decidable.

The second issue is to explore the expressivity of DLs. We believe that our approach can easily be adapted to fuzzy $\mathcal{SI}$. Additionally, if we restrict to acyclic TBoxes, we may be able to obtain a PSpace upper bound as in [2].

Finally, we want to develop an algorithm for deciding ℓ-subsumption. Notice that the residuum cannot, in general, be expressed using the t-norm, t-conorm and negation. Thus, the usual idea of reducing subsumption to satisfiability by constructing an equivalent concept cannot be applied.

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