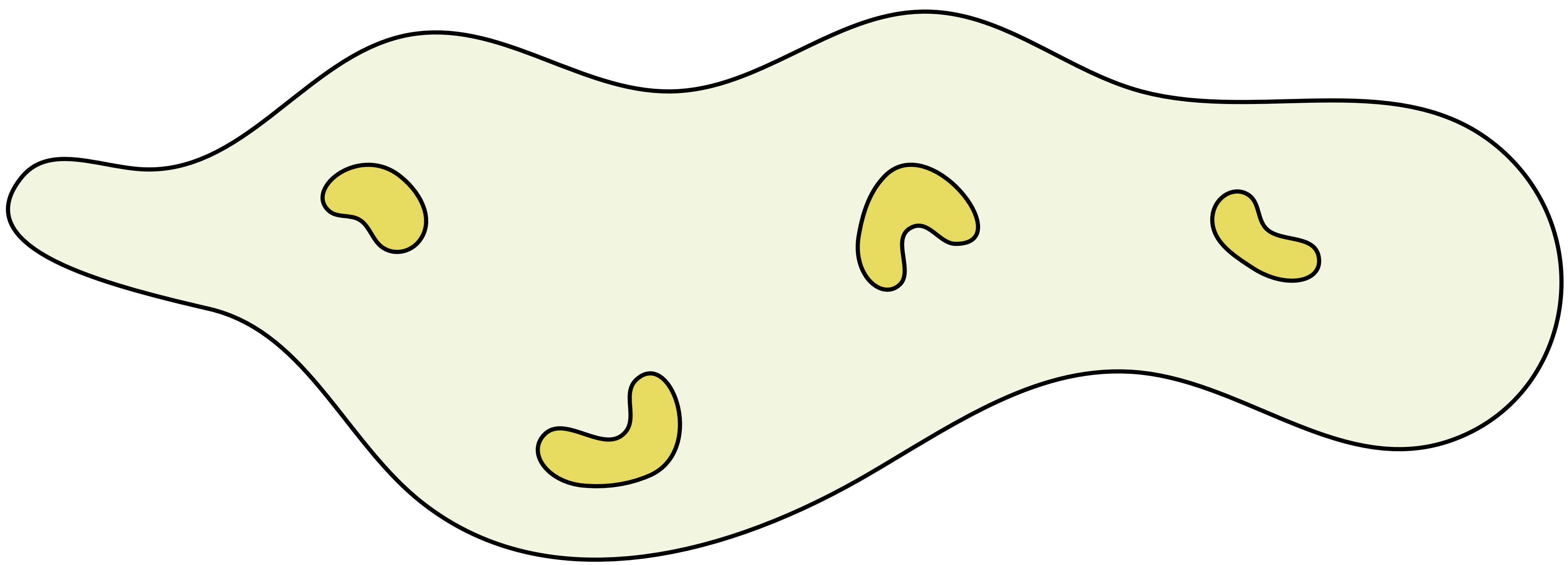
Idea: First order logic can only express "local" properties

Local = properties of nodes which are close to one another



[Some of the slides are by Diego Figueira, some of them by Anuj Dawar].

What kind of problems we study?

Definability: is the property P expressible in logic $\mathcal{L}$?

E.g. is connectivity expressible in First-Order Logic?

Expressive power: Can the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ express exactly the same properties?

Succinctness: Can $\mathcal{L}_1$ express the properties of $\mathcal{L}_2$ but shorter?

Descriptive complexity: Is there a logic characterising the complexity class $\mathcal{C}$?

Satisfiability: is there a model of a formula φ?

Model-checking (a.k.a. query evaluation): given φ and G is it the case that $G \models \varphi$?

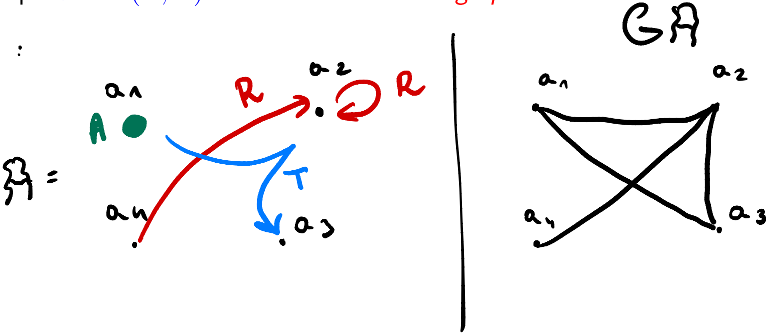
Gaifman Graphs and Neighbourhoods

On a structure $\mathbb{A}$, define the binary relation:

> $E(a_1, a_2)$ *if, and only if, there is some relation* R *and some tuple* $\mathbf{a}$ *containing both* a_1 *and* a_2 *with* $R(\mathbf{a})$.

The graph $G\mathbb{A} = (A, E)$ is called the *Gaifman graph* of $\mathbb{A}$.

Example :



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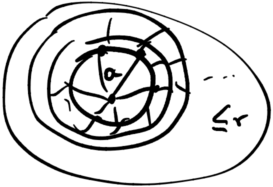
The graph $G\mathbb{A} = (A, E)$ is called the *Gaifman graph* of $\mathbb{A}$.

$dist(a, b)$ — the distance between a and b in the graph (A, E).

$\mathrm{Nbd}_r^{\mathbb{A}}(a)$ — the substructure of $\mathbb{A}$ given by the set:



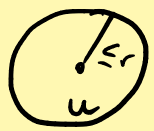
$$\{b \mid dist(a, b) \leq r\}$$

sometimes called a "ball" of radius r

Hanf locality

Definition. Two structures S_1 and S_2 are **Hanf(r, t) - equivalent**

iff for each structure B, the two numbers



$$\#u \text{ s.t. } S_1[u,r] \cong B \qquad \#v \text{ s.t. } S_2[v,r] \cong B$$

are *either the same* or *both* $\geq t$.

usually denoted with

$N_r^{S_1}(u)$ or $Neib_r^{S_1}(u)$

Hanf locality

Definition. Two structures S_1 and S_2 are **Hanf(1,1) - equivalent**

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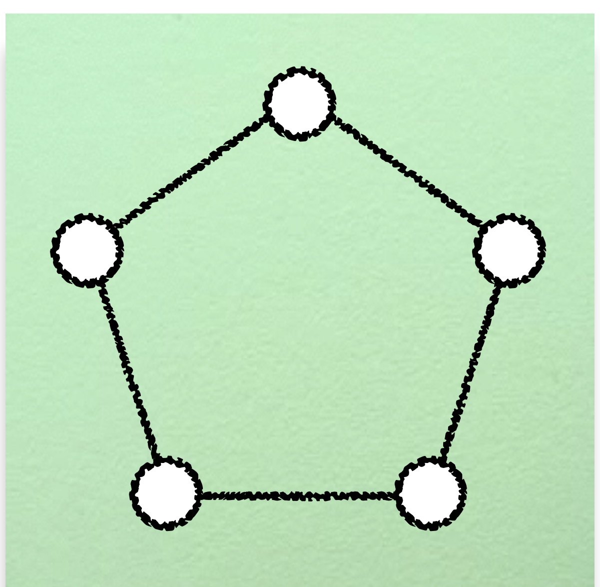
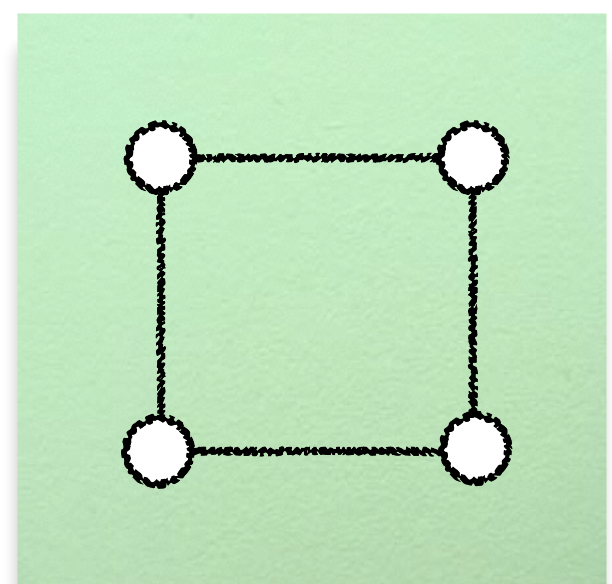
$$\#u \text{ s.t. } S_1[u,1] \cong B \qquad \#v \text{ s.t. } S_2[v,1] \cong B$$

are *either the same* or *both* ≥ 1.

or just neighbourhoods

Example. S_1, S_2 are Hanf(1, 1) - equivalent iff they have the *same balls* of radius 1





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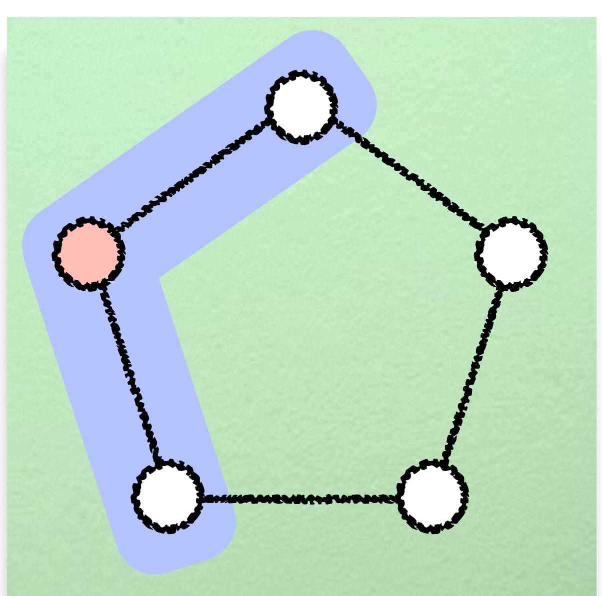

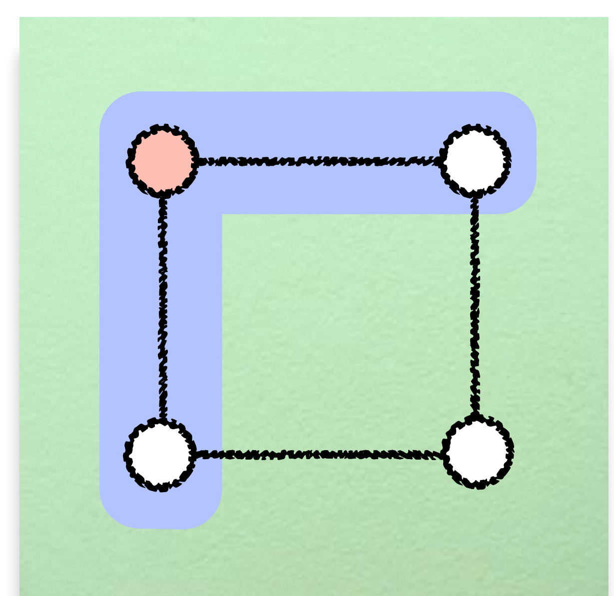

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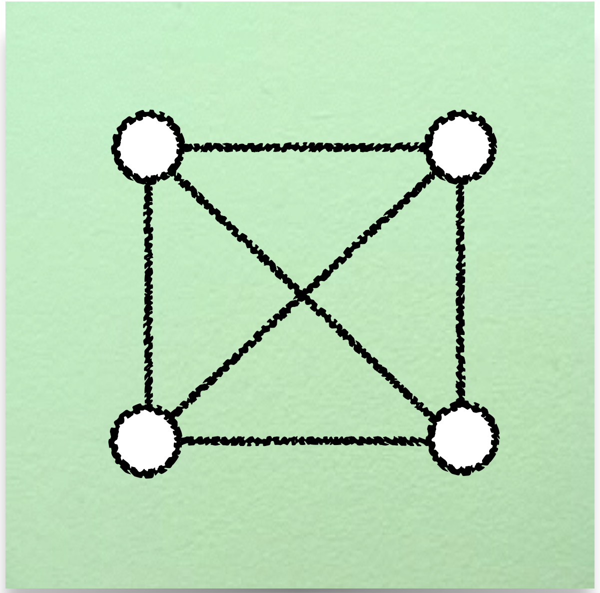
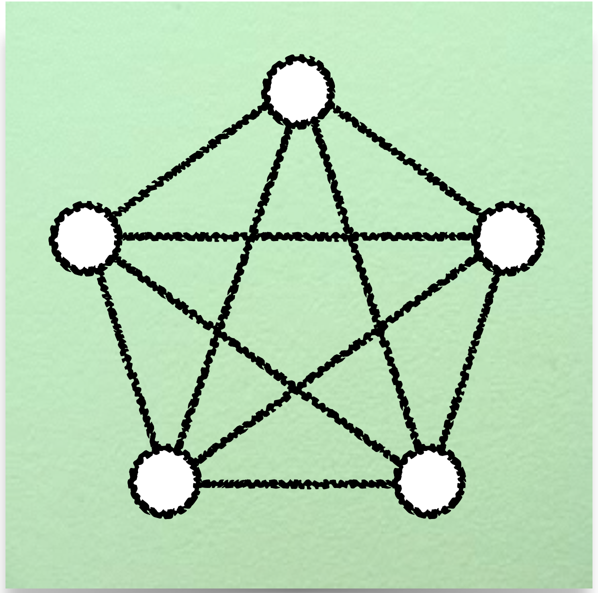
Example. K_n, K_{n+1} are **not** Hanf(1, 1) - equivalent





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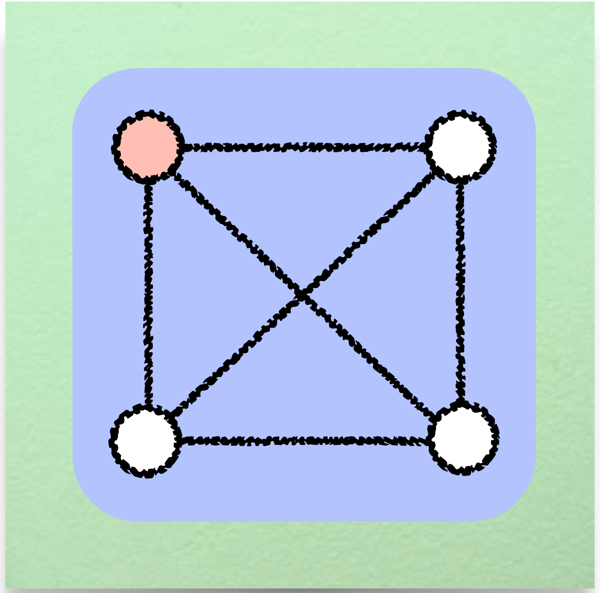

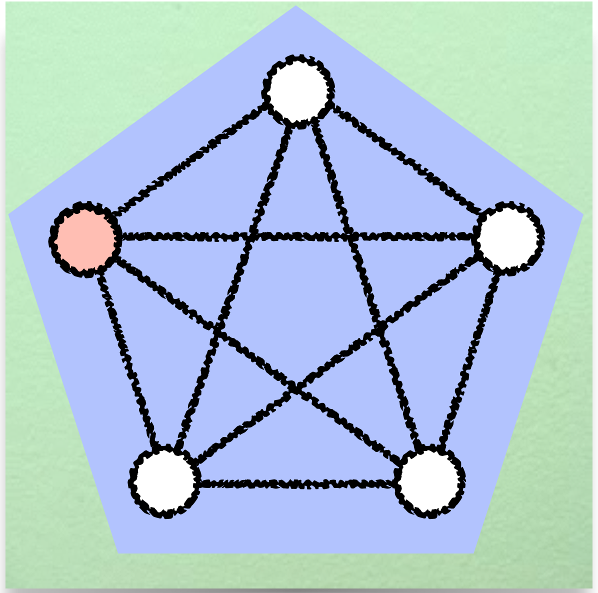

Hanf locality

Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n) whenever S_1, S_2 are Hanf$(3^n, n)$-equivalent, ~~with $r = 3^n$ and $t =$~~ n.

[Hanf '60]

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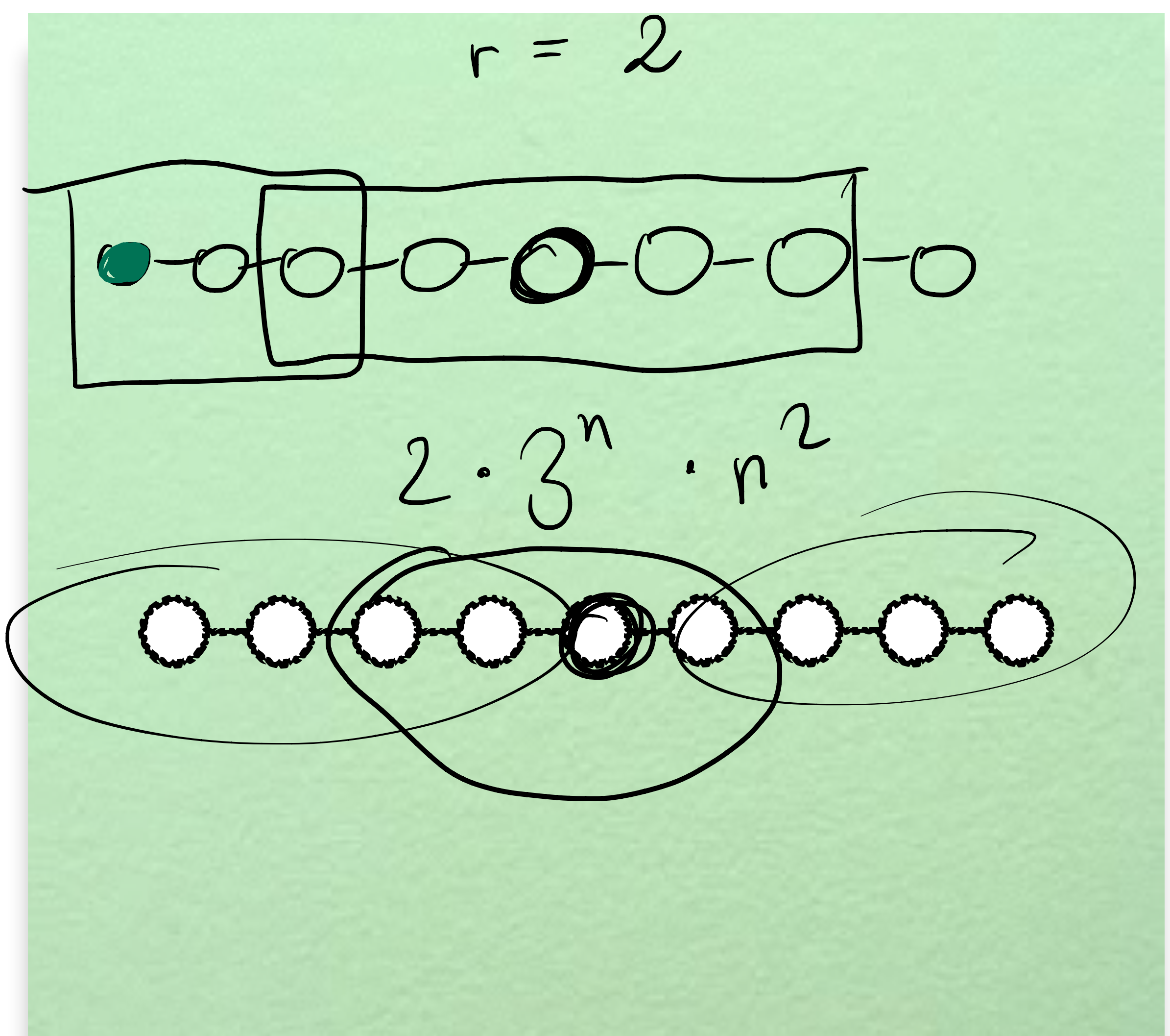

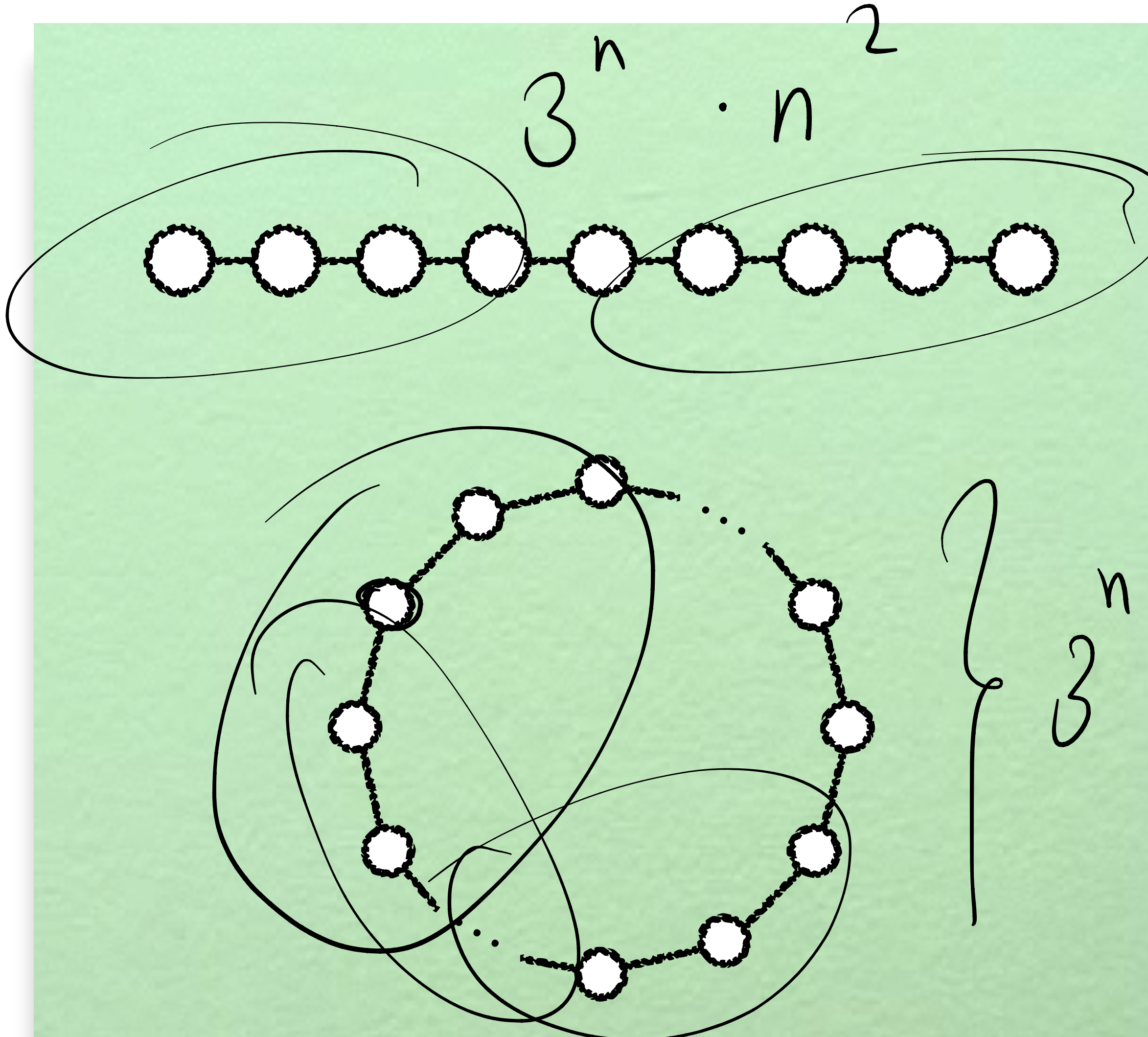

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Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n)

whenever S_1, S_2 are $\mathrm{Hanf}(r,t)$-equivalent, with $r = 3^n$ and $t = n$.

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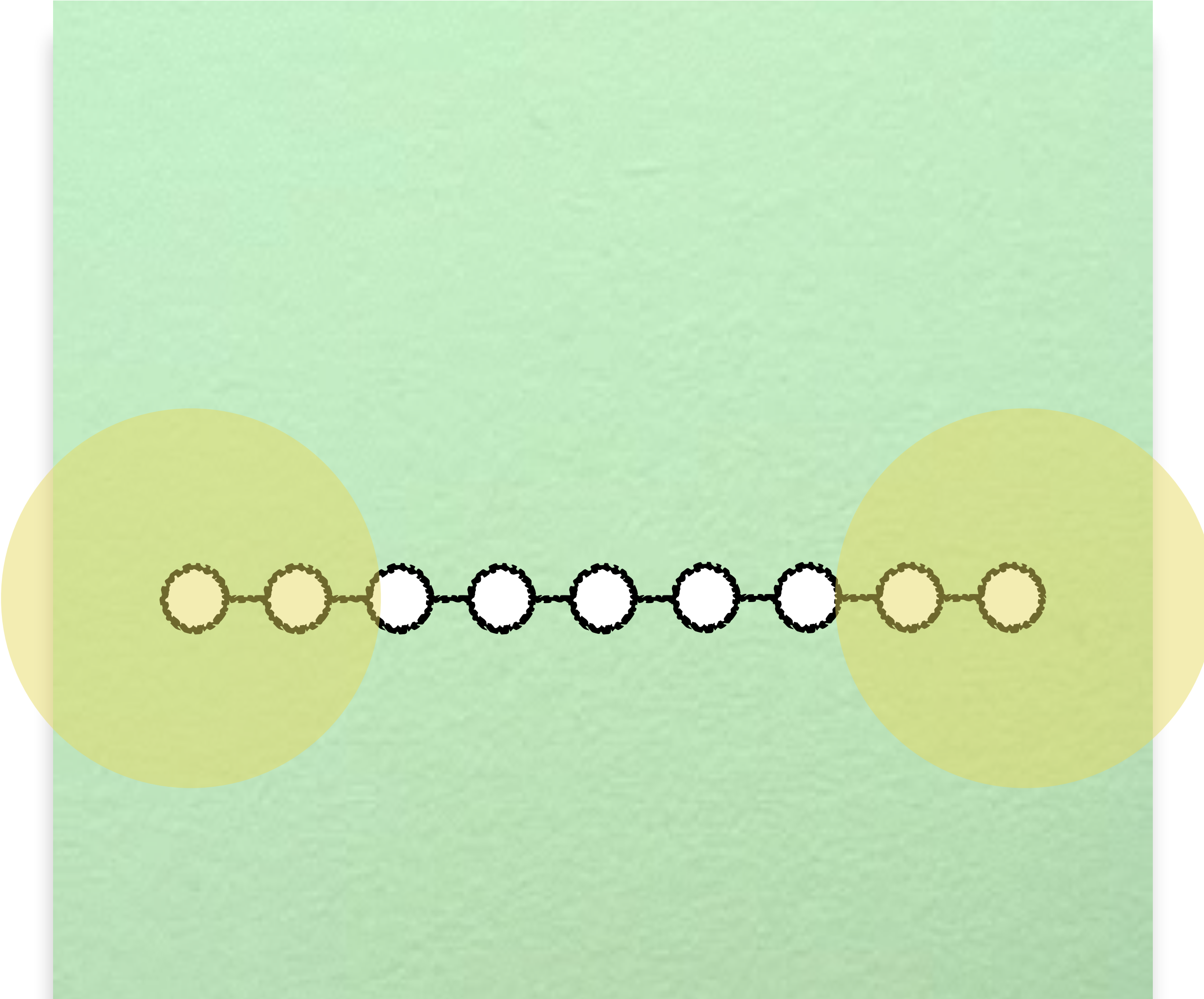

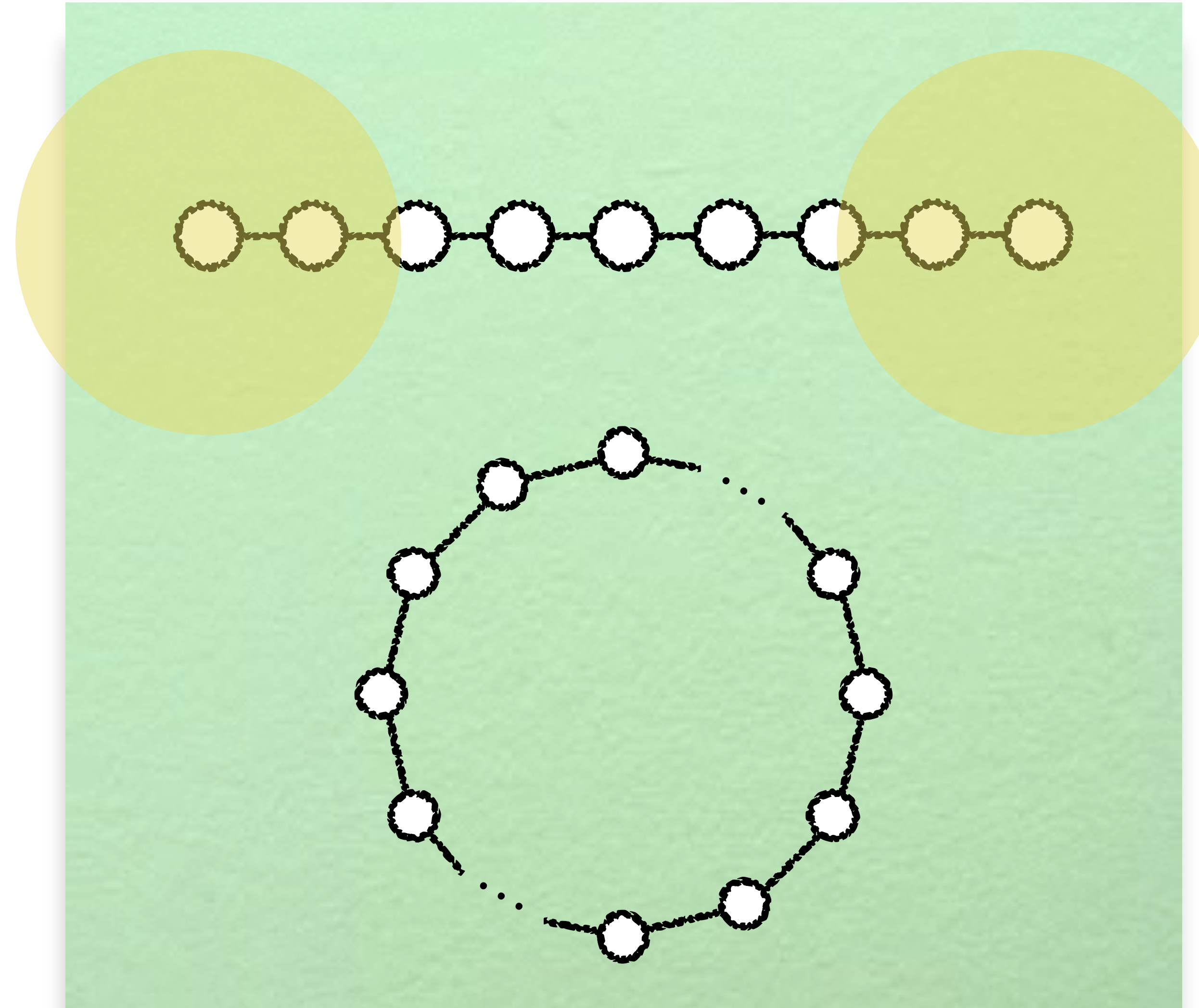

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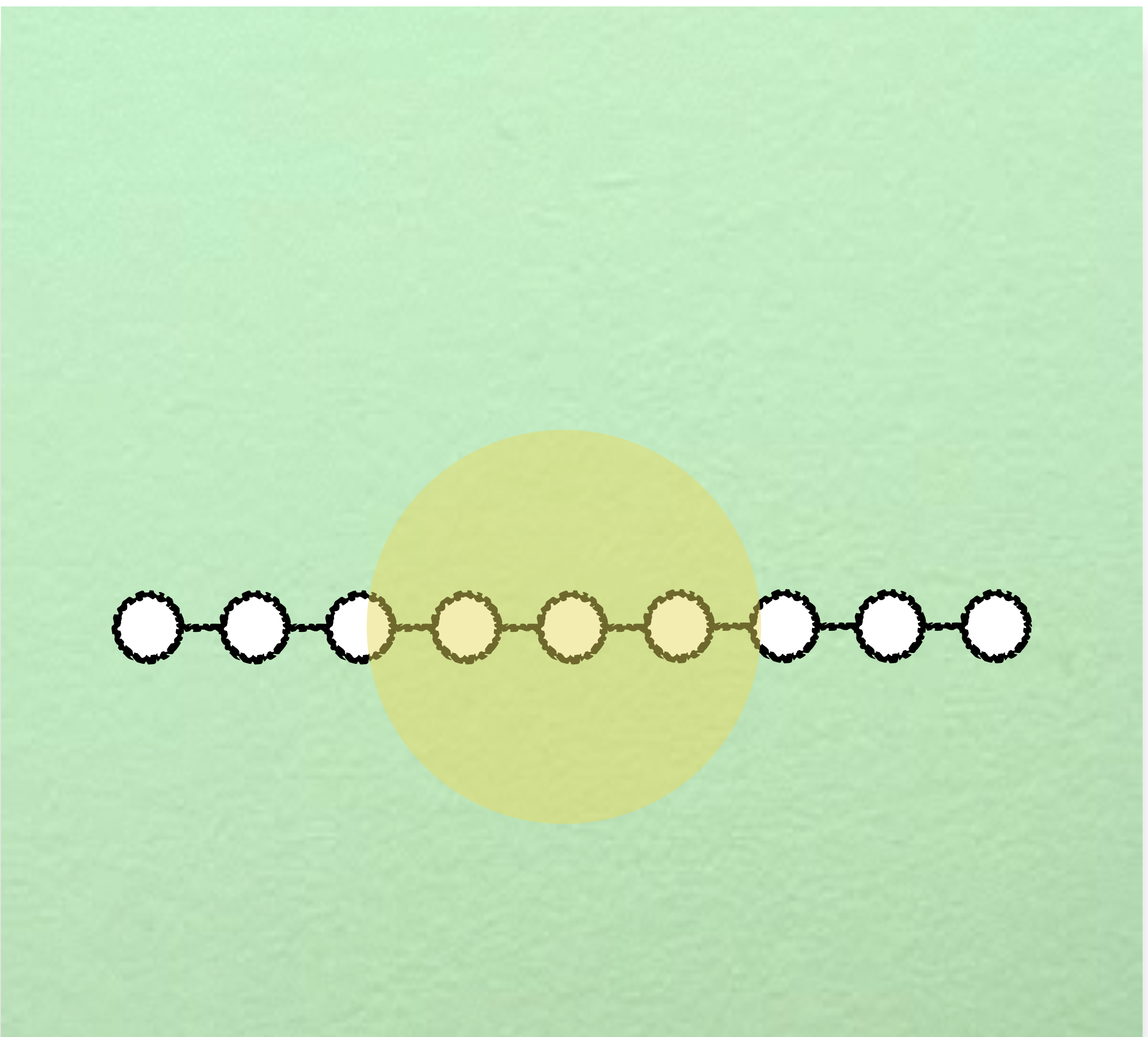

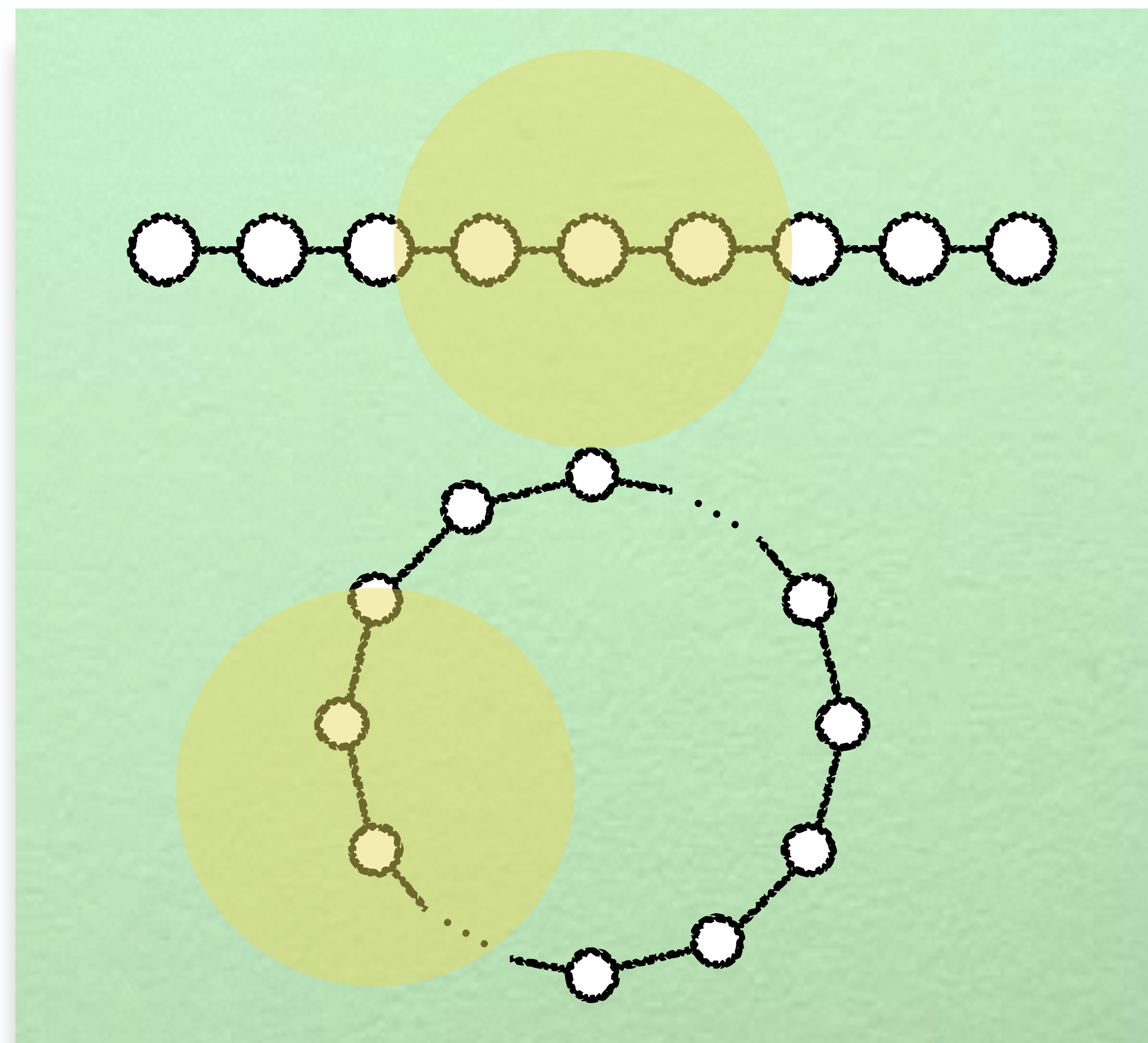

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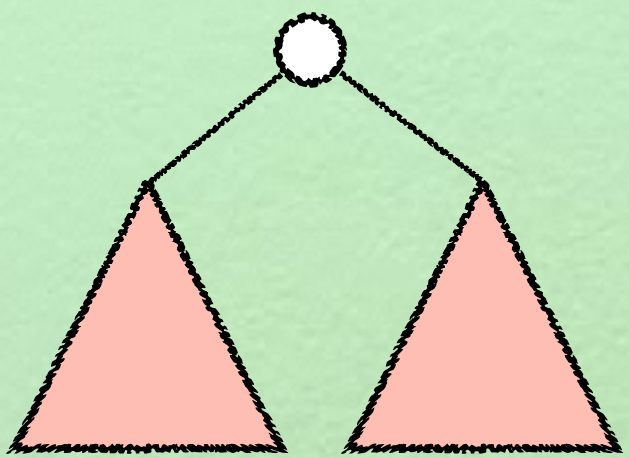
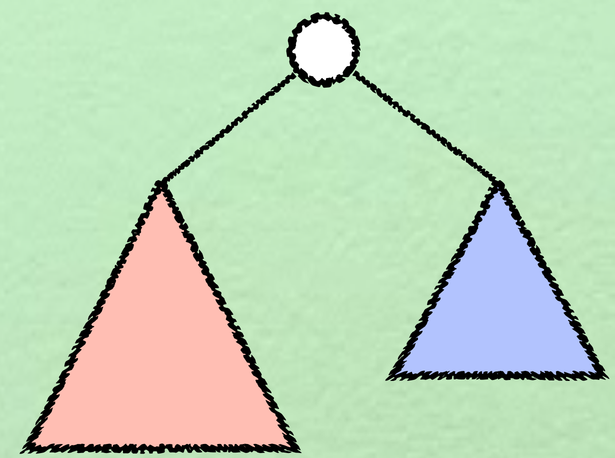
[Hanf '60]

Exercise: prove that testing whether a binary tree is *complete* is not FO-definable





Next task: How to show that FO is Hanf-local?

Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n) whenever S_1, S_2 are Hanf(r,t)-equivalent, with $r = 3^n$ and $t = n$.

[Hanf '60]

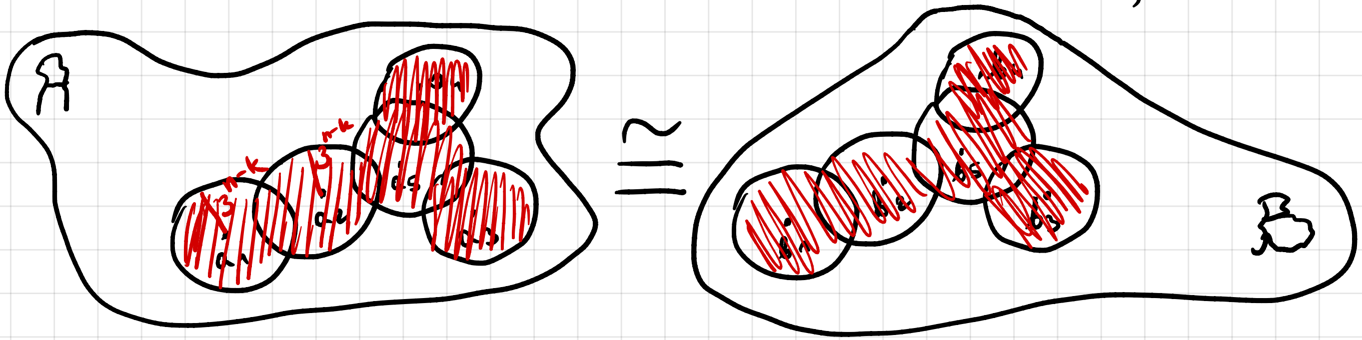
Proof idea:

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Hanf$(3^n, n)$-equivalent and let us find a winning strategy for duplicator in n-round E-F game.

! Our invariant: After k rounds we have that

$$\bigcup_{1 \leq i \leq k} N^{\mathfrak{A}}_{3^{n-k}}(a_i) \cong \bigcup_{1 \leq i \leq k} N^{\mathfrak{B}}_{3^{n-k}}(b_i),$$

where $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k$ are the selected elements.



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Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n) whenever S_1, S_2 are Hanf(r,t)-equivalent, with $r = 3^n$ and $t = n$. [Hanf '60]

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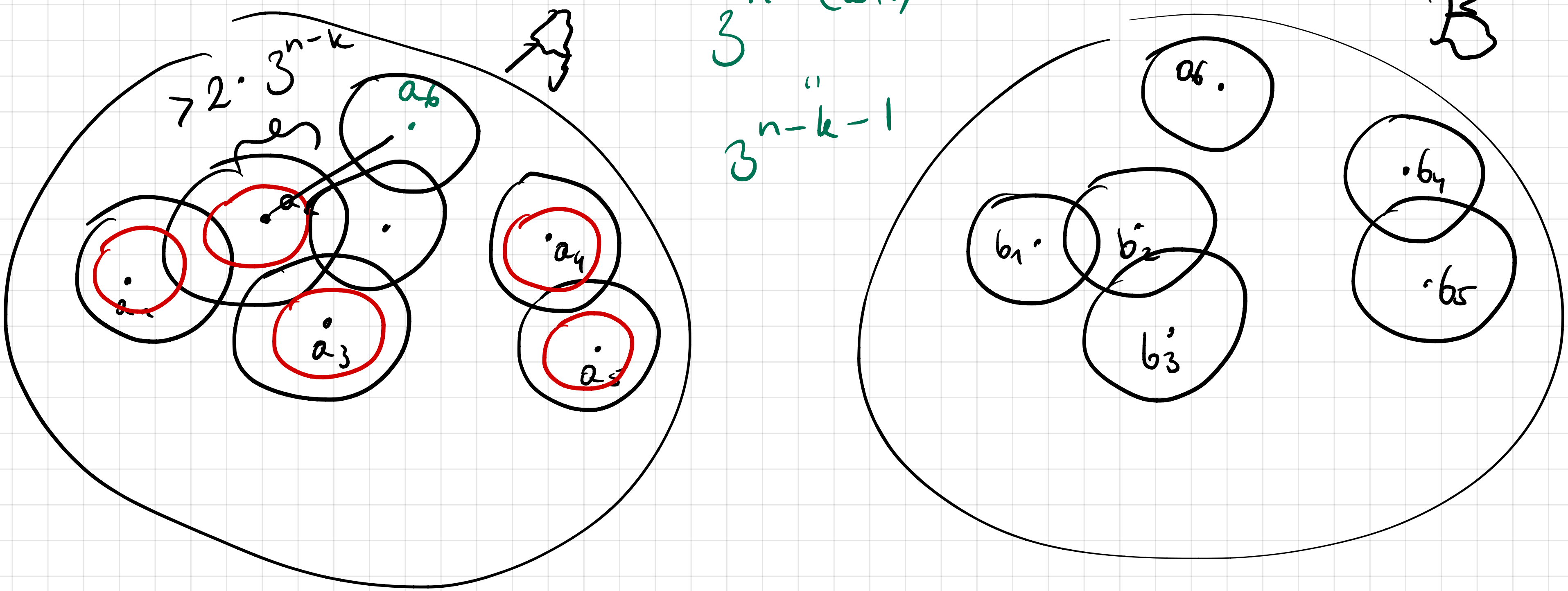
$$3^{n-(k+1)} = 3^{n-k-1}$$



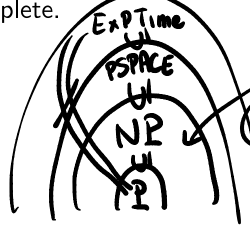
Parameterized Complexity

FPT—the class of problems of input size n and *parameter* l which can be solved in time $O(f(l)n^c)$ for some computable function f and constant c.

There is a hierarchy of *intractable* classes.

$$\mathsf{FPT} \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq \mathsf{AW}[\star]$$

The satisfaction relation for first-order logic ($\mathbb{A} \models \varphi$), parameterized by the length of φ is $\mathsf{AW}[\star]$-complete.



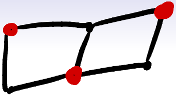
propositional logic (SAT)

$((p \wedge q) \vee \neg r) \wedge t$

0/1

$O(f(k) \cdot n^2) =$

Graph Problems

$G =$ 

Vertex cover of size k:

$$\exists x_1 \cdots \exists x_k (\forall y \forall z (E(y,z) \Rightarrow (\bigvee_{1 \leq i \leq k} y = x_i \vee \bigvee_{1 \leq i \leq k} z = x_i)$$

Vertex Cover is FPT

Independent Set:

$$\exists x_1 \cdots \exists x_k (\bigwedge_{i<j} \neg E(x_i, x_j))$$

Independent Set is $W[1]$-complete

Dominating Set:

$$\exists x_1 \cdots \exists x_k \forall y (\bigwedge_i x_i \neq y \Rightarrow \bigvee_i E(x_i, y))$$

Dominating Set is $W[2]$-complete.

Restricted Classes

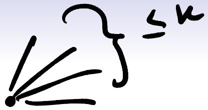
One way to get a handle on the complexity of first-order satisfaction is to consider restricted classes of structures.

Given: a first-order formula φ *and a structure* $\mathbb{A} \in \mathcal{C}$
Decide: if $\mathbb{A} \models \varphi$

For many interesting classes $\mathcal{C}$, this problem has been shown to be FPT.

The theorem of **(Courcelle 1990)** shows this for $\mathcal{T}_k$—the class of graphs of tree-width at most k, even for MSO.

Bounded Degree



$\mathcal{D}_k$—the class of structures $\mathbb{A}$ in which every element has at most k neighbours in $G\mathbb{A}$.

$O(f(k) \cdot n)$

Theorem (Seese)

For every sentence φ of FO and every k there is a linear time algorithm which, given a structure $\mathbb{A} \in \mathcal{D}_k$ determines whether $\mathbb{A} \models \varphi$.

Note: this is not true for MSO unless P = NP.

The proof is based on *locality* of first-order logic. Specifically, *Hanf's theorem*.

Motivations: why do we care about logic? Meta-Algorithms

Logical characterisation of problems leads to meta-algorithms:

Any property of "graphs" expressible in logic $\mathcal{L}$ is linear-time checkable on graphs from the class $\mathcal{C}$.

Theorem (Courcelle 1990)

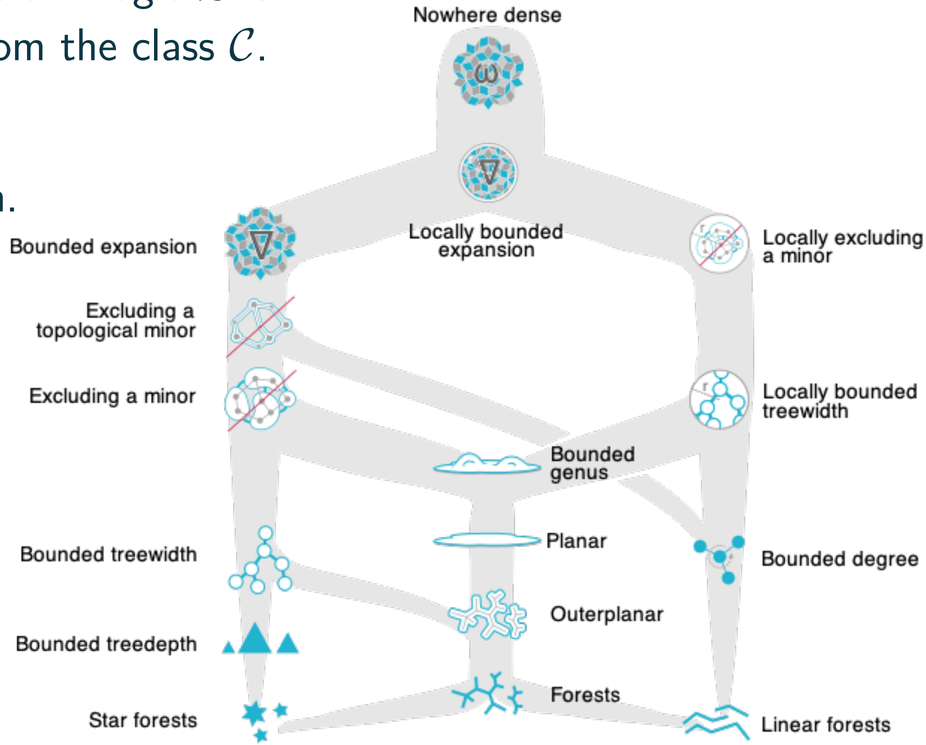
$\mathcal{L}$ = MSO, $\mathcal{C}$ = bounded-treewidth.

Theorem (Seese 1996)

$\mathcal{L}$ = FO, $\mathcal{C}$ = bounded-degree.

Theorem (Dvorák et al. 2010)

$\mathcal{L}$ = FO, $\mathcal{C}$ = bounded-expansion.



Picture by © Felix Reidl. No changes have been made.

Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(n^{1+\varepsilon})$ algorithms for $\mathcal{L}$ = FO and $\mathcal{C}$ = nowhere-dense graphs.